NOTE 3: PSEUDO-AUTOMORPHISMS

1. Definition and basic properties

Let $f : X \dashrightarrow X'$ be a bimeromorphic map. We say that f is a *pseudo-isomorphism* if f is isomorphic in codimension 1 (that is, there exist analytic closed subsets $Z \subset X$ and $Z' \subset X'$ of codimension 2 such that $f_{|X-Z}$ is an isomorphism onto $X' - Z'$). Obviously, a pseudo-isomorphism of curves is a biholomorphic map. For surfaces it is also the case, but the proof is slightly less obvious.

Proposition 1. *A pseudo-isomorphism $f : X \dashrightarrow Y$ between (smooth) compact complex surfaces is a biholomorphic map.*

Proof. Let $\Gamma \subset X \times Y$ be the graph of f. By definition, the projection $p : \Gamma \to X$ is biholomorphic outside of a finite number of points of Γ and X. In other words, p is finite and generically injective. Suppose that $x, y \in \Gamma$ are two distinct points such that $p(x) = p(y)$, then x and y have disjoint neighborhood U and V in Γ such that $p_{|U}$ and $p_{|V}$ are injective. Since X is smooth, by [3, p. 166, Theorem][1] both $p_{|U}$ and $p_{|V}$ are open, so there exists a neighborhood W of $p(x) = p(y)$ such that $p^{-1}(w)$ contains at least two points for every $w \in W$, which contradicts the property that p is generically injective. Therefore p is bijective. Since X is smooth, again by [3, p. 166, Theorem] p is a biholomorphic map. The same argument shows that the other projection $\Gamma \to Y$ is also biholomorphic, hence $f : X \dashrightarrow Y$ is a biholomorphic map. □

Therefore the notion of pseudo-isomorphisms is only interesting for manifolds of dimension 3 and on. Non-trivial flops are examples of pseudo-isomorphisms which are not holomorphic.

Example 2 (Atiyah flop)**.** Let $V \subset \mathbf{A}^4$ be the variety defined by the equation $XY - ZW = 0$ and let $\tilde{V}$ be the blow-up to V at the origin. The exceptional locus $E \subset \tilde{V}$ is $\mathbf{P}^1 \times \mathbf{P}^1$ and the two projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ can both be extended to contractions $V \to V_1$ and $V \to V_2$ to smooth complex threefolds V_1 and V_2. The induced bimeromorphic map $V_1 \dashrightarrow V_2$ is a pseudo-isomorphism which is not holomorphic.

While there exist bimeromorphic holomorphic maps which are not biholomorphic, every holomorphic pseudo-isomorphism between compact complex manifolds is necessarily biholomorphic.

Lemma 3. *Let X be a compact complex manifold and $f : X \dashrightarrow X'$ a pseudo-isomorphism. If f is holomorphic, then f is biholomorphic.*

Proof. Let $Z \subset X$ be a subvariety of codimension at least 2 such that $f_{|X-Z}$ is biholomorphic onto its image. The determinant of the differential $df : T_X \to f^*T_{X'}$ induces a section σ of the line bundle $\omega_X \otimes f^*\omega_{X'}^{\vee}$, which doesn't vanish outside of Z. As $\mathrm{codim}_X Z \geq 2$, the section σ vanishes nowhere. Therefore by the holomorphic local inversion theorem, f is locally biholomorphic. Since $\deg(f) = 1$ and X is compact, f is a biholomorphic map. □

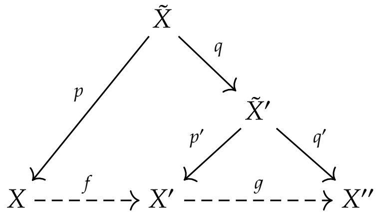
2. Pseudo-isomorphisms and H^2

In the previous lecture, we saw that a bimeromorphic map $f : X \dashrightarrow X'$ induces an isomorphism between $H^1(X, \mathbf{Z})$ and $H^1(X', \mathbf{Z})$. The next result shows that if $f : X \dashrightarrow X'$ is a pseudo-isomorphism, then it induces an isomorphism between $H^2(X, \mathbf{Z})$ and $H^2(X', \mathbf{Z})$.

Lemma 4. *A pseudo-isomorphism $f : X \dashrightarrow X'$ of compact complex manifolds induces an isomorphism $f_* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ and an isomorphism $f_* : \mathrm{NS}(X) \xrightarrow{\sim} \mathrm{NS}(X')$. Moreover, f_* is functorial in the sense that $(g \circ f)_* = g_* \circ f_*$. Finally if both X and X' are Kähler, then $f_* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is an isomorphism of Hodge structures.*

[1] The statement says that if $f : X \to Y$ is an injective holomorphic map from a reduced complex space X of pure dimension to a normal complex variety Y such that $\dim X = \dim Y$, then f maps biholomorphically onto $f(X)$.

Proof. Let
$$X \xleftarrow{p} \tilde{X} \xrightarrow{q} X'$$
be a resolution of f by a compact complex manifold $\tilde{X}$. The pushforward f_* is defined to be
$$f_* = q_* \circ p^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z}),$$
which is independent of the resolutions of f. To show that f_* is functorial, let $g : X' \dashrightarrow X''$ be another pseudo-isomorphism of compact complex manifolds and let
$$X' \xleftarrow{p'} \tilde{X}' \xrightarrow{q'} X''$$
be a resolution of g with p' a composition of blow-ups of X' along complex submanifolds. Choose



to be a resolution of $g \circ f$ where $\tilde{X}$ is a compact complex manifold. By the blow-up formula, there exist (effective) divisors $E_1, \dots, E_l$ of $\tilde{X}'$ contained in the exceptional locus of p' such that
$$H^2(\tilde{X}', \mathbf{Z}) = p'^* H^2(X', \mathbf{Z}) \oplus \sum_{i=1}^{l} \mathbf{Z}[E_i].$$
Since g is a pseudo-isomorphism and each E_i is contained in the exceptional locus of q', we have $\dim q'(E_i) \le \dim X'' - 2$. It follows that $q'_*[E_i] = 0$, so $q'_* \circ p'^* \circ p'_* = q'_*$. Therefore
$$g_* \circ f_* = (q'_* \circ p'^*) \circ (p'_* \circ q_* \circ p^*) = q'_* \circ q_* \circ p^* = (g \circ f)_*$$
which proves that f_* is functorial.

It follows from the functoriality that $(f^{-1})_* : H^2(X', \mathbf{Z}) \to H^2(X, \mathbf{Z})$ is the inverse of f_*, in particular $f_* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is an isomorphism. The same argument shows that the morphism $f_* : \mathrm{NS}(X) \to \mathrm{NS}(X')$ defined in a similar way is an isomorphism as well. Finally by construction, $f_* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is a morphism of Hodge structures if both X and X' are Kähler. □

Remark 5. To prove that $(g \circ f)_* = g_* \circ f_*$, we didn't use the assumption that f is a pseudo-isomorphism: the same equality holds if f is merely bimeromorphic.

Exercise 6.

i) In general if $f : X \dashrightarrow X'$ is only a meromorphic map, then we can also define $f_* : H^k(X, \mathbf{Z}) \to H^k(X', \mathbf{Z})$ by the same way we did in the above proof. Find examples of bimeromorphic *maps $f : X \dashrightarrow X'$ and $g : X' \dashrightarrow X''$ such that*
$$(g \circ f)_* \neq g_* \circ f_* : H^2(X, \mathbf{Z}) \to H^2(X'', \mathbf{Z}).$$

ii) Find a pseudo-isomorphism f such that f_* does not preserve the intersection product.

While a biholomorphic map preserves Kähler cones, the following theorem shows that the situation is completely opposite for non-isomorphic pseudo-isomorphisms f: the image $f_*(\mathscr{K})$ of the Kähler cone $\mathscr{K}$ is always disjoint from $\mathscr{K}$.

Theorem 7 (Fujiki [2]). *Let $f : X \dashrightarrow X'$ be a pseudo-isomorphism between compact Kähler manifolds. If X has a Kähler class α such that $f_*\alpha$ is a Kähler class of X', then f is biholomorphic.*

Proof. Let $X \xleftarrow{p} \tilde{X} \xrightarrow{q} X'$ be a resolution of f by some compact Kähler manifold $\tilde{X}$. First we prove the following

Lemma 8. *For every Kähler class $\alpha \in H^2(X, \mathbf{R})$ such that $f_*\alpha$ is also a Kähler class, we have $p^*\alpha = q^* f_*\alpha$.*

Proof. We may assume that p is a sequence of blow-ups of X along smooth centers. If $E_1, \dots E_l \subset \tilde{X}$ are the irreducible components of the exceptional locus of p, then

$$H^2(\tilde{X}, \mathbf{R}) = p^* H^2(X, \mathbf{R}) \oplus \sum_{i=1}^{l} \mathbf{R}[E_i].$$

Claim. *There exist $a_1, \dots, a_l > 0$ such that $-\sum_i a_i[E_i]$ is p-ample: namely for every curve C contracted by p,*

$$\left(-\sum_i a_i[E_i] \right) \cdot [C] > 0.$$

Proof. Up to reordering E_i, we may write p as a sequence of blow-ups along smooth centers

$$p : \tilde{X} = X_l \to X_{l-1} \to \cdots X_1 \to X$$

such that if E'_i is the exceptional divisor of $p_i : X_i \to X_{i-1}$, then E_i is the proper transformation of E'_i. Let ω_0 be a Kähler class on X, then there exists $a_1 > 0$ such that $p_1^* \omega_0 - a_1[E'_1]$ is a Kähler class. Repeating the same argument, we obtain a Kähler class $\omega := p^* \omega_0 - \sum_i a_i[E_i]$ for some $a_1, \dots, a_l > 0$. Since $p^* \omega_0 \cdot [C] = \omega_0 \cdot p_*[C] = 0$ for every curve C contracted by p, it follows that $-\sum_i a_i[E_i]$ is p-ample. □

As we have already seen, since f is a pseudo-isomorphism, the divisors E_i are also exceptional for q. So $q_*[E_i] = 0$ and since $q_* p^* : H^2(X, \mathbf{R}) \to H^2(X', \mathbf{R})$ is an isomorphism by Lemma 4, we also have a decomposition

$$H^2(\tilde{X}, \mathbf{R}) = q^* H^2(X', \mathbf{R}) \oplus \sum_{i=1}^{l} \mathbf{R}[E_i].$$

Given a Kähler class $\alpha \in H^2(X, \mathbf{R})$ such that $f_*\alpha \in H^2(X', \mathbf{R})$ is also a Kähler class, we write $p^*\alpha = q^* f_* \alpha + \sum_{i=1}^{l} b_i[E_i]$. To show that $p^*\alpha = q^* f_* \alpha$, it suffices to show that $b_i \geq 0$, since by exchanging the role of α and $f_*\alpha$ in the argument, we can also show that $-b_i \geq 0$, so $b_i = 0$ for every i. Without loss of generality we can assume that $b_1/a_1 \leq b_i/a_i$ for every i where $a_1, \dots, a_l$ are the real numbers in the above claim. Suppose to the contrary that $b_i < 0$ for some i, then since $a_i > 0$, we have $b_1/a_1 < 0$. If $C \subset E_1$ is a general irreducible curve contracted by p, then on the one hand we have $C \cdot E_i \geq 0$ for every $i \neq 1$ and since $-\sum_i a_i[E_i]$ is p-ample, we have

$$\sum_{i=1}^{l} b_i (C \cdot E_i) \geq \frac{b_1}{a_1} \sum_{i=1}^{l} a_i (C \cdot E_i) > 0.$$

On the other hand since $f_*\alpha$ is Kähler, we have $q^* f_* \alpha \cdot [C] = f_*\alpha \cdot q_*[C] \geq 0$, so

$$\sum_{i=1}^{l} b_i (C \cdot E_i) \leq q^* f_* \alpha \cdot [C] + \sum_{i=1}^{l} b_i (C \cdot E_i) = p^* \alpha \cdot [C] = \alpha \cdot p_*[C] = 0,$$

where we recall that $p_*[C] = 0$ because C is contracted by p, which leads to a contradiction. □

Now assume that $f : X \dashrightarrow X'$ is not holomorphic, then there exists a curve $C \subset \tilde{X}$ which is contracted by p but not by q. So

$$0 = p^*\alpha \cdot [C] = q^* f_* \alpha \cdot [C] = f_*\alpha \cdot q_*[C] > 0,$$

which is impossible. Hence f is holomorphic, and therefore biholomorphic by Lemma 3. □

3. K-equivalence

Let $f : X \dashrightarrow Y$ be a bimeromorphic map between complex manifolds. We say that f is a *K-equivalence* if there exists a resolution

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

of f by some complex manifold Z such that $p^* \omega_X = q^* \omega_Y$. K-equivalences are examples of pseudo-isomorphisms.

Proposition 9. *If $f : X \dashrightarrow Y$ is a K-equivalence, then f is a pseudo-isomorphism.*

Proof. Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be a resolution of f such that Z is a complex manifold and $p^*\omega_X = q^*\omega_Y$. As $p^*\omega_X = q^*\omega_Y$, we have

$$\omega_{Z/X} := \omega_Z \otimes p^*\omega_X^\vee = \omega_Z \otimes q^*\omega_Y^\vee =: \omega_{Z/Y}. \tag{3.1}$$

If $\{E_i\}_{i\in I}$ (resp. $\{E'_j\}_{j\in J}$) is the set of irreducible reduced divisors contracted by p (resp. q), then

$$\omega_{Z/X} = \mathscr{O}_Z\left(\sum_{i\in I} a_i E_i\right)$$

for some positive integers a_i and similarly $\omega_{Z/Y} = \mathscr{O}_Z\left(\sum_{j\in J} a'_j E'_j\right)$ for some $a'_j \in \mathbf{Z}_{>0}$.

Claim. *The direct image $p_*\omega_{Z/X}$ coincides with $\mathscr{O}_X$.*

Proof. Let $U \subset X$ be an open subset. It suffices to show that $H^0(U, \mathscr{O}_X) = H^0\left(p^{-1}(U), \omega_{Z/X}\right)$. For simplicity let $E = \sum_{i\in I} a_i E_i$. Obviously

$$H^0(U, \mathscr{O}_X) = H^0\left(p^{-1}(U), \mathscr{O}_Z\right) \subset H^0\left(p^{-1}(U), \mathscr{O}_Z(E)\right).$$

If $|E| = \cup_{i\in I} E_i$, then

$$H^0\left(p^{-1}(U), \mathscr{O}_Z(E)\right) \subset H^0\left(p^{-1}(U) - |E|, \mathscr{O}_Z\right) = H^0\left(U - p(|E|), \mathscr{O}_X\right) = H^0\left(U, \mathscr{O}_X\right)$$

where the last equality follows from Riemann's extension theorem because $\dim |E| \leq \dim X - 2$. □

The above claim implies that $h^0(Z, \omega_{Z/X}) = 1$, hence it follows from (3.1) that $\sum_{i\in I} a_i E_i = \sum_{j\in J} a'_j E'_j$. In particular $\{E_i\}_{i\in I} = \{E'_j\}_{j\in J}$, so the exceptional divisors of p are contracted by q and *vice versa*. Therefore f is a pseudo-isomorphism. □

Remark 10. The converse of Proposition 9 is false in general and the following example is taken from [1, 1.36]. Fix two positive integers r and s. Let

$$X = \mathbf{P}(\mathscr{O}_{\mathbf{P}^s} \oplus \mathscr{O}_{\mathbf{P}^s}(1)^{r+1}) \text{ and } X' = \mathbf{P}(\mathscr{O}_{\mathbf{P}^r} \oplus \mathscr{O}_{\mathbf{P}^r}(1)^{s+1}).$$

Let $\Sigma \subset X$ (resp. $\Sigma' \subset X'$) be the section of $X \to \mathbf{P}^s$ (resp. $X' \to \mathbf{P}^r$) corresponding to the trivial quotient. Then the blow-up of X along Σ is isomorphic to the blow-up of X' along Σ', which is

$$\tilde{X} := \mathbf{P}\left(\mathscr{O}_{\mathbf{P}^s\times\mathbf{P}^r} \oplus \mathscr{O}_{\mathbf{P}^s\times\mathbf{P}^r}(1,1)\right),$$

and the reductions of the exceptional divisors of the two blow-ups are the same, denoted by E. So on the one hand, the induced map $X \dashrightarrow X'$ is a pseudo-isomorphism, but on the other hand, $\omega_{\tilde{X}/X} = \mathscr{O}_{\tilde{X}}((s-1)E)$ and $\omega_{\tilde{X}/X'} = \mathscr{O}_{\tilde{X}}((r-1)E)$. Therefore whenever $r \neq s$, $X \dashrightarrow X'$ is not a K-equivalence.

Let D be a divisor of a compact Kähler manifold X. We call that D is *nef* if $c_1(\mathscr{O}(D)) \in H^2(X, \mathbf{R})$ is in the closure of the Kähler cone. The nefness of a divisor D implies that for every curve $C \subset X$, we have $D \cdot C \geq 0$ (the converse is true if X is projective).

Proposition 11. *Let $f : X \dashrightarrow X'$ be a bimeromorphic map between two compact Kähler manifolds. If the canonical divisors K_X and $K_{X'}$ are both nef, then f is a K-equivalence. In particular, f is a pseudo-isomorphism.*

Proof. The proof is similar to that of Lemma 8. Let $X \xleftarrow{p} \tilde{X} \xrightarrow{q} X'$ be a resolution of f such that p is a sequence of blow-ups of X along smooth centers. If $E_1, \ldots E_l \subset \tilde{X}$ are the irreducible components of the exceptional locus of p, then there exist positive integers b'_i such that

$$K_{\tilde{X}} = p^*K_X + \sum_{i\in I} b'_i E_i.$$

Similarly, there exists an effective divisor E' on $\tilde{X}$ such that $K_{\tilde{X}} = q^*K_{X'} + E'$. Let D be the largest common effective divisor contained in both $\sum_{i\in I} b'_i E_i$ and E', then the effective divisors $\sum_{i\in I} b_i E_i := \sum_{i\in I} b'_i E_i - D$ and $E := E' - D$ do not have any common irreducible component.

Assume that $\sum_{i\in I} b_i E_i \neq 0$. Recall from the proof of Lemma 8 that there exist $a_1, \ldots, a_l > 0$ such that $-\sum_i a_i E_i$ is p-ample. Without loss of generality we can assume that $b_1/a_1 \geq b_i/a_i$ for every i (so $b_1 > 0$).

Let $C \subset E_1$ be a general irreducible curve which is contracted by p. Then on the one hand we have $C \cdot E_i \geq 0$ for every $i \neq 1$ and since $-\sum_i a_i[E_i]$ is p-ample, we have

$$\sum_{i=1}^{l} b_i(C \cdot E_i) \leq \frac{b_1}{a_1} \sum_{i=1}^{l} a_i(C \cdot E_i) < 0.$$

On the other hand, since $C \subset E_1$ is a general curve and E_1 is not an irreducible component of E, it follows from the nefness of $K_{X'}$ that

$$\sum_{i=1}^{l} b_i(C \cdot E_i) = \left(p^*K_X + \sum_{i=1}^{l} b_i E_i\right) \cdot C = (q^*K_{X'} + E) \cdot C \geq 0,$$

which leads to a contradiction. Therefore $\sum_{i \in I} b_i E_i = 0$ and thus $p^*K_X = q^*K_{X'} + E$.

Now let $\tilde{X} \xleftarrow{r} \tilde{X}' \xrightarrow{\tilde{q}} X'$ be a resolution of the map $q^{-1} : X' \dashrightarrow \tilde{X}$ a a sequence of blow-ups $\tilde{q} : \tilde{X}' \to X'$ along smooth centers. Let $\tilde{p} = p \circ r$. By the same argument as above, there exists an effective divisor E'' on $\tilde{X}'$ such that $\tilde{q}^*K_X = \tilde{p}^*K_{X'} + E''$. Since $\tilde{p}^*K_X = \tilde{q}^*K_{X'} + r^*E$ and r^*E is effective, we have $r^*E = E'' = 0$. Hence $p^*K_X = q^*K_{X'}$. □

A *pseudo-automorphism* is a bimeromorphic self-map which is a pseudo-isomorphism. The group of pseudo-automorphisms of a complex manifold X is denoted by $\mathrm{PsAut}(X)$. We have the following immediate corollaries of Proposition 11.

Corollary 12. *Let X be a compact Kähler manifold. If K_X is nef (*e.g. *$c_1(X) \in H^2(X, \mathbf{Z})$ is torsion), then* $\mathrm{Bir}(X) = \mathrm{PsAut}(X)$.

In particular, combining Corollary 12 and Proposition 1 we obtain the following for surfaces.

Corollary 13. *If S is a minimal surface, then* $\mathrm{Bir}(S) = \mathrm{Aut}(S)$.

Exercise 14. *Let X be a compact Kähler manifold. Show that if ω_X or $\omega_X^\vee$ is ample (for instance, if $\rho(X) = 1$), then* $\mathrm{PsAut}(X) = \mathrm{Aut}(X)$.

Remark 15. When the manifold X in Corollary 12 is projective, then Kawamata shows that a birational self-map $f : X \dashrightarrow X$ is in fact a composition of flops [4]. It is still unknown whether a K-equivalence between projective manifolds can be decomposed as a sequence of flops.

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