NOTE 1: AUTOMORPHISMS

1. Definition, examples

Let X be a complex manifold. An automorphism of X is a biholomorphic self-map $f : X \to X$. The automorphisms of X form a group $\mathrm{Aut}(X)$, called the *automorphism group* of X.

Remark 1. If $f : X \to Y$ is a holomorphic map between complex projective varieties, then f is the analytification of a morphism of algebraic varieties. This immediately follows from Chow's theorem since it implies that the graph $\Gamma_f \subset X \times Y$ of f is an algebraic closed subset. In particular, for every complex projective manifold X, the analytification functor identifies the two automorphism groups $\mathrm{Aut}(X)$ for X considered as a variety (in algebraic geometry) or as a complex manifold.

Example 2 (Automorphisms of $\mathbf{P}^N$)**.** Let us show that $\mathrm{Aut}(\mathbf{P}^N) = \mathrm{PGL}_{N+1}(\mathbf{C})$. As $\mathrm{Aut}(\mathbf{P}^N)$ preserves the ample cone and $\mathrm{Pic}(\mathbf{P}^N) \simeq \mathbf{Z}$, $\mathrm{Aut}(\mathbf{P}^N)$ acts as the identity on $\mathrm{Pic}(\mathbf{P}^N)$. In particular if $f \in \mathrm{Aut}(\mathbf{P}^N)$ then $f^*\mathscr{O}(1) \simeq \mathscr{O}(1)$.

Let us show that conversely, if $f : \mathbf{P}^N \to \mathbf{P}^N$ is a morphism such that $f^*\mathscr{O}(1) \simeq \mathscr{O}(1)$ then $f \in \mathrm{Aut}(\mathbf{P}^N)$. Fix a basis $e_0, \dots, e_N$ of $H^0(\mathbf{P}^N, \mathscr{O}(1))$, then the global sections $f^*e_0, \dots, f^*e_N$ considered as sections of $\mathscr{O}(1)$ generate $\mathscr{O}(1)$ where f^* is the composition

$$f^* : H^0(\mathbf{P}^N, \mathscr{O}(1)) \to H^0(\mathbf{P}^N, f^*\mathscr{O}(1)) \simeq H^0(\mathbf{P}^N, \mathscr{O}(1)).$$

Conversely, given global sections $s_0, \dots, s_N$ which generate $\mathscr{O}(1)$, then for every basis $e'_0, \dots, e'_N$ of $H^0(\mathbf{P}^N, \mathscr{O}(1))$ there exists a unique morphism $g : \mathbf{P}^N \to \mathbf{P}^N$ such that $g^*\mathscr{O}(1) \simeq \mathscr{O}(1)$ and up to modifying the isomorphism $g^*\mathscr{O}(1) \simeq \mathscr{O}(1)$, we have $g^*e'_i = s_i$ for every i [4, Theorem II.7.1]. Note that in general, $s_0, \dots, s_N$ generate $\mathscr{O}(1)$ if and only if they form a basis of $H^0(\mathbf{P}^N, \mathscr{O}(1))$, so $f^*e_0, \dots, f^*e_N$ is a basis of $H^0(\mathbf{P}^N, \mathscr{O}(1))$ and the sections $e_0, \dots, e_N$ generate $\mathscr{O}(1)$. Accordingly, there exists $g : \mathbf{P}^N \to \mathbf{P}^N$ such that $g^*\mathscr{O}(1) \simeq \mathscr{O}(1)$ and $g^*f^*e_i = e_i$ for every i. By uniqueness, we have $f \circ g = \mathrm{Id}_{\mathbf{P}^N}$. As $e_0, \dots, e_N$ form a basis of $H^0(\mathbf{P}^N, \mathscr{O}(1))$, g^*f^* is the identity on $H^0(\mathbf{P}^N, \mathscr{O}(1))$, and thus $f^*g^* = g^*f^* = \mathrm{Id}$. So $f^*g^*e_i = e_i$ for every i as well and thus $g \circ f = \mathrm{Id}_{\mathbf{P}^N}$. Therefore $f \in \mathrm{Aut}(\mathbf{P}^N)$.

We have identified $\mathrm{Aut}(\mathbf{P}^N)$ with the set of morphisms $f : \mathbf{P}^N \to \mathbf{P}^N$ such that $f^*\mathscr{O}(1) \simeq \mathscr{O}(1)$. Again by [4, Theorem II.7.1], there exists an isomorphism of $\mathrm{PGL}_{N+1}(\mathbf{C})$-torsors between the set of morphisms $f : \mathbf{P}^N \to \mathbf{P}^N$ such that $f^*\mathscr{O}(1) \simeq \mathscr{O}(1)$ (here $\phi \in \mathrm{PGL}_{N+1}(\mathbf{C})$ acts by $f \mapsto \phi \circ f$) and the set of $(N+1)$-uple of global sections $(s_0, \dots, s_N)$ generating $\mathscr{O}(1)$ modulo the relations

$$(s_0, \dots, s_N) \sim (c \cdot s_0, \dots, c \cdot s_N)$$

where $c \in \mathbf{C}^\times$. Hence $\mathrm{Aut}(\mathbf{P}^N) = \mathrm{PGL}_{N+1}(\mathbf{C})$.

Example 3 (Automorphisms of a complex torus)**.** Let V be a complex vector space and $\Lambda \subset V$ a lattice. Let $T = V/\Lambda$ be the quotient, which is a complex torus. There exist two kinds of natural automorphisms of T: the translations and the automorphisms coming from $\mathrm{GL}(V)$. For the first kind of automorphisms, viewing T as a commutative Lie group, the translation of T by an element $p \in T$ defines a biholomorphic map $\tau_p : T \to T$. Obviously the translations form a subgroup of $\mathrm{Aut}(T)$ which is isomorphic to T. For the second kind of automorphisms, consider the subgroup

$$G = \{\tilde{g} \in \mathrm{GL}(V) \mid \tilde{g}(\Lambda) = \Lambda\} \subset \mathrm{GL}(V).$$

Then every element $\tilde{g} \in G$ descends to a biholomorphic *group automorphism* $g : T \to T$. By abuse of notation, let $G \subset \mathrm{Aut}(T)$ denote the subgroup of automorphisms of T induced by G. It turns out that these two kinds of automorphisms generate $\mathrm{Aut}(T)$ and we will show that $\mathrm{Aut}(T) \simeq T \rtimes G$.

First we show that every $f \in \mathrm{Aut}(T)$ can be decomposed as $\tau_p \circ g$ for some $p \in T$ and $g \in G$. Let $p = f(o)$ be the image of the origin $o \in T$. It suffices to show that $\tau_{-p} \circ f : T \to T$ is induced by G. Let $\tilde{g} : V \to V$ be the biholomorphic map lifting $\tau_{-p} \circ f$ to the universal cover of T such that $\tilde{g}(0) = 0$, then for every $\lambda \in \Lambda$ and $v \in V$, we have

$$\tilde{g}(v + \lambda) - \tilde{g}(v) \in \Lambda. \tag{1.1}$$

In particular, $v \mapsto \tilde{g}(v + \lambda) - \tilde{g}(v)$ is constant, so the first partial derivatives of $\tilde{g}$ are all Λ-periodic, thus they are constant by Liouville's theorem. Since $\tilde{g}(0) = 0$, it follows that $\tilde{g}$ is linear. As $\tilde{g}(\Lambda) \subset \Lambda$ by (1.1) and similarly $\tilde{g}^{-1}(\Lambda) \subset \Lambda$, we have $\tilde{g}(\Lambda) = \Lambda$. Hence $\tilde{g} \in G$, which shows that $\tau_{-p} \circ f \in G$. Finally assume that $f = \tau_p \circ g = \tau_{p'} \circ g'$ where $g, g' \in G$. Since $g(o) = g'(o) = o$, necessarily $p = p'$, so $g = g'$ as well.

Remark 4. If $f = \tau_p \circ g$ is the decomposition of f under the isomorphism $\mathrm{Aut}(T) \simeq T \rtimes G$, then the induced map $f_* : H_1(T, \mathbf{Z}) \to H_1(T, \mathbf{Z})$ coincides with $\tilde{g} : \Lambda \to \Lambda$ under the canonical isomorphism $\Lambda \simeq H_1(T, \mathbf{Z})$.

2. Aut(X) is a complex Lie group

In the previous examples, we note that $\mathrm{Aut}(X)$ are all complex Lie groups. Now we prove that this is the case for every compact complex manifold X.

Theorem 5. *The automorphism group* $\mathrm{Aut}(X)$ *of a compact complex manifold X is a complex Lie group of dimension* $\dim H^0(X, T_X)$.

We will prove Theorem 5 under the additional assumption that X is projective. Without the projectivity assumption, the proof of Theorem 5 is similar but the ingredients are different (see Remark 8). Essentially we need to find a natural scheme structure on $\mathrm{Aut}(X)$ (or holomorphic structure, if X is only a compact complex manifold). Since the space of morphisms $X \to X$ can be identified with the space of graphs $\Gamma \subset X \times X$ and the Hilbert scheme $\mathrm{Hilb}_{X \times X}$ of $X \times X$ is the scheme parameterizing closed subschemes of $X \times X$, we shall try to embed $\mathrm{Aut}(X)$ into $\mathrm{Hilb}_{X \times X}$ so that $\mathrm{Aut}(X)$ will inherit a scheme structure from $\mathrm{Hilb}_{X \times X}$.

For simplicity, every scheme mentioned below will be defined over $\mathbf{C}$. We refer to [2, Chapter 5] for a construction of the Hilbert scheme Hilb_X of a projective scheme X. The important property we need is that there exists a flat family of closed subschemes

$$\mathscr{C} \subset \mathrm{Hilb}_X \times X \to \mathrm{Hilb}_X$$

parameterized by Hilb_X such that for every flat family of closed subschemes $\mathscr{C}_S \subset X \times S$ of X parameterized by a scheme S, there exists a unique morphism $S \to \mathrm{Hilb}_X$ such that $\mathscr{C}_S \to S$ is isomorphic to the pullback of $\mathscr{C} \to \mathrm{Hilb}_X$ by $S \to \mathrm{Hilb}_X$. More precisely, the aforementioned property should define an isomorphism between the functor

$$\begin{aligned} F_X : Sch/k &\to Set \\ S &\mapsto \{\textit{Closed subschemes of } S \times X \textit{ flat over } S\} \end{aligned}$$

and the functor $S \mapsto \mathrm{Mor}(S, \mathrm{Hilb}_X)$. We say that the functor F_X is *represented* by the scheme Hilb_X and the subscheme $\mathscr{C} \subset \mathrm{Hilb}_X \times X$ is called the *universal closed subscheme* of X. According to the above, closed fibers of $\mathscr{C} \to \mathrm{Hilb}_X$ are identified with closed subschemes of X.

Using Hilbert schemes, we can prove that the morphisms $X \to Y$ between fixed projective schemes X and Y can also be parameterized by some scheme just as the Hilbert scheme parameterizes the closed subschemes.

Theorem 6. *Let X and Y be two projective schemes. The morphisms from X to Y are parameterized by a Zariski open subscheme* $\mathrm{Mor}(X, Y)$ *of* $\mathrm{Hilb}_{X \times Y}$. *More precisely, the functor*

$$F_{X \to Y} : S \mapsto \{\textit{S-morphisms } f : X \times S \to Y \times S\}$$

is represented by $\mathrm{Mor}(X, Y)$.

Proof. Let $\mathscr{C} \subset (\mathrm{Hilb}_{X\times Y}) \times X \times Y$ be the universal closed subscheme of $X \times Y$ and let $\pi : \mathscr{C} \to X$ be the standard projection. Set-theoretically $\mathrm{Mor}(X, Y)$ can be identified with the closed points $t \in \mathrm{Hilb}_{X\times Y}(\mathbf{C})$ such that the restriction $\pi_t : \mathscr{C}_t \to X$ of π to the subscheme $\mathscr{C}_t \subset X \times Y$ parameterized by t is an isomorphism. The following lemma applied to the projections

$$p : \mathscr{C} \to \mathrm{Hilb}_{X\times Y},$$

$$q : X \times \mathrm{Hilb}_{X\times Y} \to \mathrm{Hilb}_{X\times Y},$$

$$f : \mathscr{C} \to X \times \mathrm{Hilb}_{X\times Y}$$

implies that there exists an open subscheme $\mathrm{Mor}(X, Y) \subset \mathrm{Hilb}_{X\times Y}$ whose closed points are identified with morphisms $f : X \to Y$.

Lemma 7. *Given two flat proper families of schemes $p : \mathscr{X} \to S$ and $q : \mathscr{Y} \to S$ over S and a morphism $f : \mathscr{X} \to \mathscr{Y}$ over S. There exists an open subscheme $U \subset S$ such that*

$$U(\mathbf{C}) = \mathrm{Iso}(f) := \{s \in S(\mathbf{C}) \mid f_s : \mathscr{X}_s \to \mathscr{Y}_s \text{ is an isomorphism}\}.$$

Moreover, $f_{|\mathscr{X}_U} : \mathscr{X}_U \to \mathscr{Y}_U$ is an isomorphism where $\mathscr{X}_U = p^{-1}(U)$ and $\mathscr{Y}_U = q^{-1}(U)$.

Proof. Assume that $\mathrm{Iso}(f) \neq \emptyset$ and let $s \in \mathrm{Iso}(f)$. It suffices to show that there exists a neighborhood $U_s \subset S$ of s such that $f_{|\mathscr{X}_{U_s}} : \mathscr{X}_{U_s} \to \mathscr{Y}_{U_s}$ is an isomorphism, and then the open subscheme

$$U = \bigcup_{s\in \mathrm{Iso}(f)} U_s$$

will satisfy the desired property. Since f is proper, the dimension of the fibers of f is upper semi-continuous for the Zariski topology on $\mathscr{Y}$. As $\mathscr{Y}$ is also proper over S, there exists a neighborhood $S' \subset S$ of s such that $f_{|\mathscr{X}_{S'}} : \mathscr{X}_{S'} \to \mathscr{Y}_{S'}$ is finite. Up to replacing S with S' and f with $f_{|\mathscr{X}_{S'}}$, we may assume that f is finite.

The finite morphism f induces a morphism $f^\# : \mathscr{O}_{\mathscr{Y}} \to f_*\mathscr{O}_{\mathscr{X}}$ of coherent sheaves; let $K := \ker(f^\#)$ and $Q := \mathrm{coker}(f^\#)$. Since f_s is an isomorphism, we have $Q_{|\mathscr{Y}_s} = 0$. Again, as $\mathscr{Y}$ is proper over S, there exists a neighborhood $U_s \subset S$ of s such that $Q_{|\mathscr{Y}_{U_s}} = 0$ by Nakayama's lemma. Since $f_*\mathscr{O}_{\mathscr{X}_{U_s}}$ is $\mathscr{O}_{U_s}$-flat, we have $K_{|\mathscr{Y}_s} = \ker(\mathscr{O}_{\mathscr{Y}_s} \to f_*\mathscr{O}_{\mathscr{X}_s}) = 0$. Again by Nakayama's lemma, we have $K_{|\mathscr{Y}_{U_s}} = 0$ up to shrinking U_s. It follows that $f_{|\mathscr{X}_{U_s}} : \mathscr{X}_{U_s} \to \mathscr{Y}_{U_s}$ is an isomorphism. □

As $\mathrm{Hilb}_{X\times Y}$ represents the functor $F_{X\times Y}$, it follows that $F_{X\to Y}$ is represented by $\mathrm{Mor}(X, Y)$. □

Proof of Theorem 5 for X projective. The involution $X \times X \to X \times X$ sending (x, y) to (y, x) induces an involution $I : \mathrm{Hilb}_{X\times X} \to \mathrm{Hilb}_{X\times X}$. By construction, $\mathrm{Aut}(X)$ can be identified with the open subscheme

$$\mathrm{Mor}(X, X) \cap I(\mathrm{Mor}(X, X)) \subset \mathrm{Hilb}_{X\times X}.$$

Since $\mathrm{Mor}(X, X)$ represents the functor $F_{X\to X}$, it follows that the composition

$$\circ : \mathrm{Aut}(X) \times \mathrm{Aut}(X) \to \mathrm{Aut}(X)$$

is a morphism of schemes. Therefore $\mathrm{Aut}(X)$ is a group scheme and hence a complex Lie group. Finally, since $\mathrm{Aut}(X)$ is a open subscheme of $\mathrm{Hilb}_{X\times X}$, the tangent space of $\mathrm{Aut}(X)$ at Id_X equals the tangent space of $\mathrm{Hilb}_{X\times X}$ at the point $[\Delta] \in \mathrm{Hilb}_{X\times X}$ parameterizing the diagonal $\Delta \subset X \times X$, which is $H^0(\Delta, N_{\Delta/X\times X}) \simeq H^0(X, T_X)$. Hence $\dim \mathrm{Aut}(X) = \dim H^0(X, T_X)$. □

Remark 8. For a complex space X, there exists a complex space $\mathscr{D}(X)$ parameterizing proper complex subspaces $Y \subset X$, which is an analogue of the Hilbert schemes for projective schemes. The complex space $\mathscr{D}(X)$ is called the Douady space. To prove Theorem 5 in the general case, it suffices to replace the Hilbert schemes by the Douady spaces in the argument and replace the basic results we used in algebraic geometry by their complex analytic analogues (see for instance [3]). Note that in fact, we can prove that the automorphism group $\mathrm{Aut}(X)$ is a complex Lie group even if X is singular, namely for X an arbitrary compact complex space.

The automorphism group $\mathrm{Aut}(X)$ can have infinitely many connected components (this happens for instance when X is some complex torus; see Example 3). There exist even projective manifolds X such that $\mathrm{Aut}(X)$ is discrete and infinitely generated, and such examples have been discovered only recently in [5, 1]. Finally, as for which complex Lie groups can be realized as (the identity component) of the automorphism group, we have the following result due to M. Brion.

Theorem 9 (Brion)**.** *Let G be a connected complex Lie group. There exists a complex projective manifold X with* $\dim X = 2\dim G$ *such that* $\mathrm{Aut}^0(X) = G$.

3. The subgroup of $\mathrm{Aut}(X)$ acting trivially on $H^\bullet(X, \mathbf{Z})$

In this paragraph, we are interesting in the subgroup

$$\mathrm{Aut}_c(X) = \ker\left(\mathrm{Aut}(X) \to \mathrm{GL}(H^\bullet(X, \mathbf{Z}))\right)$$

of automorphisms of X which act as the identity on $H^\bullet(X, \mathbf{Z})$. We have the obvious inclusion $\mathrm{Aut}^0(X) \subset \mathrm{Aut}_c(X)$. When $\dim X = 1$, this inclusion is an equality.

Proposition 10. *Let $f : C \to C$ be an automorphism of a smooth projective curve C. If f^* acts on $H^1(C, \mathbf{C})$ as the identity, then $f \in \mathrm{Aut}^0(C)$.*

Proof. Let g be the genus of C. If $g = 0$, then $\mathrm{Aut}(C) = \mathrm{Aut}(\mathbf{P}^1) = \mathrm{PGL}_2$, which is connected so the statement is obviously true.

Assume that $g = 1$, so C is a torus. As f^* acts as the identity on $H^1(C, \mathbf{C})$, which is Poincaré dual to $f_* : H_1(C, \mathbf{C}) \to H_1(C, \mathbf{C})$, the restriction $f_* : H_1(C, \mathbf{Z}) \to H_1(C, \mathbf{Z})$ to the lattice $H_1(C, \mathbf{Z})$ is also the identity. Hence $f \in \mathrm{Aut}^0(C)$ according to the description of $\mathrm{Aut}(T)$ in Example 3.

Finally suppose that $g > 1$, then f has a fixed point by the holomorphic Lefschetz fixed-point formula. Let $p \in C$ be a fixed point of f. The curve C embeds into its Jacobian variety $J(C)$ and we let p be the origin of $J(C)$. Then $f : C \to C$ extends to an automorphism $F : J(C) \to J(C)$ fixing p. Since $F_* : H_1(J(C), \mathbf{C}) \to H_1(J(C), \mathbf{C})$ is isomorphic to the Poincaré dual of $f^* : H^1(C, \mathbf{C}) \to H^1(C, \mathbf{C})$, which is the identity, Remark 4 shows that F is a translation. As p is a fixed point of F, F is the identity, hence $f = \mathrm{Id}_C$. □

The inclusion $\mathrm{Aut}^0(X) \subset \mathrm{Aut}_c(X)$ can be strict starting from dimension 2.

Example 11. Let E be an elliptic curve and $c \in E$ a 2-torsion point. Let $\imath : E \times E \to E \times E$ be the involution defined by $\imath(x, y) = (-x, y + c)$. The involution $\imath$ is fixed-point-free so the quotient $X = (E \times E)/\imath$ is a smooth complex surface (which is an example of bielliptic surface). Every automorphism $f : X \to X$ can be lifted to an automorphism

$$\tilde{f} \in \mathrm{Aut}_\imath(E \times E) := \left\{\tilde{f} : E \times E \to E \times E \mid \tilde{f} \circ \imath = \imath \circ \tilde{f}\right\}$$

and $\tilde{f} : E \times E$ is a lifting of f if and only if $\imath \circ \tilde{f}$ is a lifting of f. Therefore

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}_\imath(E \times E)/\sim$$

where $\sim$ denotes the relation $\tilde{f} \sim \imath \circ \tilde{f}$. It follows that $\mathrm{Aut}^0(X)$ is the image of $\{\mathrm{Id}_E\} \times \mathrm{Aut}^0(E) \subset \mathrm{Aut}_\imath(E \times E)$ in $\mathrm{Aut}(X)$. Consider the automorphism $g : X \to X$ induced by $\tilde{g} \in \mathrm{Aut}(E \times E)$ defined by $\tilde{g}(x, y) = \tilde{g}(x + c, y)$. Since $\tilde{g}$ is a translation, $\tilde{g}$ (and thus g) acts trivially on the cohomology groups and it is easy to see that $g \notin \mathrm{Aut}^0(X)$.

When X is a compact Kähler manifold, $\mathrm{Aut}^0(X)$ is in fact a *finite index* subgroup of $\mathrm{Aut}_c(X)$. This is an immediate consequence of the following statement:

Theorem 12 (Fujiki, Lieberman)**.** *Let X be a compact Kähler manifold and $\omega \in H^2(X, \mathbf{R})$ a Kähler class. Let*

$$\mathrm{Aut}_\omega(X) = \{f \in \mathrm{Aut}(X) \mid f^*\omega = \omega\}.$$

Then $\mathrm{Aut}^0(X)$ *is a finite index subgroup of* $\mathrm{Aut}_\omega(X)$.

The following is an immediate consequence of Theorem 12

Corollary 13. $\mathrm{Aut}_\omega(X)$ *acts as a finite group on* $H^k(X, \mathbf{Z})$.

Example 11 also shows that the inclusion $\mathrm{Aut}^0(X) \subset \mathrm{Aut}_\omega(X)$ can be strict: Let ω_E be a Kähler form on E which is invariant under translations. The Kähler form $\omega_E \boxplus \omega_E$ on $E \times E$ descends to a Kähler form ω on X, which is preserved under the automorphism $g : X \to X$ in Example 11. In particular $g \in \mathrm{Aut}_\omega(X)$ but $g \notin \mathrm{Aut}^0(X)$.

Remark 14. In the next lectures, we will introduce the notion of entropy of an automorphism $f : X \to X$ and present Gromov-Yomdin's theorem, which identifies the entropy with the spectral radius of $f^* : H^\bullet(X, \mathbf{C}) \to H^\bullet(X, \mathbf{C})$. According to Gromov-Yomdin's theorem, the entropy of an automorphism $f : X \to X$ which fixes a Kähler class is always zero.

The key ingredient of the proof of Theorem 12 is Fujiki and Lieberman's improvement of Bishop's theorem, which we state now and refer to [6, Theorem 1.1] for a proof. Let X be a complex manifold, an analytic cycle of X is a formal linear combination $Z = \sum_{i=1}^k n_i \cdot Z_i$ of irreducible closed analytic subvarieties $Z_i \subset X$ with $n_i \in \mathbf{Z}_{>0}$. The support $|Z|$ of Z is defined to be the union of all Z_i. For every complex manifold X, Barlet had constructed a complex analytic space $\mathscr{B}(X)$ parameterizing analytic cycles of X with compact supports, which generalizes the notion of Chow variety.

Theorem 15 (Bishop, Fujiki-Lieberman [6, Theorem 1.1])**.** *Let X be a complex manifold and fix a Hermitian metric h on X. Let $S \subset \mathscr{B}(X)$ be a Zariski closed subset and let Z_s be the analytic cycle parameterized by $s \in S$. Suppose that*

i) $\bigcup_{s \in S} Z_s$ is contained in a compact subset of X;
ii) there exists $M > 0$ such that $\mathrm{vol}_h(|Z_s|) \leq M$ for every $s \in S$.

Then S is compact.

Proof of Theorem 12. Let
$$c : \mathscr{D}(X \times X) \to \mathscr{B}(X \times X)$$
be the Douady-Barlet morphism[1]. The restriction of c to the subspace of $\mathscr{D}(X \times X)_r \subset \mathscr{D}(X \times X)$ parameterizing irreducible reduced analytic subsets is isomorphic onto its image $\mathscr{B}(X \times X)_r \subset \mathscr{B}(X \times X)$ and $c^{-1}(\mathscr{B}(X \times X)_r) = \mathscr{D}(X \times X)_r$. As $\mathrm{Aut}_\omega(X) \subset \mathscr{D}(X \times X)_r$, it suffices to show that $c(\mathrm{Aut}_\omega(X))$ has only finitely many connected components.

As $\mathrm{Aut}_\omega(X)$ is a union of connected components of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X)$ is Zariski open in $\mathscr{D}(X \times X)$ by the proof of Theorem 5, $\mathrm{Aut}_\omega(X)$ is also Zariski open in $\mathscr{D}(X \times X)$. Since $\mathscr{D}(X \times X)_r$ is Zariski open in $\mathscr{D}(X \times X)$, $\mathrm{Aut}_\omega(X)$ is Zariski open in $\mathscr{D}(X \times X)_r$. It follows that $c(\mathrm{Aut}_\omega(X))$ is Zariski open in $\mathscr{B}(X \times X)_r$ and since $\mathscr{B}(X \times X)_r$ is a Zariski open of $\mathscr{B}(X \times X)$, $c(\mathrm{Aut}_\omega(X))$ is a Zariski open of $\mathscr{B}(X \times X)$ as well.

Let h be the Hermitian metric on $X \times X$ associated to the Kähler form $p_1^*\omega + p_2^*\omega$ where $p_i : X \times X \to X$ is th i-th projection. For every $f \in \mathrm{Aut}_\omega(X)$, let $\Gamma_f \subset X \times X$ denote the graph of f. Then
$$\mathrm{vol}_h(\Gamma_f) = \frac{1}{n!}\int_{\Gamma_f}\left(p_1^*\omega + p_2^*\omega\right)^n = \frac{1}{n!}\int_X p_{1*}\left(p_1^*\omega + p_2^*\omega\right)^n = \frac{1}{n!}\int_X (\omega + f^*\omega)^n = \frac{2^n}{n!}\int_X \omega^n,$$
which is constant with respect to f. It follows from Theorem 15 that $c(\mathrm{Aut}_\omega(X))$ is relatively compact. In particular, $c(\mathrm{Aut}_\omega(X))$ is contained in a finite union of connected components of $\mathscr{B}(X \times X)$. Hence $c(\mathrm{Aut}_\omega(X))$ has only finitely many connected components. □

Exercise 16. *Show that if $\rho(X) = 1$, then* $\mathrm{Aut}(X)$ *has only finitely many connected components.*

[1]The Douady-Barlet morphism is the complex analytic generalization of the Hilbert-Chow morphism sending a subscheme Z of a projective variety to $\sum_{i \in I} n_i Z_i$ where $Z = \cup_{i \in I} Z_i$ is the decomposition of Z into its irreducible components and n_i is the length of the zero-dimensional local ring $\mathscr{O}_{Z,Z_i}$.

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