Homework 2

Due: 10/25, 18:00

1. (**Substitution and Nonexact Differential Equation Made Exact**) [10]
Solve
$$\frac{dy}{dx} = 2 - 2e^{y} + 3e^{2x+y},\ y(0) = 0.$$

Bonus. Solve $\frac{dy}{dx} = 2 - 2e^{y} + 3e^{x+y}$, $y(0) = 0$. [10]

2. (**Method of Substitution**) [20]
Solve

(a) [10]
$$\frac{dy}{dx} = \frac{2}{x} + \left(3 - \frac{1}{x}\right) y + xy^2.$$

(b) [10]
$$\frac{dy}{dx} = 2e^{x^2} + (2x+3)y + e^{-x^2}y^2,\ y(0) = 1.$$

Hint: Choose appropriate $f(x)$ and use the substitution $u = f(x)y$ to convert the equation to the form $u' = P(u)$, where $P(u)$ is a polynomial of u.

3. (**General Solution of Homogenous Linear Differential Equations**) [10]
Find the general solutions of the following:

(a) [5]
$$y^{(4)} - 6y''' + 15y'' - 18y' + 10y = 0.$$

(b) [5]
$$(x-1)^2 y'' + (x-1)y' + 4y = 0.$$

4. (**An IVP of Homogeneous Linear DE with Constant Coefficients**) [15]
Consider the following IVP:

$$\begin{aligned} &\text{Solve} && y^{(4)} + 4y = 0 \\ &\text{subject to} && y(x_0) = 1,\ y'(x_0) = r,\ y''(x_0) = r^2,\ y(x_0) = r^3 \end{aligned}$$

(a) Find the 4 *complex* roots for the polynomial $D^4 + 4$: m_1, m_2, m_3, m_4, where $m_2 = m_1^*$, $m_4 = m_3^*$. [5]

(b) From the lecture we know that $\{e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, e^{m_4 x}\}$ is a fundamental set of solutions in the complex domain $\mathbb{C}$. Hence the general solution in the complex domain can be represented as

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x}, \;\; C_i \in \mathbb{C}, \;\; i = 1, 2, 3, 4. \qquad (1)$$

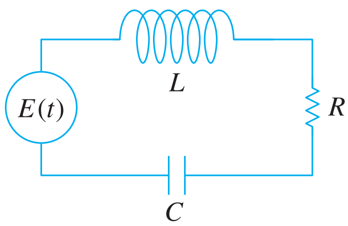
Please give the necessary and sufficient condition for y being a real-valued function, in terms of the relationships among $\{C_1, C_2, C_3, C_4\}$. [5]

(c) Use the form in (1) to find out the unique solution of the IVP. [5]

Hint: Use Cramer's Rule to solve $\{C_1, C_2, C_3, C_4\}$, and use the following fact:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i).$$

5. (**Method of Undetermined Coefficients**) [10]



Consider the above LRC series circuit. Recall from Chapter 1 that the voltage drop across the three elements are $L\frac{dI}{dt}$, IR, and $\frac{q}{C}$ respectively. Using the fact that $I = \frac{dq}{dt}$ and Kirchhoff's Law, we have

$$Lq'' + Rq' + q/C = E(t).$$

Suppose $L = 0.25$, $R = 1$, $C = 0.8$, $E(t) = e^{-t}\sin 10t + 2e^{-2t}\cos t$, $q(0) = q_0$, $I(0) = 0$. Find the current $I(t)$.

6. (**Variation of Parameters**) [10]

Find the general solution of the following DE:

$$y'' + y' - 2y = xe^{x^2}.$$

Bonus. (**Reduction of Order Two Times**) [10]

Consider a homogeneous linear third-order differential equation

$$(x^3 + 3x^2 - 3x + 1)y''' - 3(x^2 + 2x - 1)y'' + 6(x + 1)y' - 6y = 0.$$

(a) Verify that $f_1(x) = x + 1$ and $f_2(x) = x^2 + 1$ are both solutions to the above DE.

(b) Use the substitution $y = f_1(x)u_1(x)$ to convert the original DE into a second-order DE of $v_1 := u_1'$. Write down this DE, and verify that $\left(\frac{f_2(x)}{f_1(x)}\right)'$ is a solution to it.

(c) Use reduction of order to find another linearly independent solution to the derived second-order DE.

(d) From (c) derive a third solution $f_3(x)$ of the original third-order DE so that $\{f_1, f_2, f_2\}$ are linearly independent.