NOTE 2: BIMEROMORPHIC MAPS AND PRIMITIVE BIMEROMORPHIC MAPS

1. Bimeromorphic automorphisms

A *meromorphic* map $f : X \dashrightarrow Y$ between (connected) complex manifolds is an irreducible closed subvariety $\Gamma \subset X \times Y$ such that the restriction of the first projection $p : X \times Y \to X$ to Γ is generically biholomorphic. In other words, there exists a Zariski open $\Gamma_U \subset \Gamma$ such that p sends biholomorphically Γ_U to a Zariski open $U \subset X$, so the restriction of f to U defines a holomorphic map $U \to Y$. The subvariety $\Gamma \subset X \times Y$ is called the *graph* of $f : X \dashrightarrow Y$. If the restriction of the other projection $X \times Y \to Y$ to Γ is also generically biholomorphic, then we call f a *bimeromorphic* map. In this case, the same subvariety $\Gamma \subset X \times Y$ defines also a bimeromorphic map $f^{-1} : Y \dashrightarrow X$, called the inverse of f. A bimeromorphic self-map $f : X \dashrightarrow X$ is also called a bimeromorphic automorphism.

A meromorphic map $f : X \dashrightarrow Y$ is called *dominant* if the image of Γ under $X \times Y \to Y$ is a Zariski open of Y. Given meromorphic maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ and assume that f is dominant, then we can define their composition $g \circ f$ as follows: If $\Gamma_f \subset X \times Y$ and $\Gamma_g \subset Y \times Z$ denote the graph of f and g and p_{XY} denotes the projection $X \times Y \times Z \to Y \times Z$ (and we define p_{YZ} and p_{XZ} similarly), then the graph of $g \circ f$ is defined to be the irreducible component of

$$p_{XZ}\left(p_{XY}^{-1}(\Gamma_f) \cap p_{YZ}^{-1}(\Gamma_g)\right) \subset X \times Z$$

which dominates X. With this composition law and the inverse f^{-1} defined in the previous paragraph, the bimeromorphic automorphisms $f : X \dashrightarrow X$ of a complex manifold X form a group denoted by $\mathrm{Bir}(X)$.

When X and Y are projective, a meromorphic map $X \dashrightarrow Y$ is the analytification of a rational map. While meromorphic maps are the generalization of rational maps for complex manifolds, we should note that the usual equivalent descriptions of rational maps in algebraic geometry do not generalize to meromorphic maps. First of all, a morphism (in algebraic geometry) $U \to Y$ from a Zariski open $U \subset X$ of an irreducible projective variety X to another projective variety Y always extends to a rational map $X \dashrightarrow Y$, whereas the exponential map $\exp : \mathbf{C} \to \mathbf{C}$ does not extend to any meromorphic map $\mathbf{P}^1 \dashrightarrow \mathbf{P}^1$. Secondly, for irreducible projective varieties X and Y, pulling back meromorphic functions sends bijectively the set of dominant rational maps $f : X \dashrightarrow Y$ to the set of $\mathbf{C}$-algebra homomorphisms $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$ between the fields of meromorphic functions. However in complex geometry, the knowledge of the induced $\mathbf{C}$-algebra homomorphism $\mathscr{M}(Y) \to \mathscr{M}(X)$ is not sufficient to reconstruct the meromorphic map $X \dashrightarrow Y$. For example, there exists a complex torus T such that $\mathscr{M}(T) = \mathbf{C}$, but both the identity and the automorphism $\imath : T \to T$ defined by $\imath(x) = -x$ induce the identity $\mathrm{Id} : \mathscr{M}(T) \to \mathscr{M}(T)$.

Remark 1. In the previous lecture, we've seen that every automorphism group $\mathrm{Aut}(X)$ is a complex Lie group. For $\mathrm{Bir}(X)$, it still carries a natural structure of group scheme when X is a minimal projective variety [3], but in general, there might be no natural (infinite dimensional) Lie group or algebraic group structure on $\mathrm{Bir}(X)$ (see [1] for the case $X = \mathbf{P}^N$).

2. Bimeromorphic maps and the Iitaka fibrations

In this lecture, we will see some invariants and geometric structures attached to compact complex manifolds and prove that they are preserved under bimeromorphic maps. Let us start with the global sections of the canonical line bundle $\omega_X = \det T_X^\vee$ and its powers.

Lemma 2. *Let $f : \tilde{X} \to X$ be a holomorphic map between complex manifolds. If f is bimeromorphic, then for every $m \in \mathbf{Z}$,*

$$f^* : H^0(X, \omega_X^{\otimes m}) \to H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes m})$$

is an isomorphism.

Proof. Since X is smooth, there exists a subvariety $Z \subset X$ of codimension at least 2 such that if $E = f^{-1}(Z)$, then $f_{|\tilde{X}\backslash E} : \tilde{X}\backslash E \to X\backslash Z$ is an isomorphism. Define

$$g : H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes m}) \hookrightarrow H^0(\tilde{X}\backslash E, \omega_{\tilde{X}\backslash E}^{\otimes m}) = H^0(X\backslash Z, \omega_{X\backslash Z}^{\otimes m}) \simeq H^0(X, \omega_X^{\otimes m})$$

where the first map is the restriction to $\tilde{X}\backslash E$ and the last isomorphism comes from the Riemann extension theorem. Obviously $g \circ f = \mathrm{Id}$. As g is injective, g is an inverse of f. □

Lemma 3. *Let $f : X \dashrightarrow X$ be a bimeromorphic self-map of a complex manifold X. Then for every $m \in \mathbf{Z}$, f induces an automorphism*

$$f^* : H^0(X, \omega_X^{\otimes m}) \to H^0(X, \omega_X^{\otimes m})$$

by pulling back global sections.

Proof. Let $X \xleftarrow{p} \tilde{X} \xrightarrow{q} X$ be a resolution of f by a complex manifold $\tilde{X}$. By Lemma 2, p and q induce isomorphisms $p^*, q^* : H^0(X, \omega_X^{\otimes m}) \to H^0(\tilde{X}, \omega_{\tilde{X}}^{\otimes m})$. We simply define $f^* = (p^*)^{-1} \circ q^*$. □

Let X be a compact complex manifold. For every integer m, if $H^0(X, \omega_X^{\otimes m}) \neq 0$ then the linear system $|mK_X|$ defines a meromorphic map

$$\Phi_m : X \xrightarrow{|mK_X|} \mathbf{P}^N.$$

If $H^0(X, \omega_X^{\otimes m}) \neq 0$ for some $m > 0$, then the *Kodaira dimension* of X is defined to be

$$\kappa(X) = \max_{m>0} (\dim \mathrm{Im}(\Phi_m)).$$

If $H^0(X, \omega_X^{\otimes m}) = 0$ for every $m > 0$, we set $\kappa(X) = -\infty$ by convention. By Lemma 3, the Kodaira dimension is a bimeromorphic invariant, namely if X is bimeromorphic to another compact complex manifold X', then $\kappa(X) = \kappa(X')$. By a theorem of Iitaka, if $\kappa(X) \neq -\infty$, then there exists m such that $\dim \mathrm{Im}(\Phi_m) = \kappa(X)$ and a resolution of Φ_m has connected fibers. Such a map Φ_m is called an *Iitaka fibration* of X.

Example 4. Let C be a smooth projective curve of genus g. Then $\kappa(C) = -\infty$, 0, or 1 if $g = 0$, 1, or $g \geq 2$ respectively.

3. Primitive bimeromorphic maps

Definition 5. Let X be a compact complex manifold and let $f \in \mathrm{Bir}(X)$. We say that f is *primitive* if there is no commutative diagram of the form

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ B & \underset{g}{\dashrightarrow} & B \end{array}$$

where $\phi : X \dashrightarrow B$ is a dominant rational map with $0 < \dim B < \dim X$.

We will study primitive bimeromorphic self-maps of a compact complex manifold X according to its Kodaira dimension. First we show that if X has a primitive bimeromorphic self-map then $\kappa(X) \in \{-\infty, 0, \dim X\}$.

Proposition 6. *Let X be a compact complex manifold. If X has a primitive bimeromorphic self-map $f : X \dashrightarrow X$, then $\kappa(X) = -\infty, 0$ or $\dim X$.*

Proof. Using the notation as above, let m be an integer such that $\Phi_m : X \dashrightarrow \mathbf{P}^N$ is an Iitaka fibration. By Lemma 3, f induces an automorphism of $H^0(X, \omega_X^{\otimes m})$, which further induces an automorphism $g : \mathbf{P}^N \to \mathbf{P}^N$ such that $\Phi_m \circ f = g \circ \Phi_m$. Let $B = \mathrm{Im}(\Phi_m)$. By construction, given a general point $x \in X$, a section $\eta \in H^0(X, \omega_X^{\otimes m})$ vanishes at $f(x)$ if and only if $f^*\eta$ vanishes at x. So $g(B) = B$, in particular f descends to $g_{|B}$. Therefore if $0 < \kappa(X) = \dim B < \dim X$, then f is not primitive. □

Compact complex manifolds X with $\kappa(X) = \dim X$ are also called *manifolds of general type*. Note that since the Iitaka fibration of a compact complex manifold of general type is a bimeromorphic map, such a manifold is always bimeromorphic to a projective variety.

Proposition 7. *Let X be a compact complex manifold of general type.*

i) There exists a bimeromorphic map $\phi : X \dashrightarrow B$ to a projective variety B such that the group homomorphism $\Phi : \mathrm{Aut}(B) \to \mathrm{Bir}(X)$ defined by $g \mapsto \phi^{-1} \circ g \circ \phi$ is an isomorphism.

ii) $\mathrm{Bir}(X)$ is finite.

Proof. Again, let $\Phi_m : X \dashrightarrow \mathbf{P}^N$ be an Iitaka fibration and let $B = \mathrm{Im}(\Phi_m)$. Then B is projective. For every bimeromorphic map $f : X \dashrightarrow X$, the same argument as in the proof of Proposition 6 shows that there exists an automorphism $g : B \to B$ such that $\Phi_m \circ f = g \circ \Phi_m$. The map $f \mapsto g$ is the inverse of Φ, which proves *i*).

By *i*), it suffices to show that $\mathrm{Aut}(B)$ is finite. If $g : B \to B$ is an automorphism of B, then $\Phi(g)$ induces an automorphism $g' : \mathbf{P}^N \to \mathbf{P}^N$ and $g'_{|B} = g$. So $\mathrm{Aut}(B)$ can be identified with a Zariski closed subgroup of $\mathrm{PGL}_{N+1}(\mathbf{C})$ (consisting of elements $g' : \mathbf{P}^N \to \mathbf{P}^N$ such that $g'(B) = B$). If $\dim \mathrm{Aut}(B) \geq 1$, then since $\mathrm{PGL}_{N+1}(\mathbf{C})$ is an affine group (because it is the quotient of the affine group $\mathrm{SL}_{N+1}(\mathbf{C})$ by the group of $(N+1)$-th roots of unity, which is finite), $\mathrm{Aut}(B)$ has a one-parameter subgroup Σ birational to $\mathbf{P}^1$. Since the Σ-orbit of a general point of B is not a point, we deduce that B is uniruled (namely, covered by non-constant images of $\mathbf{P}^1$). As X is birational to B, X also uniruled, so $\omega_X^{\otimes m}$ have no global sections for every positive integer m. This contradicts the assumption that X is of general type. Therefore $\mathrm{Aut}(B)$ is a Zariski closed subgroup of $\mathrm{PGL}_{N+1}(\mathbf{C})$ of dimension 0 and in particular, $\mathrm{Aut}(B)$ is finite. □

Remark 8. For every positive integer n, there exists $c > 0$ such that $|\mathrm{Bir}(X)| \leq c \cdot \mathrm{vol}(X, K_X)$ for every projective manifold X of general type [2]. Here, the volume of a divisor D on X is defined to be

$$\mathrm{vol}(X, D) := \limsup_{m\to\infty} \frac{n! h^0(X, \mathscr{O}(mD))}{m^n}.$$

If D is a nef divisor, then $\mathrm{vol}(X, D) = D^n$. An explicit expression of c in terms of n is still unknown.

According to 7, compact complex manifolds X with $\kappa(X) \leq \dim(X) - 1$ are the only dynamically interesting manifolds and by Proposition 6, the dynamical systems of them can be reduced to those of a compact complex manifold X with $\kappa(X) \leq 0$.

Exercise 9.

i) Let $f : X \dashrightarrow Y$ be a dominant meromorphic map between complex manifolds. Construct $f^* : H^0(Y, \omega_Y^{\otimes m}) \to H^0(X, \omega_X^{\otimes m})$ and show that f^* is injective.

ii) Let X be a compact complex manifold of general type and $f : X \dashrightarrow X$ a meromorphic map. Show that if f is dominant, then f is bimeromorphic.

Exercise 10. *Let $f : X \dashrightarrow X$ be a birational self-map of a projective manifold. If $\rho(X) = 1$, then f is biholomorphic.*

4. Case $\kappa = 0$

Proposition 11. *Let X be a projective manifold such that $\kappa(X) = 0$. If X has a primitive birational self-map $X \dashrightarrow X$, then either X is birational to a projective complex torus* (i.e. *an abelian variety), or $b_1(X) = 0$.*

Let X be a compact Kähler manifold. By Hodge symmetry, the projection

$$H^{2n-1}(X, \mathbf{R}) \to H^{2n-1}(X, \mathbf{C})/H^{n-1}(X, \Omega_X^n) \tag{4.1}$$

is an isomorphism, and the image of the free part $H^{2n-1}(X, \mathbf{Z})_{\mathrm{free}}$ of $H^{2n-1}(X, \mathbf{Z})$ under the composition $H^{2n-1}(X, \mathbf{Z}) \to H^{2n-1}(X, \mathbf{R})$ with (4.1) is a lattice in $H^{2n-1}(X, \mathbf{C})/H^{n-1}(X, \Omega_X^n)$. The *Albanese variety* of X is defined to be the complex torus

$$\mathrm{Alb}(X) = \frac{H^{2n-1}(X, \mathbf{C})}{H^{n-1}(X, \Omega_X^n) \oplus H^{2n-1}(X, \mathbf{Z})_{\mathrm{free}}}.$$

If X is projective, then a polarization of X gives rise to a polarization of $\mathrm{Alb}(X)$, so in this case $\mathrm{Alb}(X)$ is an abelian variety. By Poincaré's duality, we have

$$\mathrm{Alb}(X) \simeq \frac{H^1(X, \mathbf{C})^\vee}{H^1(X, \mathscr{O}_X)^\vee \oplus H_1(X, \mathbf{Z})_{\mathrm{free}}} \simeq H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbf{Z})_{\mathrm{free}}.$$

Fix $x_0 \in X$. The Albanese map $a : X \to \mathrm{Alb}(X)$ is defined to be

$$\begin{aligned} a : X &\to \mathrm{Alb}(X) \\ x &\mapsto \left(\gamma \mapsto \int_{x_0}^{x} \gamma\right). \end{aligned} \tag{4.2}$$

Up to composing a with a translation, the Albanese map does not depend on the choice of $x_0 \in X$.

Lemma 12. *A bimeromorphic map $f : X \dashrightarrow X'$ between compact complex manifolds induces an isomorphism $f_* : H^1(X, \mathbf{Z}) \to H^1(X', \mathbf{Z})$. If X and X' are Kähler, then f_* is an isomorphism of Hodge structures.*

Proof. Let us first show that if $g : \tilde{X} \to X$ is a bimeromorphic holomorphic map from a compact Kähler manifold $\tilde{X}$, then $g_* : H^1(\tilde{X}, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ is an isomorphism, whose inverse is $g^* : H^1(\tilde{X}, \mathbf{Z}) \to H^1(X, \mathbf{Z})$. Resolving $g^{-1} : X \dashrightarrow \tilde{X}$ by a sequence of blow-ups $g' : \tilde{X}' \to X$ along smooth centers, we obtain a factorization $g' : \tilde{X}' \xrightarrow{\tau} \tilde{X} \xrightarrow{g} X$. Since τ and g are surjective and generically of degree 1, the pushforwards $\tau_* : H^1(\tilde{X}', \mathbf{Z}) \to H^1(\tilde{X}, \mathbf{Z})$ and $g_* : H^1(\tilde{X}, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ are surjective. As $g'_* : H^1(\tilde{X}, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ is an isomorphism by the blow-up formula, $g_* : H^1(\tilde{X}, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ is injective. It follows from $g_* \circ g^* = \mathrm{Id}$ that g^* is the inverse of g_*.

Now let

$$X \xleftarrow{p} \tilde{X} \xrightarrow{q} X'$$

be a resolution of f obtained by a sequence of blow-ups $p : \tilde{X} \to X$ along smooth centers. The map $f_* : H^1(X, \mathbf{Z}) \to H^1(X', \mathbf{Z})$ is defined to be $f_* = q_* \circ p^*$. Since q_* and p^* are isomorphisms, so is f_*. Since two resolutions of f are always dominated by a third one, a similar argument shows that the definition of f_* is independent of the resolutions. Finally if X and X' are Kähler, then by construction $\tilde{X}$ is also Kähler. It follows that q_* and p^* are morphisms of Hodge structures, and so is the composition $f_* = q_* \circ p^*$. □

Remark 13. For compact complex manifolds, the fundamental group is in fact a bimeromorphic invariant. This is stronger than Lemma 12 because $H^1(X, \mathbf{Z})$ is the Poincaré dual of $H_1(X, \mathbf{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$.

It follows from Lemma 12 that a bimeromorphic automorphism $f : X \dashrightarrow X$ of a compact Kähler manifold, giving rise to a morphism of Hodge structures $f_* : H^1(X, \mathbf{Z}) \to H^1(X, \mathbf{Z})$, induces an automorphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ of Lie groups. Moreover, if $a : X \to \mathrm{Alb}(X)$ is the Albanese map defined by $x \mapsto \int_{x_0}^{x}$ for some $x_0 \in X$ such that $f(x_0)$ is well-defined, then the diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ \downarrow a & & \downarrow a \\ \mathrm{Alb}(X) & \xrightarrow{\tau \circ F} & \mathrm{Alb}(X) \end{array} \tag{4.3}$$

is commutative where $\tau : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ is the translation by $-a(f(x_0))$. In other words, bimeromorphic automorphism preserves the Albanese map.

Proposition 11 is now a direct consequence of the following theorem due to Kawamata.

Theorem 14 (Kawamata [6])**.** *Let X be a projective manifold. If $\kappa(X) = 0$, then the Albanese map $X \to \mathrm{Alb}(X)$ is surjective with connected fibers.*

Proof of Proposition 11. Since the Albanese maps in (4.3) are surjective, if f is primitive then $\dim \mathrm{Alb}(X) = \dim X$ or $b_1(X) = 2 \dim \mathrm{Alb}(X) = 0$. If $\dim \mathrm{Alb}(X) = \dim X$, then since $a : X \to \mathrm{Alb}(X)$ is surjective with connected fibers, a is a bimeromorphic map. □

5. Uniruled manifolds

Let X be a compact complex variety. We say that X is *rationally connected* if for any general pair of points $x, y \in X$, there exists a connected compact curve $C \subset X$ such that $x, y \in C$ and whose normalization is a union of $\mathbf{P}^1$. Here by convention, 0-dimensional compact complex varieties are also rationally connected. A meromorphic map $g : X \dashrightarrow B$ is called *almost holomorphic* if there exists a dense Zariski open subset $U \subset X$ such that $g_{|U}$ is holomorphic and proper (so the fiber of g over a point $b \in g(U)$ is well-defined, which is $g_{|U}^{-1}(b)$). The following theorem asserts that every compact Kähler manifold X admits a unique almost holomorphic fibration in rationally connected subvarieties of maximal dimension.

Theorem 15 (Campana, Kollár-Miyaoka-Mori)**.** *There exists a unique dominant almost holomorphic map $g : X \dashrightarrow B$ such that a general fiber of g is rationally connected and if $X \dashrightarrow B'$ is another almost holomorphic map whose general fiber is rationally connected, then* $\dim B \leq \dim B'$. *Moreover, if $b \in B$ is a very general point (namely, if $b \in B$ is in a non-empty countable intersection of Zariski open subsets) and C is a rational curve intersecting $f^{-1}(b)$, then $C \subset f^{-1}(b)$.*

We refer to [8, Chapter IV.5] for a proof of Theorem 15 and further discussions. The almost holomorphic map $X \dashrightarrow B$ in Theorem 15 is called the *maximal rationally connected fibration* of X (or *MRC-fibration*). We easily observe that X is rationally connected if and only if $\dim B = 0$. Also, X is uniruled if and only if $\dim B < \dim X$.

Proposition 16. *Bimeromorphic self-maps preserve the MRC-fibration. In particular, if $f : X \dashrightarrow X$ is a primitive bimeromorphic self-map of a uniruled compact Kähler manifold, then X is rationally connected.*

Proof. Let $g : X \dashrightarrow B$ be the MRC-fibration of X and let $F = g^{-1}(b)$ where $b \in B$ is a very general point. Since F is rationally connected, its image $f(F)$ is also rationally connected. By the second statement of Theorem 15, $f(F)$ has to be a fiber of g, and from this we obtain a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X \\ \downarrow g & & \downarrow g \\ B & \underset{f'}{\dashrightarrow} & B \end{array} \tag{5.1}$$

for some bimeromorphic map $f' : B \dashrightarrow B'$. Finally if X is uniruled, then $\dim B < \dim X$. As f is primitive, then necessarily $\dim B = 0$, therefore X is rationally connected. □

6. (Conjectural) manifolds supporting a primitive bimeromorphic self-map

Let X be a projective manifold. The Minimal Model Program (MMP) predicts that X is either uniruled or birational to a minimal variety X', namely a normal $\mathbf{Q}$-factorial projective variety X' with at worst terminal singularities such that the canonical divisor $K_{X'}$ is nef (*i.e.* $K_{X'} \cdot C \geq 0$ for every curve $C \subset X'$). Moreover, the abundance conjecture predicts that $|mK_{X'}|$ is based-point-free for $m \gg 0$. In other words, the Iitaka fibration of X' is a morphism. If $\kappa(X) = 0$ (resp. $\kappa(X) = -\infty$), then the conjunction of the MMP and the abundance conjecture implies that $K_{X'}$ is numerically trivial (resp. X is uniruled). Both the MMP and the abundance conjecture are known to hold for projective threefolds (and more generally, for compact Kähler threefolds). See [7, 4] for related discussions.

Combining Proposition 7, Proposition 11, and Proposition 16, we obtain the following conjectural birational classification of projective manifolds which have a primitive birational self-map of infinite order.

Corollary 17 (D.-Q. Zhang [9])**.** *Let X be a projective manifold and assume the MMP and the abundance conjecture for X. If X has a primitive birational self-map $f : X \dashrightarrow X$ of infinite order, then X is birational to one of the following:*

i) an abelian variety;

ii) a normal $\mathbf{Q}$-factorial variety X' with at worst terminal singularities and numerically trivial $K_{X'}$ such that $h^1(X', \mathscr{O}_{X'}) = 0$;

iii) a rationally connected manifold.

Remark 18. For a projective variety X' as in *ii*), thanks to recent work of Druel, Greb-Guenancia-Kebekus, and Höring-Peternell [5], we know more precisely that up to a finite quasi-étale (*i.e.* étale in codimension 1) cover, X' is a product of Calabi-Yau varieties and irreducible symplectic varieties.

References

[1] Jérémy Blanc and Jean-Philippe Furter. Topologies and structures of the Cremona groups. *Ann. of Math. (2)*, 178(3):1173–1198, 2013.

[2] Christopher D. Hacon, James McKernan, and Chenyang Xu. On the birational automorphisms of varieties of general type. *Ann. of Math. (2)*, 177(3):1077–1111, 2013.

[3] Masaki Hanamura. On the birational automorphism groups of algebraic varieties. *Compositio Math.*, 63(1):123–142, 1987.

[4] Andreas Höring and Thomas Peternell. Bimeromorphic geometry of Kähler threefolds. arxiv:1701.01653, 2017. to appear in "Proceedings of 2015 Summer Institute on Algebraic Geometry".

[5] Andreas Höring and Thomas Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. *Inventiones mathematicae*, page 1, January 2019.

[6] Yujiro Kawamata. Characterization of abelian varieties. *Compositio Math.*, 43(2):253–276, 1981.

[7] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[8] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Math.* Springer-Verlag, Berlin, 1996.

[9] De-Qi Zhang. Dynamics of automorphisms on projective complex manifolds. *Journal of Differential Geometry*, 82(3):691–722, 2009.