NOTE 4: DYNAMICAL DEGREES AND ENTROPY

In this lecture, we will introduce and discuss some dynamical-system-theoretic invariants associated to an endomorphism $f : X \to X$ of a compact Kähler manifold X such as the dynamical degree and the entropy. Before we start, we first remark that such an endomorphism, if surjective, is always finite.

Proposition 1. *A surjective endomorphism $f : X \to X$ of a compact Kähler manifold is finite.*

Proof. Since $f : X \to X$ is a surjective endomorphism, it is generically finite. So $f_* \circ f^* : H^\bullet(X, \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$ is an isomorphism (which is the multiplication by $\deg(f)$). It follows that $f_* : H^\bullet(X, \mathbf{Q}) \to H^\bullet(X, \mathbf{Q})$ is surjective, hence an isomorphism. If f is not finite. Then there exists a subvariety $Y \subset X$ such that $\dim Y > \dim f(Y)$. So $f_*[Y] = 0$ and therefore $[Y] = 0$, which is impossible because X is a compact Kähler manifold. □

1. Dynamical degrees: definition

Let $f : X \to X$ be a surjective endomorphism of a compact Kähler manifold. The topological degree of f is defined to be the cardinal of $f^{-1}(x)$ where x is a general point of X. This notion can be generalized to the notion of dynamical degree which we now define. Fix a Kähler class ω of X and let $0 \le p \le \dim X$ be an integer. Let

$$\delta_p(f) = \int_X f^*\omega^p \wedge \omega^{n-p}.$$

The *p-th dynamical degree* is defined to be

$$d_p(f) = \lim_{k\to\infty} \left(\delta_p(f^k)\right)^{\frac{1}{k}}$$

where f^k denotes the k-th iterate of f. The above limit exists thanks to the following lemma.

Lemma 2. *There exists $C > 0$ which depends only on X and ω such that for all surjective endomorphisms $f, g : X \to X$, we have*

$$\delta_p(f \circ g) \le C\delta_p(f) \cdot \delta_p(g).$$

Exercise 3. *Here are some examples of dynamical degrees.*

i) Show that $d_0(f) = 1$.

ii) Show that if $p = \dim X$, then d_p is the topological degree of f.

iii) Let $f : \mathbf{P}^n \to \mathbf{P}^n$ be an endomorphism of $\mathbf{P}^n$. Then there exists homogeneous polynomials $f_0, \dots, f_n$ in $n+1$ variables of degree d such that $f(x) = [f_0(x), \dots, f_n(x)]$ and we call d the algebraic degree *of f. Show that $d = d_1(f)$. (Hint: what is the degree of the pre-image of a hyperplane of $\mathbf{P}^n$?)*

In order to prove Lemma 2, we shall first introduce the notion of smooth positive (k, k)-forms on a compact Kähler manifold and recall some of their basic properties. The reader is referred to [2, Chapter III.1 and III.2] for further details, in which the more general (and natural) notion of positive currents is defined. For the purpose of our lecture, we only need to work with smooth positive (k, k)-forms.

Let X be a complex manifold and u a smooth (k, k)-form on X. We say that u is *strongly positive* if for every $p \in X$, u_p lies in the convex cone generated by

$$(i\theta_1 \wedge \overline{\theta_1}) \wedge \cdots \wedge (i\theta_k \wedge \overline{\theta_k})$$

where each θ_l is a linear form on $T_{X,p} \otimes \mathbf{C}$ of type $(1, 0)$ (here, $T_{X,p}$ is the real tangent space at $p \in X$ of X as a smooth manifold). A smooth (l, l)-form v on X is called *positive* if for every strongly positive (k, k)-form u on X with $k + l = \dim X$, there exists a continuous function $f : X \to \mathbf{R}_{\ge 0}$ such that

$$u \wedge v = f \cdot \mathrm{vol}_X$$

where vol_X is a volume form of X. Let

$$(\mathrm{S})\mathrm{Pos}^k(X) = \left\{[\alpha] \in H^{k,k}(X, \mathbf{R}) \mid \alpha \text{ is a closed (strongly) positive } (k,k)\text{-form}\right\}.$$

The subsets $\mathrm{SPos}^k(X)$ and $\mathrm{Pos}^k(X)$ are both closed convex cone of $H^{k,k}(X, \mathbf{R})$ with nonempty interior and which does not contain any nonzero linear subspace. Clearly, $\mathrm{SPos}^k(X) \subset \mathrm{Pos}^k(X)$ and the wedge products of Kähler forms lie in the interior of $\mathrm{SPos}^k(X)$. Also, both $\mathrm{SPos}^k(X)$ and $\mathrm{Pos}^k(X)$ are preserved under proper pushforwards.

Lemma 4. *For every $\alpha \in \mathrm{Pos}^{n-k}(X)$, let $\|\alpha\| = \int_X \alpha \wedge \omega^k$. There exists $C > 0$ (which depends only on X) such that*

$$\|f_*\alpha\| \leq C\|\alpha\| \cdot \delta_k(f).$$

Proof. By definition, $\mathrm{Pos}^{n-k}(X)$ is in the dual (by the Poincaré duality) of $\mathrm{SPos}^k(X)$ in $H^{k,k}(X, \mathbf{R})$. For every Kähler class ω, since ω^k is in the interior of $\mathrm{SPos}^k(X)$, we have $\|\alpha\| = \int_X \alpha \wedge \omega^k > 0$ for every $\alpha \in \mathrm{Pos}^{n-k}(X)$. If follows that the subset of $\mathrm{Pos}^{n-k}(X)$ consisting of elements $\beta \in \mathrm{Pos}^{n-k}(X)$ such that $\|\beta\| = 1$ is bounded, so there exists $C \gg 0$ such that

$$\omega^{n-k} - \frac{1}{C}\frac{\alpha}{\|\alpha\|} \in \mathrm{Pos}^{n-k}(X)$$

for every $\alpha \in \mathrm{Pos}^{n-k}(X)$. Thus $C\|\alpha\| \cdot f_*(\omega^{n-k}) - f_*\alpha \in \mathrm{Pos}^{n-k}(X)$, so

$$\|f_*\alpha\| = \int_X f_*\alpha \wedge \omega^k \leq C\|\alpha\| \left(\int_X f_*(\omega^{n-k}) \wedge \omega^k\right) = C\|\alpha\| \cdot \delta_k(f).$$

□

Proof of Lemma 2. Since $[\omega]^{n-k} \in \mathrm{Pos}^{n-k}(X)$, we have $g_*([\omega]^{n-k}) \in \mathrm{Pos}^{n-k}(X)$ as well. Applying Lemma 4 to $\alpha = g_*([\omega]^{n-k})$ yields

$$\delta_k(f \circ g) = \int_X f_* g_*(\omega^{n-k}) \wedge \omega^k \leq C\left(\int_X g_*(\omega^{n-k}) \wedge \omega^k\right)\delta_k(f) = C\delta_k(f) \cdot \delta_k(g).$$

□

Exercise 5. *Show that $d_p(f)$ is independent of the choice of the Kähler class ω.*

Remark 6. In the definition of dynamical degree, instead of $\delta_p(f) = \int_X f^*\omega^p \wedge \omega^{n-p}$, if we define

$$\delta_p(f) = \int_X f^*(\omega_1 \wedge \cdots \wedge \omega_p) \wedge (\omega_{p+1} \wedge \cdots \wedge \omega_n)$$

where the ω_i's are Kähler forms, then the limit $\lim_{k\to\infty} \left(\delta_p(f^k)\right)^{\frac{1}{k}}$ still exists and coincides with $d_p(f)$ by the same type of argument.

2. The sequence of dynamical degrees is log-concave

Let S be a smooth projective surface. The Hodge index theorem for S implies that if C and D are two nef divisors on S, then

$$(C \cdot D)^2 \geq (C^2)(D^2).$$

Here we state a generalization of the above inequality for compact Kähler manifolds of arbitrary dimension and refer to [3, Section 5] for a proof. The projective case was due to Khovanskii and Teissier and the general case due to Demailly.

Theorem 7. *Let X be a compact Kähler manifold of dimension n. For all $\omega_1, \ldots, \omega_n \in \overline{\mathscr{K}(X)}$, we have*

$$\omega_1 \wedge \cdots \wedge \omega_n \geq (\omega_1^n)^{\frac{1}{n}} \cdots (\omega_n^n)^{\frac{1}{n}}.$$

The following is a particular case of the Khovanskii-Teissier-Demailly inequality:

Corollary 8. *Let ω_1 and ω_2 be two nef classes on a compact Kähler manifold X of dimension n. If we define $\delta_p = \omega_1^p \wedge \omega_2^{n-p}$, then we have the following log-concave inequality:*

$$\delta_p^2 \geq \delta_{p-1} \cdot \delta_{p+1}.$$

It follows from Corollary 8 that the sequence $\{d_p(f)\}$ of dynamical degrees is log-concave:

Corollary 9. *Let $f : X \to X$ be an endomorphism of a compact Kähler manifold. We have*

$$d_p(f)^2 \geq d_{p-1}(f) \cdot d_{p+1}(f).$$

Exercise 10. *Note that since $\delta_p(f) \geq 0$, we have $d_p(f) \geq 0$. Show that $d_p(f) \geq 1$ for every p. (Hint: Compute $d_0(f)$ then use the log-concavity of $d_p(f)$ to conclude.)*

3. Dynamical degrees as spectral radii and comparison to the logarithmic volume growth

Let $f : X \to X$ be an endomorphism of a compact Kähler manifold X. Since $f^* : H^\bullet(X, \mathbf{C}) \to H^\bullet(X, \mathbf{C})$ is a morphism of Hodge structures, we have $f^*(H^{p,p}(X)) \subset H^{p,p}(X)$ for every p. Let $r_p(f)$ denote the spectral radius of $f^*_{|H^{p,p}(X)}$.

Proposition 11. *Let $f : X \to X$ be an endomorphism of a compact Kähler manifold X. We have $d_p(f) = r_p(f)$.*

Proof. Fix a Kähler class ω on X. Choose a norm $N : H^{p,p}(X) \to \mathbf{R}$ such that for every $\lambda \in H^{p,p}(X)$, we have

$$N(\lambda) \geq \left| \int_X \lambda \wedge \omega^{\dim X - p} \right|.$$

Then

$$r_p(f) = \lim_{k\to\infty} \sup_{\lambda \in H^{p,p}(X)} N\Big((f^k)^*\lambda\Big)^{\frac{1}{k}} \geq \lim_{k\to\infty} \left(\int_X (f^k)^*\omega^p \wedge \omega^{\dim X - p} \right)^{\frac{1}{k}} = d_p(f).$$

To prove that $r_p(f) \leq d_p(f)$, let $\{e_i\}$ (resp. $\{f_i\}$) be a basis of $H^{p,p}(X, \mathbf{R})$ (resp. $H^{n-p,n-p}(X, \mathbf{R})$) such that $e_i \in \mathrm{SPos}^p(X)$ (resp. $f_i \in \mathrm{Pos}^{n-p}(X)$) for every i. Let $C, C' > 0$ such that $C \cdot \omega^p - e_i \in \mathrm{SPos}^p(X)$ and $C' \cdot \omega^{n-p} - f_i \in \mathrm{Pos}^{n-p}(X)$ for every i. Then

$$\left| \int_X (f^k)^* e_i \wedge f_j \right| = \int_X (f^k)^* e_i \wedge f_j \leq C \cdot C' \int_X (f^k)^* \omega^p \wedge \omega^{n-p}$$

for every i and j. Therefore $r_p(f) \leq d_p(f)$. □

We can also compare the dynamical degrees with the growth of the volume of the graph of f^k. Fix a Kähler metric ω on X and let X^k be endowed with the metric $\omega_k := \mathrm{pr}_1^*\omega \oplus \cdots \oplus \mathrm{pr}_k^*\omega$. Let

$$\Gamma_k = \left\{ \Big(x, f(x), \dots f^{k-1}(x)\Big) \mid x \in X \right\} \subset X^k,$$

and let

$$\mathrm{lov}(f) = \limsup_{k\to\infty} \frac{1}{k} \log \mathrm{vol}(\Gamma_k)$$

where the volume is defined with respect to the metric ω_k.

Exercise 12. *Let $f : X \to X$ be an endomorphism of a compact Kähler manifold. Show that*

$$\mathrm{lov}(f) = \max_{0 \leq p \leq n = \dim X} \log d_p(f).$$

Hint: First show that

$$\mathrm{vol}(\Gamma_k) = \frac{1}{n!} \int_{\Gamma_k} \omega_k^n = \frac{1}{n!} \sum_{0 \leq j_1, \dots, j_n \leq k-1} \int_X (f^{j_1})^*\omega \wedge \cdots \wedge (f^{j_n})^*\omega$$

and therefore $\mathrm{lov}(f) \geq d_p(f)$ for every $0 \leq p \leq \dim X$. To prove the other inequality, prove by induction that for every $\varepsilon > 0$, there exists $c > 0$ such that for every $j_1 \geq \cdots \geq j_n \geq 0$,

$$\int_X (f^{j_1})^*\omega \wedge \cdots \wedge (f^{j_n})^*\omega \leq c \left(\max_{0 \leq p \leq n} d_p + \varepsilon \right)^{j_1}.$$

Remark 13. The dynamical degrees can be defined more generally for meromorphic dominant self-maps $f : X \dashrightarrow X$ of a compact complex manifold and in this case, we still have $\mathrm{lov}(f) = \max_{0 \leq p \leq \dim X} \log d_p(f)$ [4].

4. Topological entropy

Let (X, d) be a compact metric space and $f : X \to X$ a continuous map (for the topology induced by d). For any positive integer n and any pair of points $x, y \in X$ define

$$d_f^n(x, y) = \max_{i=0,\ldots,n} d\left(f^i(x), f^i(y)\right),$$

which is a metric on X. For every $\varepsilon > 0$, let $N(f, n, \varepsilon)$ be the minimal number of balls of radius ε for the metric d_f^n covering X. Let

$$h(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log N(f, n, \varepsilon).$$

The topological entropy is defined to be

$$h(f) = \lim_{\varepsilon\to 0} h(f, \varepsilon) \in \mathbf{R} \cup \{\infty\}.$$

The limit exists because $\varepsilon \mapsto h(f, \varepsilon)$ is non-decreasing. Roughly speaking, a map f has high entropy if for every pair of points $x, y \in X$ which are closed to each other, the growth of the distance between them after iterations of f is fast.

Exercise 14. *Prove the following statements:*

i) The topological entropy $h(f)$ does not depend on the metric d if the induced topology is the same.
ii) If $Y \subset X$ is a subset such that $f(Y) = Y$, then $h(f_{|Y}) \le h(f)$.
iii) If there exist a surjective continuous map $\phi : X \to B$ to another metric space B such that f descends to a continuous map $g : B \to B$ (namely, there exists $g : B \to B$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ B & \xrightarrow{g} & B \end{array}$$

is commutative), then $h(g) \le h(f)$.
iv) If f is of finite order, then $h(f) = 0$.
v) Let X and Y be metric spaces. Given continuous maps $f : X \to X$ and $g : Y \to Y$, we have $h(f\times g) = h(f)+h(g)$.

In the Kähler situation, we have the following upper bound of the entropy due to Gromov.

Proposition 15 (Gromov [6])**.** *Let $f : X \to X$ be a holomorphic map of a compact Kähler manifold X. We have*

$$h(f) \le \mathrm{lov}(f).$$

In particular, $h(f)$ is always finite.

Sketch of the proof. Fix a Kähler metric ω and endow X^n with the product metric. Let $\mathrm{dens}(f, n, \varepsilon) := \liminf_{z\in\Gamma_n} \mathrm{vol}(B(z, \varepsilon) \cap \Gamma_n)$ and

$$\mathrm{ldens}(f) := \lim_{\varepsilon\to 0} \lim_{n\to\infty} \left(\frac{1}{n} \log \mathrm{dens}(f, n, \varepsilon)\right).$$

Since $\mathrm{vol}(\Gamma_n) \ge N(f, n, 2\varepsilon) \cdot \mathrm{dens}(f, n, \varepsilon)$, we have

$$\mathrm{lov}(f) \ge h(f) + \mathrm{ldens}(f),$$

so it suffices to prove that $\mathrm{ldens}(f) \ge 0$. To this end, we prove the auxiliary result that for every $K > 0$, $\varepsilon > 0$, and $n \in \mathbf{Z}_{>0}$, there exists a constant $C > 0$ such that $\mathrm{vol}(B_\varepsilon \cap V) \ge C$ for every Riemannian manifold M with sectional curvature $\le K$ and every minimal submanifold (in the sense that they have vanishing mean curvature) $V \subset M$ of dimension n. Since complex submanifolds of a Kähler manifold are minimal, we can apply the above result and obtain that $\lim_{n\to\infty} \left(\frac{1}{n} \log \mathrm{dens}(f, n, \varepsilon)\right) \ge 0$. □

Every continuous map $f : X \to X$ induces a linear map $f^* : H^\bullet(X, \mathbf{C}) \to H^\bullet(X, \mathbf{C})$ by pulling back cohomological classes. Let $r(f)$ denote the spectral radius of f^*. In the Kähler situation, we have a lower bound of $h(f)$ in terms of $r(f)$ due to Gromov and Yomdin.

Theorem 16 (Gromov-Yomdin [5])**.** *Let X be a compact Kähler manifold and $f : X \to X$ a holomorphic map. Then*

$$h(f) \geq \log r(f).$$

Combining Proposition 11, Exercise 12, Proposition 15, and Theorem 16, we obtain a chain of equalities for $h(f)$ and the upper bound and the lower bound of $h(f)$ mentioned earlier turn out to coincide.

Corollary 17. *Let X be a compact Kähler manifold and $f : X \to X$ a surjective holomorphic map. Then*

$$h(f) = \mathrm{lov}(f) = \max_{0\leq p\leq \dim X} \log d_p(f) = \max_{0\leq p\leq \dim X} \log r_p(f) = \log r(f).$$

Exercise 18.

i) *What is the entropy of the automorphism of a curve?*

ii) *Let $f : X \to X$ be a surjective endomorphism of a compact Kähler manifold. Show that $h(f) > 0$ if and only if $d_1(f) > 1$. In particular, an automorphism $f : X \to X$ of a projective manifold with $\rho(X) = 1$ has vanishing entropy.*

Remark 19. If $f : X \dashrightarrow X$ is only a dominant meromorphic map, then by [4]

$$h(f_{|U}) \leq \mathrm{lov}(f) = \max_{0\leq p\leq \dim X} d_p(f).$$

where $U = X \backslash \cup_{k\in \mathbf{Z}} f^k(I_f)$ and $I_f \subset X$ is the indeterminacy locus of f. The dynamical degrees $d_p(f)$ are birational invariants but not $h(f)$ [7]. In particular, the above inequality can be strict.

5. Endomorphisms fixing a Kähler ray

The work of Gromov and Yomdin (or more precisely Corollary 17) provides a way to compute the entropy of an endomorphism of a compact Kähler manifold and in some cases, the computation is easy. As an example, we compute the entropy of an endomorphism fixing a Kähler ray.

Proposition 20. *Let X be a compact Kähler manifold and $f : X \to X$ a surjective holomorphic map such that $f^*\omega = q\omega$ for some Kähler class ω and some real number $q > 0$. Then f is finite and* $\deg f = q^{\dim X}$ *(in particular, $q > 1$). Moreover, $h(f) = \dim X \cdot \log q$.*

Proof. First we prove the following statement observed by Serre.

Lemma 21 (Serre)**.** *Under the same hypothesis of the proposition, the absolute value of the eigenvalues of $f^* : H^k(X,\mathbf{C}) \to H^k(X,\mathbf{C})$ is $q^{k/2}$.*

Proof. Let Q be a bilinear form on $H^k(X,\mathbf{C})$ defined by

$$Q_k(\alpha,\beta) = \int L^{n-k}(\alpha) \wedge \beta = \int \omega^{n-k} \wedge \alpha \wedge \beta.$$

and let

$$H_k(\alpha,\beta) = \begin{cases} \sqrt{-1} \cdot Q_k(\alpha,\bar\beta) \text{ if } k \text{ is odd;} \\ Q_k(\alpha,\bar\beta) \text{ if } k \text{ is even.} \end{cases}$$

Then H_k is a Hermitian form. Let $g_k : H^k(X,\mathbf{C}) \xrightarrow{f^*} H^k(X,\mathbf{C}) \xrightarrow{\cdot q^{-k/2}} H^k(X,\mathbf{C})$. Then $H_k(g_k(\alpha), g_k(\beta)) = H_k(\alpha,\beta)$ and g_k preserves the Hodge decomposition and the Lefschetz decomposition. As these two decompositions are orthogonal with respect to H_k and the restriction of H_k to each of the summands $H^{p,q}_{\mathrm{prim}}$ is either definite positive or definite negative, the restriction of g_k to $H^{p,q}_{\mathrm{prim}}$ is a unitary transformation of $H^{p,q}_{\mathrm{prim}}$. Therefore the eigenvalues of g_k has absolute value 1, which proves the lemma. □

Therefore $\deg f = q^{\dim X}$. As $q > 1$, by Corollary 17 we have

$$h(f) = \max_{0\leq k\leq \dim X} \log q^k = \dim X \cdot \log q.$$

□

Remark 22. By Proposition 20, we see that if $f : X \to X$ is an endomorphism preserving a Kähler ray, then $h(f) = 0$ if and only if f is an automorphism. The "if" direction also follows from Fujiki-Lieberman's theorem together with Corollary 17. The "only if" part has a far reaching generalization which holds for every smooth compact oriented manifold due to Misiurewicz and Przytycki [8, Theorem 8.3.1]: they showed that for every $f : X \to X$ self-map of class $\mathscr{C}^1$ of a smooth compact oriented manifold, we have $h(f) \geq |\log \deg(f)|$.

Remark 23. Let $f : X \to X$ be an endomorphism of a compact Kähler manifold. By Proposition 20, if f is an automorphism and $r_1(f) \neq 1$, then f^* has no eigenvectors in $\mathscr{K}(X)$. However since f^* preserves $\mathscr{K}(X)$, according to a Perron-Frobenius-type theorem [1], we can always find $\alpha \in \overline{\mathscr{K}(X)} \backslash \{0\}$ such that $f^*\alpha = r_1(f)\alpha$. If $r_1(f) \neq 1$ (or equivalently, $h(f) > 0$ by Exercise 18), then necessarily $\alpha \in \partial \overline{\mathscr{K}(X)}$.

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