Modern Algebra I

A first course in representation theory

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Contents

Convention

We assume the axiom of choice together with its consequence, e.g. the Zorn lemma.

Theorem 0.1 (Zorn's lemma)**.** *Let Σ be a nonempty partially ordered set. Assume that every totally ordered subset of Σ has an upper bound in Σ. Then Σ has a maximal element.*

LECTURE 1

Group actions and examples

"Numbers measure size, groups measure symmetry."

— M. A. Armstrong (Groups and symmetry)

1. Group actions

Let G be a group.

1.1. The category of G-sets. A *G-action* on a set X is a group homomorphism $G \to \mathrm{Bij}(X)$, often denoted by $G \circlearrowright X$. Explicitly, it is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned} \tag{1.1}$$

such that $x \mapsto g \cdot x$ is bijective self-map of X for every g, and

$$g \cdot (h \cdot x) = (gh) \cdot x$$

for any $g, h \in G$ and $x \in X$.

A set X endowed with a G-action is also called a *G-set*. A morphism of G-sets is a map $f : X \to Y$ between G-sets such that

$$g \cdot f(x) = f(g \cdot x)$$

for all $g \in G$ and $x \in X$. If $\alpha : G \to \mathrm{Bij}(X)$ and $\beta : G \to \mathrm{Bij}(Y)$ are the group homomorphisms defining the G-actions on X and Y, then a morphism $f : X \to Y$ of G-sets gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Bij}(X) & \xrightarrow{\phi \mapsto f \circ \phi \circ f^{-1}} & \mathrm{Bij}(f(X)) \\ & {}_{\alpha}\nwarrow \quad \nearrow_{\beta} & \\ & G & \end{array}$$

An *isomorphism* of G-sets is a bijective morphism of G-sets.

1.2. Example: the set of cosets. Let $H \leqslant G$ be a subgroup. For any $g \in G$ and any (left-)coset $g'H$, define

$$g \cdot (g'H) := (gg')H.$$

Exercise 1.1. Verify that this defines a group action $G \circlearrowright G/H$.

Proposition 1.2. *Let H and H' be two subgroups of G. The following assertions are equivalent.*

(i) $G/H \simeq G/H'$ *as G-sets;*
(ii) *H is conjugate to H'.*

Proof. Recall that for any $g, g' \in G$, we have

$$gH = g'H \quad \text{if and only if} \quad g^{-1}g' \in H; \tag{*}$$

we will repeatedly use this fact after. First we prove the following statement.

Claim. *Let $\phi : G/H \to G/H'$ be a morphism of G-sets. Then ϕ is surjective and we have*

$$H \subset gH'g^{-1}$$

for some $g \in G$.

PROOF. We have $\phi(H) = gH'$ for some $g \in G$, so

$$\phi(g'H) = \phi(g' \cdot H) = g' \cdot \phi(H) = g'gH'$$

for every $g' \in G$, which shows that ϕ is surjective. For every $h \in H$, the cosets

$$\phi(H) = gH' \quad \text{and} \quad \phi(hH) = hgH'$$

are equal, so $g^{-1}hg \in H'$ by (*), which proves that $H \subset gH'g^{-1}$. □

Exercise 1.3. In the above Claim, show that $H = gH'g^{-1}$ if and only if ϕ is injective. Complete the proof of Proposition 1.2.

□

1.3. Stabilizers. Let $G \circlearrowright X$ be a group action and let $x \in X$. The subset $G \cdot x \subset X$ is called the *orbit* of x. The *stabilizer* of an element $x \in X$ is the subgroup

$$\mathrm{Stab}(x) := \{ g \in G \mid g \cdot x = x \}.$$

The G-action $G \circlearrowright X$ restricts to a G-action on $G \cdot x$. Define

$$\begin{aligned} \mu : G/\mathrm{Stab}(x) &\to G \cdot x \\ g \cdot \mathrm{Stab}(x) &\mapsto g \cdot x. \end{aligned} \tag{1.2}$$

This is a well-defined map: if $g \cdot \mathrm{Stab}(x) = g' \cdot \mathrm{Stab}(x)$, then $g^{-1}g' \in \mathrm{Stab}(x)$, so

$$g \cdot x = g \cdot (g^{-1}g') \cdot x = g' \cdot x.$$

We verify that μ is a morphism of G-sets.

Theorem 1.4 (Orbit-stabilizer theorem). *μ defines an isomorphism of G-sets*

$$G/\mathrm{Stab}(x) \xrightarrow{\sim} G \cdot x.$$

PROOF. We verify easily that μ is surjective. Now suppose that $g, g' \in G$ are two elements such that $g \cdot x = g' \cdot x$. Then $g^{-1}g' \in \mathrm{Stab}(x)$, so $g \cdot \mathrm{Stab}(x) = g' \cdot \mathrm{Stab}(x)$. Thus μ is injective. □

A G-action $G \circlearrowright X$ is called *transitive* if is has exactly one orbit.[1] The following statement is an immediate consequence of Theorem 1.4.

Corollary 1.5. *A G-action $G \circlearrowright X$ is transitive if and only if $X \simeq G/H$ as G-sets for some subgroup $H \leqslant G$.*

1.4. Partition a G-set into orbits. Let $G \circlearrowright X$ be a group action. For every $x, y \in X$, we have either

$$G \cdot x = G \cdot y \quad \text{or} \quad G \cdot x \cap G \cdot y = \varnothing.$$

Thus the set of orbits

$$\mathrm{Orb}(G \circlearrowright X) := \{ G \cdot x \mid x \in X \}$$

forms a partition of X. Together with Corollary 1.5, we deduce that

$$X \simeq \bigsqcup_{G \cdot x \,\in\, \mathrm{Orb}(G \circlearrowright X)} G/\mathrm{Stab}(x) \tag{1.3}$$

as G-sets. In particular, we have the following statement:

Corollary 1.6 (Burnside's lemma). *Assume that both G and X are finite, then*

$$\frac{|X|}{|G|} = \sum_{G \cdot x \,\in\, \mathrm{Orb}(G \circlearrowright X)} \frac{1}{|\mathrm{Stab}(x)|}.$$

Informally, the above formula provides in some sense a more correct way of counting "X/G" by taking into account the symmetry of the objects, than just counting the number of orbits. For instance,

[1] Thus an empty G-set is not transitive.

if G acts on a point $X = \{*\}$, then the "number" of X/G should be 1 divided by $|G|$ (the number of symmetries of the point).

1.5. Burnside rings. Let G be a finite group. The *Burnside ring* of G is the $\mathbf{Z}$-module $B(G)$ defined as follows. First let

$$\Lambda(G) := \mathbf{Z}[\text{Isomorphism classes of finite } G\text{-sets}]$$

be the free $\mathbf{Z}$-module generated by all isomorphism classes finite G-sets. For any G-set X, let $[X]$ denote its class in $\Lambda(G)$. Define the product

$$[X_1] \cdot [X_2] = [X_1 \times X_2]$$

for any pair of G-sets X_1 and X_2, then extend it linearly to the whole $\Lambda(G)$. This turns $\Lambda(G)$ into a commutative ring with $1 = [G \circlearrowright \{*\}]$.

Now consider the subgroup of $\Lambda(G)$ generated by

$$[X_1 \sqcup X_2] - [X_1] - [X_2],$$

which forms an ideal (of relations) $\mathscr{R}$ of $\Lambda(G)$. The Burnside ring is defined as

$$B(G) := \Lambda(G)/\mathscr{R}.$$

For instance, we have $[G \circlearrowright \varnothing] = 0$ in $B(G)$.

The $\mathbf{Z}$-module structure of $B(G)$ is easy to describe.

Exercise 1.7. Show that as $\mathbf{Z}$-modules,

$$B(G) \simeq \bigoplus \mathbf{Z}[G/H] \tag{1.4}$$

where the direct sum runs through all conjugacy classes of subgroups of G, and H is a representative for each conjugacy class.

The product operation of $B(G)$ is more complicated.

Exercise 1.8. Describe $B(\mathfrak{S}_3)$ as a ring. For instance, describe $B(\mathfrak{S}_3)$ using (1.4), and compute the multiplication table of the generators.

1.6. Double cosets. Let G be any group and let $K, H \leqslant G$ be subgroups of G. How to describe the G-orbits of $(G/K) \times (G/H)$?

The *double cosets* KgH with g goes through G form a partition of G. Let $K\backslash G/H$ be the set of double cosets.

Exercise 1.9. Show that

$$K\backslash G/H \to \operatorname{Orb}(G \circlearrowright (G/K) \times (G/H))$$

sending KgH to $G \cdot (K, gH)$, and

$$K\backslash G/H \to \operatorname{Orb}(K \circlearrowright (G/H))$$

sending KgH to $K \cdot (gH)$ are both bijections.

2. Symmetry group

2.1. The linear isometry group on $\mathbf{R}^n$. Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. The Euclidean inner product $(\bullet|\bullet)$ on $\mathbf{R}^n$ defines a metric on $\mathbf{R}^n$, and a linear transformation $g \in \mathrm{GL}(n, \mathbf{R})$ is an isometry if and only if g preserves $(\bullet|\bullet)$, namely g is in the orthogonal group $O(n, \mathbf{R})$.

The orthogonal group $O(n, \mathbf{R})$ has two connected components: these are the preimages of the determinant

$$\det : O(n, \mathbf{R}) \to \{\pm 1\}.$$

The kernel of det is called the *special orthogonal group*, denoted by $\mathrm{SO}(n, \mathbf{R})$. Equivalently, $\mathrm{SO}(n, \mathbf{R})$ is the subgroup of *orientation-preserving* elements of $O(n, \mathbf{R})$. Formally this means that if we fix any basis

$e_1, \dots, e_n$ of $\mathbf{R}^n$, an element $g \in O(n, \mathbf{R})$ is in $\mathrm{SO}(n, \mathbf{R})$ if and only if

$$g(e_1 \wedge \cdots \wedge e_n) = e_1 \wedge \cdots \wedge e_n \in \bigwedge^n \mathbf{R}^n.$$

Elements of $\mathrm{SO}(n, \mathbf{R})$ are also called *rotations*.

2.2. Decomposition group, inertia group. Let $G \circlearrowright X$ be a group action on a set X. Let $Y \subset X$ be a subset. The *decomposition group* and the *inertia group* of Y with respect to $G \circlearrowright X$ are defined as

$$\mathrm{Dec}(Y) := \{ g \in G \mid g(Y) = Y \},$$
$$\mathrm{Ine}(Y) := \{ g \in G \mid g(y) = y \text{ for all } y \in Y \} = \bigcap_{y \in Y} \mathrm{Stab}(y).$$

For instance, the subgroup $O(n, \mathbf{R}) \subset \mathrm{GL}(n, \mathbf{R})$ is also the decomposition of a sphere $S^{n-1} \subset \mathbf{R}^n$ centered at the origin, with respect to the linear action $\mathrm{GL}(n, \mathbf{R}) \circlearrowright \mathbf{R}^n$. The inertia group of a line L in $\mathbf{R}^3$ for the action $\mathrm{SO}(3, \mathbf{R}) \circlearrowright \mathbf{R}^3$ consists of the rotations with axis L.

In these lectures, the decomposition group of $Y \subset \mathbf{R}^n$ with respect to $O(n, \mathbf{R}) \circlearrowright \mathbf{R}^n$ is called the *symmetry group* of Y. Replacing $O(n, \mathbf{R})$ with $\mathrm{SO}(n, \mathbf{R})$, we call the resulting decomposition group the *rotational symmetry group* (or *chiral* symmetry group) of Y.

Exercise 2.1. What is the symmetry group and the rotational symmetry group of a plane $\mathbf{R}^2 \subset \mathbf{R}^3$?

2.3. The symmetry of regular polygon. Let $\Pi \subset \mathbf{R}^2$ be a regular n-gon centered at the origin. Then the rotation ρ with angle $2\pi/n$ preserves Π. The reflection σ with respect to the line passing through the origin and a vertex of Π (or the midpoint of an edge) also preserves Π.

Exercise 2.2.

(1) Show that the rotational symmetry group of Π is the cyclic group of rotations with with angle $\frac{2\pi}{n} \cdot \mathbf{Z}$.

(2) Show that the symmetry group of Π is generated by ρ and σ, and consists of rotations with angle $\frac{2\pi}{n} \cdot \mathbf{Z}$ and reflections. The latter group is called the *dihedral group* and is denoted by D_n. What is the order of D_n?

(3) Show that the dihedral group is defined by generators and relations as follows:

$$D_n = < r, s \mid r^n = s^2 = (rs)^2 = 1 > .$$

(4) Classify the finite subgroups of $\mathrm{SO}(2, \mathbf{R})$ and $O(2, \mathbf{R})$.

2.4. Tetrahedraon. Let $T \subset \mathbf{R}^3$ be a tetrahedron centered at origin.

Exercise 2.3. By considering group actions on the vertices of T, show that:

(1) The symmetry group of T is isomorphic to the permutation group $\mathfrak{S}_4$.

(2) The rotational symmetry group of T is isomorphic to the alternating group $\mathfrak{A}_4$. (Hint: otherwise, it contains a transposition, so an element fixing two vertices of T.)

2.5. Cube and octahedron. Let $C \subset \mathbf{R}^3$ be a cube centered at origin.

Exercise 2.4.

(1) Show that the rotational symmetry group of C is isomorphic to the permutation group of the set of four diagonals of C.

(2) Find a tetrahedron T inscribed in C, and identify $\mathfrak{A}_4 \subset \mathfrak{S}_4$ as the subgroup preserving T.

(3) Show that a cube and an octahedron have the same symmetry group. (Hint: cube and octahedron are "dual" solids.)

2.6. Dodecahedron and icosahedron. Let $D \subset \mathbf{R}^3$ be a dodecahedron centered at origin.

Exercise 2.5.

(1) We define a *needle* to be a vertex of D together with an edge adjacent to it. Show that the rotational symmetry group G of D acts freely and transitively on the set of needles. Deduce that $|G| = 60$.
(2) Show that there are exactly 5 tetrahedra inscribed in D. Show that permutations of these tetrahedra yield an injective homomorphism $G \hookrightarrow \mathfrak{S}_5$.
(3) Show that for each pair $i \neq j$, there exists a unique pair of vertices of $T_i \cup T_j$ such that the line L passing through them is a diagonal of D.
(4) Deduce that the rotations of D preserving T_i and T_j yields cyclic permutations of the three remaining tetrahedra. Conclude that the image of $G \hookrightarrow \mathfrak{S}_5$ does not contain any transposition.
(5) Show that $\mathfrak{A}_n$ is the unique subgroup of index 2 of $\mathfrak{S}_n$, and conclude that $G \simeq \mathfrak{A}_5$. (Hint: show that the only surjective homomorphism $\mathfrak{S}_n \twoheadrightarrow \{\pm 1\}$ onto $\{\pm 1\}$ is the signature homomorphism; recall that transpositions are all conjugates and they generate $\mathfrak{S}_n$.)
(6) Show that a dodecahedron and an icosahedron have the same symmetry group.

2.7. Finite subgroups of $\mathrm{SO}(3, \mathbf{R})$.

Theorem 2.6. *A finite subgroup of* $\mathrm{SO}(3, \mathbf{R})$ *is isomorphic to one of the following.*

(1) $\mathbf{Z}/n\mathbf{Z}$ *(rotations of a regular n-gon).*
(2) D_n *(rotations + reflections of a regular n-gon).*
(3) $\mathfrak{A}_4$ *(rotations of a tetrahedron).*
(4) $\mathfrak{S}_4$ *(rotations of a cube or an octahedron).*
(5) $\mathfrak{A}_5$ *(rotations of a dodecahedron or an icosahedron).*

Exercise 2.7. The aim of this exercise is to prove the above theorem. Let $G \leqslant \mathrm{SO}(3, \mathbf{R})$ be a finite subgroup.

(1) Consider the G-action on the unit sphere S^2. Show that the stabilizer of a point $p \in S^2$ is a cyclic group $\mathbf{Z}/m_p\mathbf{Z}$ and that $m_{h \cdot p} = m_p$ for every $h \in G$.
(2) A point $p \in S^2$ is called a *pole* if p is fixed by some nontrivial $g \in G$. Let $p_1, \ldots, p_k$ be representatives of the orbits of $G \circlearrowright \{\text{poles}\}$ and let $m_i := m_{p_i}$. Show that
$$|G| - 1 = \frac{1}{2} \sum_{i=1}^{k} \frac{|G|}{m_i}(m_i - 1).$$
(3) Prove Theorem 2.6.
(4) Prove the following corollary.

Corollary 2.8 (Euclid). *There exist exactly five regular polyhedra in* $\mathbf{R}^3$.

3. Counting flags via group actions

3.1. Flag varieties. Let $\mathbf{k}$ be a field and let V be an n-dimensional $\mathbf{k}$-vector space. A *flag variety* is the set of nested linear subspaces of some fixed dimensions (draw picture)

$$\mathrm{Fl}(i_1 < i_2 < \cdots < i_k, V) := \left\{ V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_k} \subset V \;\middle|\; V_{i_j} \text{ linear subspace of } V \text{ with } \dim V_{i_j} = i_j \right\}.$$

Grassmannians

$$\mathrm{Gr}(m, V) := \{ W \subset V \mid \dim W = m \}$$

are particular examples of flag varieties.

The linear group action $\mathrm{GL}(V) \circlearrowright V$ induces a $\mathrm{GL}(V)$-action on each flag variety. Since scalar matrices stabilize each flag, $\mathrm{GL}(V) \circlearrowright \mathrm{Fl}(\cdots, V)$ descends to $\mathrm{PGL}(V) \circlearrowright \mathrm{Fl}(\cdots, V)$.

Exercise 3.1. Show that the $\mathrm{GL}(V)$-actions on flag varieties are transitive.

3.2. Counting flags. Now let $\mathbf{k} = \mathbf{F}_q$ be a finite field of cardinal q. Then flag varieties are finite sets. Using the $\mathrm{GL}(V)$-actions on flag varieties we can count the number of flags.

Exercise 3.2.

(1) Show that
$$|\mathrm{GL}(n, \mathbf{F}_q)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}).$$
(2) Count the number of stabilizers of an m-dimensional $W \subset V$ for the $\mathrm{GL}(V)$-action.

(3) Deduce that
$$|\mathrm{Gr}(m, V)| = \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{m-1})}{(q^m - 1)(q^m - q)\cdots(q^m - q^{m-1})}.$$

As a consequence, the fraction in the above formula is an integer.

We can count the cardinal number of any flag variety in a similar way.

3.3. Asides: q-analog. Regarding q as a formal variable, the q-analog of an integer n is defined as
$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}.$$
When $q \in \mathbf{R}$, the "classical n" is obtained by taking the limit
$$\lim_{q \to 1} [n]_q = n.$$
Likewise, we define
$$[n]_q! := [1]_q [2]_q \cdots [n]_q.$$
Then
$$|\mathrm{Gr}(m, V)| = \binom{n}{m}_q := \frac{[n]_q!}{[n-m]_q! [m]_q!}.$$

3.4. Examples: $\mathrm{PGL}(3, \mathbf{F}_2)$**, or the finite simple group with 168 elements.** Let $V = (\mathbf{F}_2)^3$. Then V has 7 lines and 7 planes, which correspond to 7 points and 7 lines in the projectivization
$$\mathbf{P}(V) := (V - 0)/\mathbf{k}^\times.$$

Exercise 3.3. Draw the incidence relations of these points and lines.

We can also identify the set of 7 lines in $\mathbf{P}(V)$ (together with the $\mathrm{PGL}(3, \mathbf{F}_2)$-action) with the *dual projective space*
$$\mathbf{P}(V^\vee) := (V^\vee - 0)/\mathbf{k}^\times.$$

Exercise 3.4. Let $V \xrightarrow{\sim} V^\vee$ be the isomorphism defined by the standard basis of $V = (\mathbf{F}_2)^3$. This induces a bijection $\mathbf{P}(V) \xrightarrow{\sim} \mathbf{P}(V^\vee)$. Let $p \in \mathbf{P}(V)$ and let $\ell \in \mathbf{P}(V^\vee)$ be its image.

(1) Let $M \in \mathrm{GL}(3, \mathbf{F}_2)$. Show that
$$M \cdot p = p \quad \text{if and only if} \quad {}^t M \cdot \ell = \ell.$$
(2) Let $H \leqslant \mathrm{GL}(3, \mathbf{F}_2)$ be the stabilizer of p for the $\mathrm{GL}(3, \mathbf{F}_2)$-action on $\mathbf{P}(V)$. Show that H is not conjugate to ${}^t H$.

(3) Show that the group actions
$$\mathrm{PGL}(3, \mathbf{F}_2) \circlearrowright \mathbf{P}(V) \quad \text{and} \quad \mathrm{PGL}(3, \mathbf{F}_2) \circlearrowright \mathbf{P}(V^\vee)$$
are *not isomorphic*.

3.5. Examples: $\mathrm{PGL}(2, \mathbf{F}_5)$ **and** $\mathfrak{A}_5$**.** Let V be a two-dimensional $\mathbf{k}$-vector space. We have an identification
$$\mathbf{P}(V) = \{\, [a : b] \mid a, b \in \mathbf{k}, (a, b) \neq (0, 0) \,\} / \sim$$
where $(a, b) \sim (a', b')$ whenever $(\lambda a, \lambda b) = (a', b')$ for some $\lambda \in \mathbf{k}^\times$. Sending $[a : b]$ to a/b (with $\infty := a/0$) defines another identification
$$\mathbf{P}(V) \simeq \mathbf{k} \cup \{\infty\}.$$

Exercise 3.5. Show that $\mathrm{PGL}(2, \mathbf{k}) \circlearrowright \mathbf{P}(V)$ is identified with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az+b}{cz+d}$$

through $\mathbf{P}(V) \simeq \mathbf{k} \cup \{\infty\}$.

Now let $\mathbf{k} = \mathbf{F}_5$. Consider the labelling of faces on a dodecahedron as follows:

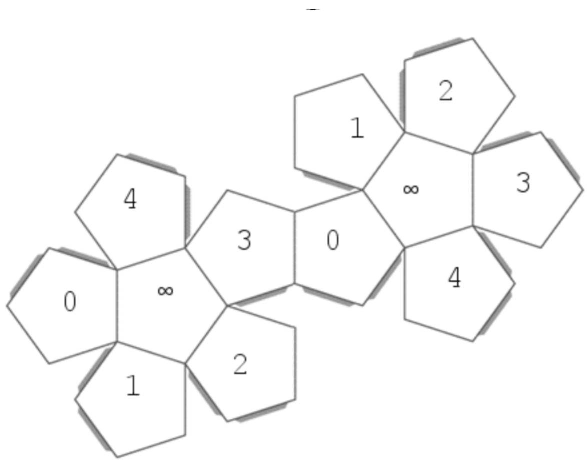


FIGURE 1. Unfolded dodecahedron from D. Speyer's answer in [5]

Exercise 3.6.

(1) Show that the action of the symmetry group G of a dodecahedron D on the labelled faces defines a faithful G-action on $\mathbf{F}_5 \cup \{\infty\}$.

(2) Show that this G-action is isomorphic to $\mathrm{PGL}(2, \mathbf{F}_5) \circlearrowright \mathbf{F}_5 \cup \{\infty\}$. Deduce that

$$\mathrm{PGL}(2, \mathbf{F}_5) \simeq \mathfrak{A}_5.$$

LECTURE 2

Quiver representations

4. Definitions and examples

4.1. Objects. A *quiver* Q is a finite directed graph. Suppose that V (resp. E) is the set of vertices (resp. edges) of Q. A *representation of* Q over a field $\mathbf{k}$ is a collection of data (V_i, f_α) consisting of

- finite dimensional $\mathbf{k}$-vector spaces V_i, one for each $i \in V$;
- $\mathbf{k}$-linear transformations $\rho_e : V_{t(\alpha)} \to V_{h(\alpha)}$, one for each edge $\alpha \in E$; here $t(\alpha)$ and $h(\alpha)$ denote the tail and the head of α respectively.

4.2. Morphisms. A *morphism* $\phi : (V_i, f_\alpha) \to (V'_i, f'_\alpha)$ of representations of Q is a collection of $\mathbf{k}$-linear transformations $\phi_i : V_i \to V'_i$ indexed by $i \in V$ such that

$$f'_\alpha \circ \phi_{h(\alpha)} = \phi_{t(\alpha)} \circ f_\alpha$$

for every $\alpha \in E$. A morphism ϕ of representations of Q is called *isomorphism* if each ϕ_i is an isomorphism. If each $\phi_i : V_i \to V'_i$ is the inclusion of a subspace of V'_i, then we call (V_i, f_α) a *sub-representation* of (V'_i, f'_α).

The category of representations of Q over a field $\mathbf{k}$ is denoted by $\mathrm{Rep}(Q, \mathbf{k})$.

4.3. Indecomposable representations. A nonzero representation (V_i, f_α) of Q is called *indecomposable* if (V_i, f_α) is not isomorphic to the direct sum

$$(V'_i, f'_\alpha) \oplus (V''_i, f''_\alpha) := (V'_i \oplus V''_i, f'_\alpha \oplus f''_\alpha)$$

of non-trivial representations of Q. Here, we say that a representation (V_i, f_α) is trivial if $V_i = 0$ for all vertex i.

We will prove the following theorem in the future.

Theorem 4.1 (Krull–Schmidt theorem). *Every representation (V_i, f_α) of Q can be decomposed into a finite direct sum of indecomposable representations:*

$$(V_i, f_\alpha) = \bigoplus_{j=1}^{n} (V_i^{(j)}, f_\alpha^{(j)}).$$

Moreover, the decomposition is unique, up to permutation of the index j and up to isomorphism of each factor $(V_i^{(j)}, f_\alpha^{(j)})$.

4.4. A_1-quiver and A_2-quiver. Consider the quiver $Q = \bullet$. Then $\mathrm{Rep}(Q, \mathbf{k})$ is equivalent to the category of $\mathbf{k}$-vector spaces. It has only one isomorphism class of indecomposable representation, which is $\mathbf{k}$.

Consider the quiver $Q = \bullet \to \bullet$. Then a representation of Q is nothing but a $\mathbf{k}$-linear morphism $\phi : V \to W$ of finite dimensional $\mathbf{k}$-vector spaces. If $V' \subset V$ is a supplement of $\ker(\phi)$ and $W' \subset W$ a suppliment of $\mathrm{Im}(\phi)$, then

$$(V \xrightarrow{\phi} W) \simeq (\ker \phi \to 0) \oplus (V' \xrightarrow{\phi} \mathrm{Im}(\phi)) \oplus (0 \to W').$$

Note that $V' \xrightarrow{\phi} \mathrm{Im}(\phi)$ is an isomorphism, so each $V \xrightarrow{\phi} W$ is isomorphic to a direct sum of copies of

$$(\mathbf{k} \to 0), \quad (\mathbf{k} \xrightarrow{\mathrm{Id}} \mathbf{k}), \quad (0 \to \mathbf{k}).$$

The above representations are indecomposable.

4.5. A_3-quivers. Consider a quiver Q whose underlying undirected graph is •—•—•. We want to decompose a representation of Q into indecomposable representations, in particular finding all indecomposable representations. There are two cases to be studied:

$$\bullet \to \bullet \leftarrow \bullet, \quad \bullet \to \bullet \to \bullet.$$

Let's start with the first case. First of all by splitting away the kernels, a quiver representation of the form $\bullet \to \bullet \leftarrow \bullet$ is isomorphic to a direct sum of quiver representations of the form

$$\bullet \hookrightarrow \bullet \hookleftarrow \bullet, \quad \mathbf{k} \to 0 \leftarrow 0, \quad 0 \to 0 \leftarrow \mathbf{k}.$$

By splitting away the common subspace of $\bullet \hookrightarrow \bullet \hookleftarrow \bullet$, the later is isomorphic to a direct sum of quiver representations of the form

$$U \hookrightarrow V \hookleftarrow W, \quad \mathbf{k} = \mathbf{k} = \mathbf{k}$$

with $U \cap W = 0$. If V' is a supplement of $U \oplus W$ in V, then

$$(U \hookrightarrow V \hookleftarrow W) = (U = U \hookleftarrow 0) \oplus (0 \hookrightarrow W = W) \oplus (0 \hookrightarrow V' \hookleftarrow 0).$$

Hence every quiver representation of Q is isomorphic to a direct sum of

$$(0 \to 0 \leftarrow \mathbf{k}),\ (0 \to \mathbf{k} \leftarrow 0),\ (\mathbf{k} \to 0 \leftarrow 0),\ (0 \to \mathbf{k} = \mathbf{k}),\ (\mathbf{k} = \mathbf{k} \leftarrow 0),\ (\mathbf{k} = \mathbf{k} = \mathbf{k}),$$

which are all irreducible.

Exercise 4.2. Do the case $\bullet \to \bullet \to \bullet$ and compare with the previous result.

4.6. Jordan quiver. The Jordan quiver is a quiver Q with one loop $\circlearrowright \bullet$. A representation of Q is the same as an endomorphism $\phi : V \to V$ of a finite dimensional $\mathbf{k}$-vector space V. Two endomorphisms $\phi_1, \phi_2 \in \mathrm{End}(V)$ define isomorphic representation of Q if and only if ϕ_1 is conjugate to ϕ_2.

A Jordan quiver thus have infinitely many indecomposable representations: for instance if $\mathbf{k}$ is infinite, then

$$\mathbf{k} \xrightarrow{\times \lambda} \mathbf{k}$$

for $\lambda \in \mathbf{k}$ are non-isomorphic indecomposable representations.

Exercise 4.3. Construct infinitely many non-isomorphic indecomposable representations of Q when $\mathbf{k}$ is a finite field.

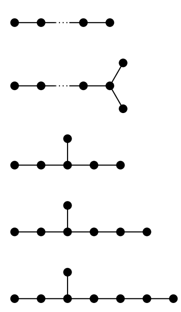
5. Dynkin quivers

5.1. Gabriel's theorem. Let Q be a quiver.

Theorem 5.1 (Gabriel)**.** *Let $\mathbf{k}$ be any field and let Q be a quiver. The following assertions are equivalent.*

(1) $\mathrm{Rep}(Q, \mathbf{k})$ *has only finitely many isomorphism classes of indecomposable representations.*
(2) The underlying undirected graph of Q is one of the ADE Dynkin diagrams:

A_n :
D_n :
E_6 :
E_7 :
E_8 :



A quiver as in the above theorem is called a *quiver of finite type* or a *Dynkin quiver*. Note that whether or not a quiver Q is of finite type only depends on the underlying undirected graph.

We will prove (1) $\implies$ (2) in Theorem 5.1 assuming that $\mathbf{k}$ is algebraically closed.

5.2. Conjugation. Let Q be a quiver, and let V (resp. E) be the set of vertices (resp. edges) of Q. For every vertex i of Q, we fix a $\mathbf{k}$-vector space V_i. Let

$$\mathrm{Rep}(Q, \mathbf{k}; V_i) = \prod_{\alpha \in E} \mathrm{Hom}_{\mathbf{k}}(V_{t(\alpha)}, V_{h(\alpha)})$$

be the space of quiver representations of Q of the form (V_i, f_α). Then

$$G := \prod_{i \in V} \mathrm{GL}(V_i) \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i)$$

defined by conjugation:

$$(g_i)_{i \in V} \cdot (V_i, f_\alpha) = (V_i, g_{h(\alpha)} \circ f_\alpha \circ g_{t(\alpha)}^{-1}).$$

By construction, two representations (V_i, f_α), (V'_i, f'_α) of Q lie in the same G-orbit if and only if they are isomorphic.

Let $d_i = \dim V_i$; we call $\mathbf{d} := (d_i)_{i \in V}$ the dimension vector of (V_i, f_α).

Lemma 5.2. *Assume that* $\mathbf{k}$ *is algebraically closed. Let* W *be a* $\mathbf{k}$*-vector space. Suppose that* W *admits a linear action* $G \circlearrowright W$ *by* $G = \prod_{i \in V} \mathrm{GL}(V_i)$ *with only finitely many orbits. Then*

$$\dim W \leqslant \sum_{i \in V} d_i^2.$$

The idea of the proof is simple with some algebraic geometry.

Proof (you may just concentrate on the idea if you haven't followed Modern Algebra II). Since $G \circlearrowright W$ has only finitely many orbits, at least one of them $G \cdot x$ is Zariski dense in W (namely, not contained in any proper subsets of W defined as the zero locus of a system of polynomials with coefficients in $\mathbf{k}$). Since $G \to G \cdot x$ is a surjective morphism of affine varieties over $\mathbf{k}$, we have

$$\sum_{i \in V} d_i^2 = \dim G \geqslant \dim(G \cdot x) = \dim W.$$

□

Proposition 5.3. *Suppose that* $\mathrm{Rep}(Q, \mathbf{k})$ *has only finitely many isomorphism classes of indecomposable representations. Then*

$$\sum_{\alpha \in E} d_{t(\alpha)} d_{h(\alpha)} < \sum_{i \in V} d_i^2.$$

as long as d_i *is not all* 0.

Proof. Since $\mathrm{Rep}(Q, \mathbf{k})$ has only finitely many isomorphism classes of indecomposable representations, Theorem 4.1 implies that $\mathrm{Rep}(Q, \mathbf{k}; V_i)$ has only finitely many isomorphism classes of representations of Q. In particular, the conjugation action $G \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i)$ has only finitely many orbits.

Consider the linear action

$$G \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i) \oplus \mathbf{k}$$

defines as the direct sum of the conjugation action $G \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i)$ and the action $G \circlearrowright \mathbf{k}$ defined by the determinant

$$(g_i)_{i \in V} \cdot \lambda = \left(\prod_{i \in V} \det(g_i) \right) \cdot \lambda.$$

For every $\mu \in \mathbf{k}^\times$, the actions $(g_i)_{i \in V} \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i)$ and $(\mu \cdot g_i)_{i \in V} \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i)$ are the same, but $(g_i)_{i \in V} \circlearrowright \mathbf{k}$ and $(\mu \cdot g_i)_{i \in V} \circlearrowright \mathbf{k}$ differ by a factor of $\mu^{\sum_i d_i}$. Since $\sum_i d_i > 0$, it follows that for every $(V_i, f_\alpha) \in \mathrm{Rep}(Q, \mathbf{k})$, the pairs

$$((V_i, f_\alpha); \lambda) \in \mathrm{Rep}(Q, \mathbf{k}; V_i) \oplus \mathbf{k}$$

lie in the same G-orbit whenever $\lambda \neq 0$. This implies that $G \circlearrowright \mathrm{Rep}(Q, \mathbf{k}; V_i) \oplus \mathbf{k}$ also has only finitely many orbits.

We apply Lemma 5.2 to conclude. □

5.3. Cartan matrix. Let Γ be a finite undirected graph and let V be the set of vertices. Define the adjacent matrix $A(\Gamma) = (a_{ij})_{i,j\in V}$ as follows:

$$a_{ij} = \begin{cases} \text{the number of edges between } i \text{ and } j & \text{if } i \neq j \\ 2 \cdot (\text{the number of loops at } i) & \text{if } i = j. \end{cases} \tag{5.1}$$

Let

$$C(\Gamma) = 2 \cdot \mathrm{Id} - A(\Gamma).$$

It is a symmetric matrix, so defines a quadratic form q_Γ on $\mathbf{R}^V$. Explicitly,

$$\frac{1}{2} q_\Gamma(d_i; i \in V) = \sum_{i \in V} d_i^2 - \sum_{\alpha \in E} d_{t(\alpha)} d_{h(\alpha)} \tag{5.2}$$

Corollary 5.4. *If Γ underlies a quiver of finite type, then q_Γ is definite positive.*

When the quadratic form q_Γ is definite positive, we call $C(\Gamma)$ the *Cartan matrix* of Γ.

Proof. By Proposition 5.3, we have $q_\Gamma(\mathbf{d}) > 0$ for every nonzero $\mathbf{d} \in \mathbf{Z}^V_{\geqslant 0}$. Since q is a quadratic form, the same holds for every nonzero $\mathbf{d} \in \mathbf{Q}^V_{\geqslant 0}$. As q is continuous and $\mathbf{Q}$ is dense in $\mathbf{R}$, the same holds for every nonzero $\mathbf{d} \in \mathbf{R}^V_{\geqslant 0}$. Finally for every nonzero $\mathbf{d} \in \mathbf{R}^V$, by (5.2) we have

$$q_\Gamma(\mathbf{d}) \geqslant q_\Gamma(|d_i|; i \in V) > 0.$$

□

Exercise 5.5.

(1) Compute $C(\Gamma)$ for a cycle graph Γ. Show that $\det C(\Gamma) = 0$.
(2) Deduce that for any finite graph Γ, if q_Γ is definite positive, then Γ is a tree.
(3) Compute $C(\Gamma)$ for the ADE Dynkin diagrams. Show that the associated quadratic form q_Γ is definite positive.

(Hint: use Sylvester's criterion.)

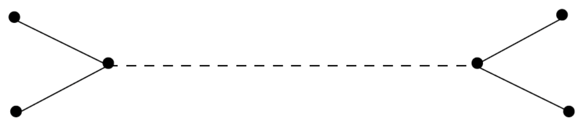
5.4. ADE Dynkin diagrams. Let Γ be a finite undirected graph.

Theorem 5.6. *The quadratic form q_Γ is definite positive if and only if Γ is an ADE Dynkin diagram.*

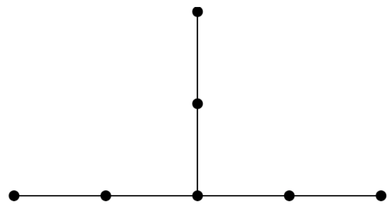
The "if" part is covered by Exercise 5.5. The following exercise proves the "only if" part.

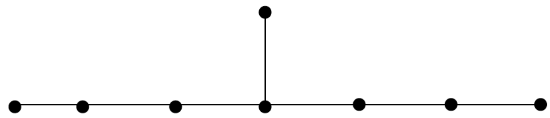
Exercise 5.7. Suppose that Γ is a tree such that q_Γ is definite positive.

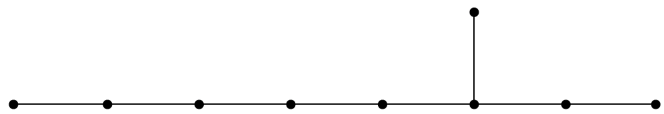
(1) Show that the associated quadratic form of the graph



is not definite positive. Deduce that Γ cannot contain a vertex with at least four incoming edges, or two vertices which have each of them at least three incoming edges.
(2) Show that Γ does not contain the following graphs:







(3) Conclude.

The implication (1) $\implies$ (2) of Theorem 5.1 then follows from Corollary 5.4 and Theorem 5.6 (for $\mathbf{k}$ algebraically closed).

LECTURE 3

Categories

6. Definitions and examples

6.1. Categories. A (locally small) category $\mathscr{C}$ consists of

- A class $\mathrm{Ob}(\mathscr{C})$ (or $\mathscr{C}$ by abuse of notation) of objects;
- For all $X, Y \in \mathrm{Ob}(\mathscr{C})$ a set of morphisms

$$\mathrm{Hom}(X, Y) = \{\phi : X \to Y\};$$

- A collection of maps

$$\begin{aligned} \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) &\xrightarrow{\circ} \mathrm{Hom}(X, Z) \\ (f, g) &\mapsto g \circ f. \end{aligned} \tag{6.1}$$

  for all $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$,

subject to the following conditions:

(1) The sets $\mathrm{Hom}(X, Y)$ are pairwise disjoint;
(2) For every $X \in \mathrm{Ob}(\mathscr{C})$, there exists $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$ such that

$$\mathrm{Id}_X \circ f = f \quad \text{and} \quad g \circ \mathrm{Id}_X = g$$

for all $f \in \mathrm{Hom}(Y, X)$ and $g \in \mathrm{Hom}(X, Y)$;
(3) For all morphisms f, g, h in $\mathscr{C}$, we have

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever the compositions are defined.

Exercise 6.1. Show that Id_X is unique for every $X \in \mathrm{Ob}(\mathscr{C})$.

A morphism $\phi : X \to Y$ in $\mathbf{C}$ is called an *isomorphism* if there exists $\psi : Y \to X$ such that

$$\phi \circ \psi = \mathrm{Id} \quad \text{and} \quad \psi \circ \phi = \mathrm{Id}.$$

In this case, we say that X and Y are isomorphic.

6.2. Examples. Fix a field $\mathbf{k}$.

(1) The category of sets: objects are sets and morphisms are maps between sets.
(2) The category of groups: objects are groups and morphisms are group homomorphisms.
(3) The category of $\mathbf{k}$-vector spaces: objects are $\mathbf{k}$-vector spaces and morphisms are linear transformations.
(4) Fix a group G. We've defined the category of G-sets before.
(5) The category of field extensions $L/\mathbf{k}$ over $\mathbf{k}$. A morphism from $L/\mathbf{k}$ to $L'/\mathbf{k}$ is a morphism of $\mathbf{k}$-algebras $L \to L'$.
(6) Sometimes morphisms are not maps in the set-theoretical sense. For instance, fix a topological space X. We can consider the category whose objects are points of X, and morphisms $p \to q$ between two points $p, q \in X$ are paths from p to q up to reparameterization. Composition is defined by concatenation.

6.3. Functors. A *covariant functor* $F : \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of

- A map

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{D}) \\ X &\mapsto F(X) \quad ; \end{aligned} \tag{6.2}$$

- A map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)) \\ \phi &\mapsto F(\phi) \end{aligned} \tag{6.3}$$

  for every $X, Y \in \mathrm{Ob}(\mathscr{C})$,

such that

$$F(\phi \circ \psi) = F(\phi) \circ F(\psi) \quad \text{and} \quad F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$$

for all $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms ϕ and ψ such that $\phi \circ \psi$ is defined.

Exercise 6.2. Show that F sends isomorphisms to isomorphisms.

A *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$ is defined similarly, with

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, Y) &\to \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X)) \\ \phi &\mapsto F(\phi) \end{aligned} \tag{6.4}$$

for every $X, Y \in \mathrm{Ob}(\mathscr{C})$.

6.4. Examples. Again, we fix a field $\mathbf{k}$.

(1) Fix $N \in \mathbf{Z}_{>0}$

$$\begin{aligned} \mathrm{GL}_N : \mathrm{Fields}/\mathbf{k} &\to \mathrm{Groups} \\ L/\mathbf{k} &\mapsto \mathrm{GL}_N(L) \end{aligned} \tag{6.5}$$

is a covariant functor.

(2) Forgetful functor: for instance

$$\mathbf{k}\text{-vector spaces} \to \mathrm{Groups} \to \mathrm{Sets}$$

are covariant functors.

(3) For any category $\mathscr{C}$ and any $A \in \mathrm{Ob}(\mathscr{C})$,

$$\begin{aligned} \mathrm{Hom}(\bullet, A) : \mathscr{C} &\to \mathrm{Sets} \\ B &\mapsto \mathrm{Hom}(B, A) \end{aligned} \tag{6.6}$$

is a contravariant functor.

(4) Let $\mathrm{Vect}_{\mathbf{k}}$ denote the category of $\mathbf{k}$-vector spaces. Taking dual

$$\begin{aligned} (\bullet)^\vee : \mathrm{Vect}_{\mathbf{k}} &\to \mathrm{Vect}_{\mathbf{k}} \\ V &\mapsto V^\vee \end{aligned} \tag{6.7}$$

is a contravariant functor.

6.5. Full functors, faithful functors. A (covariant) functor $F : \mathscr{C} \to \mathscr{D}$ is called

- *full* if $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is surjective;
- *faithful* if $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is injective.

Exercise 6.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a fully faithful functor. For every $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ show that f is an isomorphism if and only if $F(f)$ is an isomorphism.

6.6. Subcategories. A *subcategory* $\mathscr{D} \subset \mathscr{C}$ of a category $\mathscr{C}$ is a category $\mathscr{D}$ such that

- $\mathrm{Ob}(\mathscr{D}) \subset \mathrm{Ob}(\mathscr{C})$;

- $\mathrm{Hom}_{\mathscr{D}}(X,Y) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathscr{D})$, which is compatible with compositions and identities.

We call $\mathscr{D} \subset \mathscr{C}$ a *full* subcategory if $\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all $X, Y \in \mathrm{Ob}(\mathscr{D})$.

7. Equivalence of categories

We start with a guiding example. Fix a filed $\mathbf{k}$ and consider the category $\mathrm{Vect}_{\mathbf{k},f}$ of finite dimensional $\mathbf{k}$-vector spaces. Let $\mathscr{N}$ be the full subcategory of $\mathrm{Vect}_{\mathbf{k},f}$ with

$$\mathrm{Ob}(\mathscr{N}) = \left\{ \mathbf{k}^N \mid N \in \mathbf{Z}_{\geqslant 0} \right\}.$$

We want $\mathscr{N} \hookrightarrow \mathrm{Vect}_{\mathbf{k},f}$ to be an equivalence of categories.

As a first attempt, we start with a definition, which is actually too strong to be useful. We say that two categories $\mathscr{A}$ and $\mathscr{B}$ are *isomorphic* if there exist functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{B}$ such that $F \circ G = \mathrm{Id}_{\mathscr{B}}$ and $G \circ F = \mathrm{Id}_{\mathscr{A}}$. If two categories $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, then F and G define a *bijection* between $\mathrm{Ob}(\mathscr{A})$ and $\mathrm{Ob}(\mathscr{B})$. For instance the category $\mathscr{C}$ above is not isomorphic to $\mathrm{Vect}_{\mathbf{k},f}$.

A more natural definition is the following.

Definition 7.1. A functor $F : \mathscr{A} \to \mathscr{B}$ is called an *equivalence* of categories if

- F is fully faithfull;
- F is essentially surjective: namely for any $Y \in \mathscr{B}$, there exists $X \in \mathscr{A}$ such that $F(X)$ is *isomorphic* to Y.

Exercise 7.2. Show that the category $\mathscr{N}$ is equivalent to $\mathrm{Vect}_{\mathbf{k},f}$.

Example 7.3. Let $F : \mathscr{C} \to \mathscr{D}$ be a fully faithful functor. Then $\mathscr{C}$ is equivalent to a full subcategory of $\mathscr{D}$.

7.1. Natural transformations. Let $F, G : \mathscr{A} \to \mathscr{B}$ be a two functors. A natural transformation $f : F \to G$ is a collection of morphisms $f(X) : F(X) \to G(X)$ for each $X \in \mathrm{Ob}(\mathscr{A})$ such that for every morphism $\phi : X \to Y$ in $\mathscr{A}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{f(X)} & G(X) \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle G(\phi)} \\ F(Y) & \xrightarrow{f(Y)} & G(Y) \end{array}$$

commutes.

Natural transformations are the morphisms in the category $\mathrm{Funct}(\mathscr{A}, \mathscr{B})$ of functors from $\mathscr{A}$ and $\mathscr{B}$.

Exercise 7.4. Show that the bidual functor $(\bullet)^{\vee\vee} : \mathrm{Vect}_{\mathbf{k},f} \to \mathrm{Vect}_{\mathbf{k},f}$ is isomorphic to the identity functor $\mathrm{Id} : \mathrm{Vect}_{\mathbf{k},f} \to \mathrm{Vect}_{\mathbf{k},f}$.

7.2. Equivalence of categories: an equivalent definition. Let $\mathscr{A}$ and $\mathscr{B}$ be categories.

Theorem 7.5. *The following assertions are equivalent.*

(1) $\mathscr{A}$ and $\mathscr{B}$ are equivalent.
(2) There exist functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G \simeq \mathrm{Id}_{\mathscr{B}}$ and $G \circ F \simeq \mathrm{Id}_{\mathscr{A}}$.

We refer to [**1**, Theorem II.2.7] for a proof. By Theorem 7.5, we see that both isomorphism of categories and equivalence of categories are about the existence of some functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$. For the equivalence of categories, instead of $F \circ G = \mathrm{Id}$, we only require $F \circ G \simeq \mathrm{Id}$.

Exercise 7.6. Show that the category $\mathscr{N}$ is equivalent to $\mathrm{Vect}_{\mathbf{k},f}$, using Theorem 7.5 (2) instead of the definition.

7.3. Example: Galois theory. Let $\mathbf{k}$ be a field. Fix a separable closure $\mathbf{k}^s$ of $\mathbf{k}$. The absolute Galois group $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ is defined to be the group of of automorphisms of $\mathbf{k}^s$ as a $\mathbf{k}$-algebra. For every finite Galois extension $L/\mathbf{k}$ in $\mathbf{k}^s$, an automorphism of $\mathbf{k}^s$ fixing $\mathbf{k}$ restricts to an automorphism of L, which gives rise to a group homomorphism.

$$\mathrm{Gal}(\mathbf{k}^s/\mathbf{k}) \to \mathrm{Gal}(L/\mathbf{k}).$$

Exercise 7.7. Show that

$$\mathrm{Gal}(\mathbf{k}^s/\mathbf{k}) = \varprojlim_{L/\mathbf{k}} \mathrm{Gal}(L/\mathbf{k}),$$

where the projective limit runs through all finite extensions $L/\mathbf{k}$ in $\mathbf{k}^s$. Explicitly, $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ is the subgroup

$$\mathrm{Gal}(\mathbf{k}^s/\mathbf{k}) \leqslant \prod_{L/\mathbf{k}} \mathrm{Gal}(L/\mathbf{k})$$

consisting of $(g_L) \in \prod_{L/\mathbf{k}} \mathrm{Gal}(L/\mathbf{k})$ such that for every pair of finite Galois extensions $L/\mathbf{k}$ and $L'/\mathbf{k}$ such that $L' \subset L$, the image of g_L under $\mathrm{Gal}(L/\mathbf{k}) \to \mathrm{Gal}(L'/\mathbf{k})$ is $g_{L'}$.

Exercise 7.8. Let $\mathbf{k}$ be a finite field. Show that

$$\mathrm{Gal}(\mathbf{k}^s/\mathbf{k}) \simeq \hat{\mathbf{Z}} := \varprojlim_{n \in \mathbf{Z}_{>0}} \mathbf{Z}/n\mathbf{Z}.$$

(Hint, recall that every finite extension of $\mathbf{F}_p$ for a prime number p is of the form $\mathbf{F}_p \to \mathbf{F}_{p^n}$.)

The absolute Galois group $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ is thus a profinite group, and is endowed with the profinite topology (namely, the topology induced from the product topology of $\prod_{L/\mathbf{k}} \mathrm{Gal}(L/\mathbf{k})$, with each $\mathrm{Gal}(L/\mathbf{k})$ endowed with the discrete topology). This makes $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ into a topological group (i.e. the product $G \times G \to G$ and the inverse $G \to G$ are both continuous.)

Exercise 7.9. Show that open subgroups of $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ have finite index.

Remark 7.10. In general, finite index subgroups of $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ are not open. See Proposition 7.29 in `https://www.jmilne.org/math/CourseNotes/FT.pdf` for some construction of such subgroups in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ using the axiom of choice.

A $\mathbf{k}$-algebra L is called *finite étale* if $L \simeq \mathbf{k}_1 \times \cdots \times \mathbf{k}_n$ where each $\mathbf{k}_i$ is a finite separable extension of $\mathbf{k}$. Note that if $\mathbf{k}$ is perfect, this is equivalent to the condition that L is reduced (i.e. L has no non-trivial nilpotent elements).

Theorem 7.11 (Galois–Grothendieck correspondence)**.** *We have an equivalence of categories:*

$$\begin{aligned} \{ \text{ Finite étale } \mathbf{k}\text{-algebras } \} &\xrightarrow{\sim} \{ \text{ Finite sets with a continuous } \mathrm{Gal}(\mathbf{k}^s/\mathbf{k})\text{-action } \} \\ L &\mapsto \{ L \to \mathbf{k}^s \text{ morphisms of } \mathbf{k}\text{-algebras } \}. \end{aligned} \tag{7.1}$$

Here for any topological group (e.g. $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$ endowed with the profinite topology), we say that a G-action on a finite set S is continuous if the map $G \times S \to S$ is continuous with S endowed with the discrete topology.

Exercise 7.12. Show that a $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$-action on a finite set S is continuous if and only if the stabilizer of each element of S is open in $\mathrm{Gal}(\mathbf{k}^s/\mathbf{k})$.

Exercise 7.13. Deduce the Galois–Grothendieck correspondence from the Galois theory (for finite extensions) you've learned in undergraduate algebra.

Remark 7.14. (For those who know some algebraic geometry) For every $\mathbf{k}$-algebra L, $\mathbf{k}$-algebra morphisms $L \to \mathbf{k}^s$ are $\mathbf{k}^s$-points of $Z_L := \mathrm{Spec}(L)$ and vice versa. So $L \mapsto Z_L(\mathbf{k}^s)$ is another way of describing the Galois–Grothendieck correspondence.

LECTURE 4

Representations of associative algebras

Group representations and quiver representations are different, but they have a common generalization: we can regard both of them as representations of associative algebras. The later also encompasses Lie algebra representations (which we wouldn't be able to explain this semester). This also explains why group representations and quiver representations share some similar properties. Working with representations of associative algebras allows us to prove these properties for both situations at the same time.

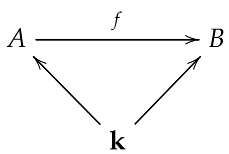
We first explain how group representations and quiver representations arise as examples of representations of associative algebras.

8. Associative k-algebras

Throughout these notes, unless otherwise specified a ring A is always assumed to be unital (i.e. there exists an element $1 \in A$ such that $1 \cdot x = x \cdot 1 = x$ for any $x \in A$). We don't assume that A is commutative in general. A morphism of rings $f : A \to B$ always maps 1 to 1.

Let $\mathbf{k}$ be a field.

8.1. Associative algebras. An *(associative)* $\mathbf{k}$*-algebra* is a ring A together with a ring homomorphism $\mathbf{k} \to A$, whose image is contained in the center $Z(A)$ of A. A morphism between two $\mathbf{k}$-algebras A and B is a ring homomorphism $f : A \to B$ which commutes with the structural morphisms:



Here are some examples of $\mathbf{k}$-algebras.

- (Endomorphism algebra) The endomorphism ring $\mathrm{End}(V)$ of a $\mathbf{k}$-vector space is a $\mathbf{k}$-algebra.
- (Free algebra) Let S be a set. The free associated algebra $\mathbf{k}\langle S\rangle$ generated by S is defined as follows. As a $\mathbf{k}$-vector space, $\mathbf{k}\langle S\rangle$ is the $\mathbf{k}$-vector space generated by all finite words

$$c_1 \cdots c_n \, ; \quad c_1, \ldots c_n \in S.$$

  The product of two words is defined by concatenation, which extends $\mathbf{k}$-linearly to a product on $\mathbf{k}\langle S\rangle$. The unit of $\mathbf{k}\langle S\rangle$ is the empty word.
- (Group ring) Let G be a group. The group ring $\mathbf{k}[G]$ with coefficient in $\mathbf{k}$ is defined as follows. As a $\mathbf{k}$-vector space, $\mathbf{k}[G]$ is the $\mathbf{k}$-vector space generated by the symbols

$$X^g \qquad (g \in G).$$

  For every $g, h \in G$, define

$$X^g \cdot X^h = X^{gh},$$

  and extends $\mathbf{k}$-linearly to a product on $\mathbf{k}[G]$.
- (Path algebra) Let Q be a quiver; the set of vertices and edges are denoted by V and E respectively. The path algebra $\mathbf{k}Q$ of Q with coefficients in $\mathbf{k}$ is defined as follows. As a $\mathbf{k}$-vector space, $\mathbf{k}Q$ is the $\mathbf{k}$-vector space generated by the directed paths in Q (namely finite sequences $e_n \cdots e_1$ of E with $h(e_j) = t(e_{j+1})$ for all j), including the trivial paths p_i for each $i \in V$.

The product PP' of Given two paths P and P', define the product

$$PP' = \begin{cases} \text{the concatenation of } P' \text{ by } P & \text{if the head of } P' \text{ is the tail of } P \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

The above definition extends $\mathbf{k}$-linearly to a product on $\mathbf{k}Q$, turning $\mathbf{k}Q$ into a $\mathbf{k}$-algebra. Note that the unit of $\mathbf{k}Q$ is $\sum_{i \in V} p_i$.

8.2. Ideals and quotients. Let A be an associative $\mathbf{k}$-algebra. A left-ideal (resp. right-ideal) of A is an additive subgroup $I \subset A$ such that $aI \subset I$ (reps. $Ia \subset I$) for all $a \in A$. A *two-sided ideal* is a subset $I \subset A$ which is both a left-ideal and a right-ideal. For instance, the kernel

$$\ker f \subset A$$

of a morphism of $\mathbf{k}$-algebra $f : A \to B$ is a two-sided ideal.

Let $I \subset A$ be a two-sided ideal. Since I is in particular a $\mathbf{k}$-linear subspace of A, we have the quotient $\mathbf{k}$-vector space A/I; let $\pi : A \to A/I$ be the quotient map. For every $x, y \in A$ and $\lambda \in \mathbf{k}$, define

$$\pi(x)\pi(y) := \pi(xy).$$

Exercise 8.1. Show that $\pi(x)\pi(y)$ is well defined, namely

$$\pi(xy) = \pi(x'y')$$

if $\pi(x) = \pi(x')$ and $\pi(y) = \pi(y')$.

This makes A/I into an associative $\mathbf{k}$-algebra, and $\pi : A \to A/I$ into a morphism of $\mathbf{k}$-algebras.

Exercise 8.2. Let $f : A \to B$ be a morphism of $\mathbf{k}$-algebras. Show that $f(A) \simeq A/\ker(f)$ as $\mathbf{k}$-algebras.

8.3. Generators and relations. Let A be a $\mathbf{k}$-algebra and let $S \subset A$ be a subset. The *$\mathbf{k}$-subalgebra generated by S* is the smallest $\mathbf{k}$-subalgebra $B \subset A$ containing S. If $B = A$, then we say that S is a set of *generators* of A.

Suppose that S is a set of generators of A. Then we have a surjective morphism of $\mathbf{k}$-algebras

$$\mathbf{k}\langle S \rangle \to A$$

sending $s \in S$ to $s \in A$. The kernel I is also called the ideal of relations. By Exercise 8.2, we have $A \simeq \mathbf{k}\langle S \rangle / I$. In particular, every $\mathbf{k}$-algebra is the quotient of a free algebra by a two-sided ideal (of relations).

Exercise 8.3. Let Q be a quiver, with the set of vertices and edges denoted by V and E respectively. Show that $\mathbf{k}Q$ is defined by generators and relations as follows.

- Generators: E and $\{ p_i \mid i \in V \}$ (trivial paths)
- Relations: the two-sided ideal "generated by"
  (1) $\sum_{i \in V} p_i = 1$;
  (2) $p_i^2 = p_i$, $p_i p_j = 0$ if $i \neq j$.
  (3) For every $e \in E$ $ep_i = e$ if i is the tail of e and 0 otherwise.
  (4) $p_i e = e$ if i is the head of e and 0 otherwise.

9. Representations of associative algebras

Let $\mathbf{k}$ be a field and let A be an associative $\mathbf{k}$-algebra.

9.1. Objects and morphisms. A *representation of A* (or a *left A-module*) is a $\mathbf{k}$-vector space V together with a $\mathbf{k}$-algebra homomorphism

$$\rho : A \to \mathrm{End}(V).$$

For every $a \in A$ and $v \in V$, we also write

$$a \cdot v := \rho(a)(v).$$

Informally, a left A-module is a $\mathbf{k}$-vector space V, but we enlarge the coefficients from $\mathbf{k}$ to A.

Let V and W be two left A-modules. A morphism from V to W is a $\mathbf{k}$-linear map $f : V \to W$ such that for every $a \in A$ and $v \in V$, we have

$$f(a \cdot v) = a \cdot f(v).$$

The space of morphisms between V and W is denoted by $\mathrm{Hom}_A(V, W)$.

The category of left A-modules is denoted by A-Mod.

9.2. Right modules, bimodules. The opposite algebra A^{op} is defined to be $A^{\mathrm{op}} := A$ as a $\mathbf{k}$-vector space, with $a \cdot b$ in A^{op} defined to be $b \cdot a$ in A. A right A-module is defined to be a left A^{op}-modules V. If $\rho : A^{\mathrm{op}} \to \mathrm{End}(V)$ is the structural morphism, we also write

$$v \cdot a := \rho(a) \cdot v$$

for every $a \in A$ and $v \in V$. The category of right A-modules is denoted by Mod-A.

Exercise 9.1. Show that $A^{\mathrm{op}} \simeq \mathrm{End}_A(A)$ as $\mathbf{k}$-algebras.

Let B be another $\mathbf{k}$-algebra. An *(A,B)-bimodule* is a $\mathbf{k}$-vector space V equipped with a left A-module structure and a right B-module structure, such that

$$(av) \cdot b = a \cdot (vb)$$

for every $a \in A$, $b \in B$ and $v \in V$.

Remark 9.2. Two $\mathbf{k}$-algebras A and B are called *Morita equivalent* if we have an equivalence of categories

$$A\text{-Mod} \simeq B\text{-Mod}.$$

In general, a $\mathbf{k}$-algebra A is not Morita equivalent to its opposite A^{op}. The Brauer group $\mathrm{Br}(\mathbf{k})$ is in bijection with the Morita equivalence classes of central simple algebras over $\mathbf{k}$, and $A \mapsto A^{\mathrm{op}}$ corresponds to taking inverse on $\mathrm{Br}(\mathbf{k})$. In general $\mathrm{Br}(\mathbf{k})$ contains nontrivial elements of order not equal to 2 (e.g. when $\mathbf{k} = \mathbf{Q}$). We would provide more explanations in the future if time permits.

Unless otherwise specified, we use the term "A-module" for left A-module.

9.3. Example: group representations. Let G be a group. A group representation of G over $\mathbf{k}$ is a group action $G \circlearrowright V$ on a $\mathbf{k}$-vector space such that $g : V \to V$ is $\mathbf{k}$-linear for every $g \in G$. A morphism between two representations $G \circlearrowright V$ and $G \circlearrowright W$ over $\mathbf{k}$ is a $\mathbf{k}$-linear map $V \to W$ of G-sets. The category of group representation of G over $\mathbf{k}$ is denoted by $\mathrm{Rep}(G, \mathbf{k})$

We can associate every group representation $\rho : G \to \mathrm{GL}(V)$ a morphism of $\mathbf{k}$-algebras

$$\begin{aligned} \mathbf{k}[G] &\to \mathrm{End}(V) \\ \sum_{g \in G} \lambda_g \cdot g &\mapsto \sum_{g \in G} \lambda_g \cdot \rho(g), \end{aligned} \tag{9.1}$$

where $\lambda_g \in \mathbf{k}$ and $\lambda_g = 0$ for all but finitely many $g \in G$.

Exercise 9.3. Show that the above construction extends to an equivalence of categories

$$\mathrm{Rep}(G, \mathbf{k}) \simeq \mathbf{k}[G]\text{-Mod}.$$

9.4. Example: quiver representations. Let Q be a quiver. We can associate every quiver representation (V_i, f_α) a morphism of $\mathbf{k}$-algebras. For every path $p = e_n \cdots e_1$ of Q, define

$$\rho(p) := f_{e_n} \circ \cdots \circ f_{e_1} : V_{t(e_1)} \to V_{h(e_n)},$$

and $\rho(p_i) := \mathrm{Id}_{V_i}$ for each trivial path p_i at the vertex i.

$$\begin{aligned} \mathbf{k}Q &\to \mathrm{End}(\oplus_i V_i) \\ \sum_{p \text{ path}} \lambda_p \cdot p &\mapsto \sum_{p \text{ path}} \lambda_p \cdot \rho(p), \end{aligned} \tag{9.2}$$

where $\lambda_p \in \mathbf{k}$ and $\lambda_p = 0$ for all but finitely many paths p.

Exercise 9.4. Show that the above construction extends to an equivalence of categories

$$\mathrm{Rep}(Q, \mathbf{k}) \simeq \mathbf{k}Q\text{-Mod}.$$

9.5. Weyl algebra and polynomial differential operators. The (first) Weyl algebra is defined by

$$A_1 := \frac{\mathbf{k}\langle x, y\rangle}{xy - yx - 1}.$$

It is also the algebra of polynomial differential operators on $\mathbf{k}[t]$: defining

$$x \cdot f = tf, \quad y \cdot f = \partial_t f,$$

gives rise to an A_1-module structure on $\mathbf{k}[t]$ thanks to the Leibniz rule.

Using the A_1-representation on $\mathbf{k}[t]$, we can prove the following.

Proposition 9.5. *The elements $x^i y^j$ for all $i, j \geqslant 0$ form a basis of the $\mathbf{k}$-vector space A_1.*

Proof assuming $\mathbf{k} = \mathbf{C}$. As $yx = xy - 1$, we can show by induction that A_1 is generated by these $x^i y^j$ as a $\mathbf{C}$-vector space.

Now let $D \in A_1$ and write

$$D = \sum_{j=0}^{n} P_j(x) y^j$$

where $P_j \in \mathbf{C}[x]$ with $P_n \neq 0$. Suppose that $D = 0$. Then $D \cdot t^N = 0$ for all positive integer N. In other words, we have for N large,

$$\sum_{j=0}^{n} N(N-1)\cdots(N-j+1) \cdot P_j(t) t^{N-j} = 0$$

in $\mathbf{C}[t]$, so

$$\sum_{j=0}^{n} N(N-1)\cdots(N-j+1) \cdot P_j(t) t^{-j} = 0$$

in $\mathbf{C}[t, t^{-1}]$. Considering the limit of the coefficients of Laurent polynomials, we have

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{j=0}^{n} N(N-1)\cdots(N-j+1) \cdot P_j(t) t^{-j} = P_n(t) t^{-n},$$

so $P_n(t) = 0$, which is a contradiction. ☐

Exercise 9.6. Prove Proposition 9.5 for any field $\mathbf{k}$ (Hint: consider N as a variable and let A_1 acting on $\mathbf{k}[N, t] \cdot t^N$. Each element of $\mathbf{k}[N, t] \cdot t^N$ is a linear combination of t^{n+N} with coefficients in $\mathbf{k}[N]$, and $y \cdot t^{n+N} := (n + N) t^{n-1+N}$.)

Likewise the nth Weyl algebra is defined by

$$A_n := \frac{\mathbf{k}\langle x_i, y_j \mid 1 \leqslant i, j \leqslant n\rangle}{x_i y_j - y_j x_i - \delta_{ij},\ \ x_i x_j - x_j x_i,\ \ y_i y_j - y_j y_i}$$

where δ_{ij} is the Kronecker delta. It is the algebra of polynomial differential operators on $\mathbf{k}[t_1, \ldots, t_n]$.

The following conjecture is still open.

Conjecture 9.7 (Dixmier conjecture)**.**

$$\mathrm{End}(A_n) = \mathrm{Aut}(A_n).$$

9.6. Example: Regular representation. The *(left) regular representation* of a $\mathbf{k}$-algebra A is

$$\begin{aligned} A &\to \mathrm{End}(A) \\ a &\mapsto (v \mapsto a \cdot v). \end{aligned} \tag{9.3}$$

For instance, if G is a group, then the G-representation which corresponds to the regular representation is

$$G \circlearrowright \bigoplus_{g \in G} \mathbf{k} \cdot e_g$$

with $g \cdot e_h = e_{gh}$.

9.7. Example: quotients by left-ideals. Let $I \subset A$ be a left-ideal. The multiplication by $\mathbf{k}$ defines $\mathbf{k}$-vector space structures on A and I. The quotient A/I is thus a $\mathbf{k}$-vector space. Let $\pi : A \to A/I$ be the quotient map.

For every $a, b \in A$, define

$$a \cdot \pi(b) := \pi(ab).$$

Since I is a left-ideal, the product $a \cdot \pi(b)$ is well defined, which defines a left A-module structure on A/I.

Exercise 9.8. Let V be an A-module and let $v \in V$. Let

$$I = \mathrm{Ann}(v) := \{ a \in A \mid a \cdot v = 0 \}$$

be the annihilator of v. Show that I is a left-ideal of A and that

$$A/I \simeq A \cdot v$$

as left A-modules.

9.8. Some basic properties of A-Mod.

- $\mathrm{Hom}_A(V, W)$ is an abelian group for every A-modules V, W, and compositions are bi-additive.
- The zero vector space is also an A-module, called the *trivial* or *zero* A-module.
- The direct sum $V \oplus W$ of two A-modules is still an A-module. This turns A-Mod into an additive category.
- For every $f \in \mathrm{Hom}_A(V, W) \subset \mathrm{Hom}_{\mathbf{k}}(V, W)$, the kernel $\ker f$ (resp. $\mathrm{coker} f$) inherits an A-module structure from V (resp. W). In particular, given an A-submodule $W \subset V$ (namely a $\mathbf{k}$-linear subspace such that $A \cdot W = W$), the quotient V/W inherits a A-module structure from V.

Exercise 9.9. Show that the A-submodules of A are the left-ideals of A.

9.9. Faithful representation. A representation $\rho : A \to \mathrm{End}_{\mathbf{k}}(V)$ of A is called *faithful* if ρ is injective. Any representation $\rho : A \to \mathrm{End}_{\mathbf{k}}(V)$, is a faithful representation over $A/\ker \rho$.

9.10. Generators and relations. Let V be a A-module and let $S \subset V$ be a subset. The *A-submodule generated by S* is the smallest A-subalgebra $W \subset V$ containing S. If $W = V$, then we say that S is a set of *generators* of V. In this case, if S is finite, then we say that V is *finitely generated*.

Suppose that S is a set of generators of W. Then we have a surjective morphism of A-algebras

$$\bigoplus_S A \to V$$

from the direct sum of regular A-representations, sending $1 \in A$ in the "$s \in S$ factor" to $s \in A$. The A-modules of the form $\bigoplus_S A$ are called *free* A-modules (of rank $|S|$). Thus every A-module V is the quotient of a free A-module by a A-submodule (of relations). If V is finitely generated, we can choose the free A-module to have finite rank.

10. Irreducible modules, indecomposable modules

10.1. Exact sequences. A sequence of morphisms of A-modules

$$U \xrightarrow{f} V \xrightarrow{g} W$$

is called *exact* if $\ker(g) = \mathrm{Im}(f)$. A sequence of morphisms of A-modules of the form

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0 \tag{10.1}$$

which is exact at each term is called a *short exact sequence*. For instance,

$$0 \to U \xrightarrow{u \mapsto (u,0)} U \oplus W \xrightarrow{(u,w) \mapsto w} W \to 0 \tag{10.2}$$

is a short exact sequence.

We say that a short exact sequence (10.1) *splits* if there exists a morphism $h : W \to V$ such that

$$g \circ h = \mathrm{Id}.$$

Exercise 10.1. Show that (10.1) splits if and only if (10.1) is isomorphic to (10.2). In other words, U (or equivalently W) is a direct summand of V.

Exercise 10.2.

(1) For $A = \mathbf{k}$, show that every short exact sequence of A-modules splits.
(2) For $A = \mathbf{k}[X]$, show that

$$0 \to \mathbf{k}[X] \xrightarrow{f \mapsto X \cdot f} \mathbf{k}[X] \xrightarrow{f \mapsto f(0)} \mathbf{k} \to 0$$

is a short exact sequence of A-modules which does not split.

10.2. Irreducible modules and indecomposable modules. A nonzero A-module V is called *irreducible* (resp. *indecomposable*) if V has no A-submodules (resp. direct summand) different from V and 0. Irreducible modules are indecomposable, but the converse does not hold in general (see e.g. Exercise 10.2). Irreducible A-modules are also called *simple* A-modules.

Exercise 10.3. Let V and W be indecomposable A-modules. Suppose that $f : V \to W$ and $g : W \to V$ are two morphisms such that $f \circ g = \mathrm{Id}_W$. Show that f and g are isomorphisms.

Proposition 10.4. *An A-module V is simple if and only if V is isomorphic to $A/\mathfrak{m}$ for some maximal left-ideal $\mathfrak{m}$.*

Proof. Let $\mathfrak{m} \subset A$ be a maximal left-ideal and let $\pi : A \to A/\mathfrak{m}$ be the projection. Let $W \subset A/\mathfrak{m}$ be an A-submodule. Then $\pi^{-1}(W) \subset A$ is a left-ideal, so either $\pi^{-1}(W) = \mathfrak{m}$ or $\pi^{-1}(W) = A$. Hence W is either 0 or $A/\mathfrak{m}$.

Let V be a simple A-module. Choose a nonzero element $v \in V$, we then have $A \cdot v = V$. So if $\mathfrak{m} := \mathrm{Ann}(v)$, then $A/\mathfrak{m} \simeq V$ as left A-modules. For every ideal $I \subset A$ containing $\mathfrak{m}$, the quotient A/I is a quotient of $A/\mathfrak{m} \simeq V$, so is either 0 or $A/\mathfrak{m}$. Hence $I = A$ or $I = \mathfrak{m}$, which shows that $\mathfrak{m}$ is a maximal ideal. □

10.3. Schur's lemma.

Proposition 10.5 (Schur's lemma). *Let $f : V \to W$ be a nonzero morphism of A-modules.*

(1) If V is irreducible, then f is injective.
(2) If W is irreducible, then f is surjective.

Proof. Since $f \neq 0$, we have $\ker f \neq V$ and $\mathrm{coker} f \neq W$. If V (resp. W) is irreducible, then the submodule $\ker f \subset V$ (resp. $\mathrm{coker} f \subset W$) is zero, so f is injective (resp. surjective). □

As a consequence, the endomorphism ring $D := \mathrm{End}_A(V)$ of an irreducible A-module is a division ring, namely every nonzero element $f \in D$ has a multiplicative inverse: an element $f^{-1} \in D$ such that $ff^{-1} = f^{-1}f = 1$. Division rings are also called *skew-fields*. Thus an irreducible A-module is naturally a left module over the division ring D.

To some extent, a division ring behaves like a field.

Exercise 10.6. Let D be a division $\mathbf{k}$-algebra. Prove the following properties.

(1) The regular representation of D is irreducible.
(2) Every D-module V is free.

Corollary 10.7. *Assume that* $\mathbf{k}$ *is algebraically closed. Let* V *be an irreducible* A*-module such that* $\dim_{\mathbf{k}} V < \infty$ *and let* $f : V \to V$ *be a morphism of* A*-modules. Then* $f = \lambda \cdot \mathrm{Id}$ *for some* $\lambda \in \mathbf{k}$*. As a consequence,*

$$\mathrm{End}_A(V) \simeq \mathbf{k}.$$

Proof. Since $\mathbf{k}$ is algebraically closed and $\dim_{\mathbf{k}} V < \infty$, the $\mathbf{k}$-linear map f has a nonzero eigenvector $v \in V$. Suppose that $f(v) = \lambda v$. Then $f - \lambda \mathrm{Id} : V \to V$ is a morphism of A-modules which is not injective. As V is irreducible as an A-module, we have $f = \lambda \mathrm{Id}$ by Schur's lemma. □

Exercise 10.8. Reformulate Corollary 10.7 for group representations and quiver representations.

Corollary-Exercise 10.9. *Suppose that* $\mathbf{k}$ *is algebraically closed and* A *is a commutative* $\mathbf{k}$*-algebra. If* V *is an irreducible* A*-module with* $\dim_{\mathbf{k}} V < \infty$*, then* $V \simeq \mathbf{k}$ *as* $\mathbf{k}$*-vector spaces.*

Exercise 10.10. Find a counterexample of Corollary 10.7 if we drop the assumption that $\mathbf{k}$ is algebraically closed. (Hint, you may consider the regular representation of the $\mathbf{R}$-algebra $\mathbf{C}$.)

10.4. Finite dimensional division algebras over R. Commutative division algebras over $\mathbf{R}$ are fields extensions of $\mathbf{R}$, and those with finite degree are $\mathbf{R}$ and $\mathbf{C}$.

An example of noncommutative division algebras over $\mathbf{R}$ is the *quaternion algebra* $\mathbf{H}$ over $\mathbf{R}$. As a vector space, $\mathbf{H}$ is 4-dimensional, generated by 1 and the symbols i, j, k. The product on $\mathbf{H}$ is defined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Exercise 10.11. Write down the multiplication table of $1, i, j, k$.

Proposition 10.12. *A finite dimensional division algebra* D *over* $\mathbf{R}$ *is isomorphic to either* $\mathbf{R}$, $\mathbf{C}$, *or* $\mathbf{H}$.

Proof. If D is commutative, we've already seen that D is isomorphic to either $\mathbf{R}$ or $\mathbf{C}$.

Assume that D is noncommutative. Then there exists $x \in D \backslash \mathbf{R}$. As the subalgebra $\mathbf{R}[x] \subset D$ generated by x is commutative, we have $\mathbf{R}[x] \simeq \mathbf{C}$. Consider $i := \sqrt{-1} \in \mathbf{C} \subset D$ and the $\mathbf{R}$-linear map $\phi : D \to D$ defined by

$$\phi(y) = i \cdot y \cdot i^{-1}.$$

As $\phi^2 = \mathrm{Id}$, we have an $\mathbf{R}$-linear decomposition

$$D = D_+ \oplus D_-$$

where $D_\pm$ is the eigenspace of ϕ of eigenvalue ± 1. For every nonzero $z \in D_-$, the left-multiplication by z defines an isomorphism from D_+ to D_-, so $\dim_{\mathbf{R}} D_+ = \dim_{\mathbf{R}} D_-$. Note that for every $y \in D_+$, the subalgebra $\mathbf{C}[y] \subset D$ is commutative, thus $\mathbf{C}[y] = \mathbf{C}$, which implies $D_+ = \mathbf{C}$. It follows that $\dim_{\mathbf{R}} D = 4$.

Let $z \in D_-$ be a nonzero element. Then $z^2 \in D_+ = \mathbf{C}$. Since D is a division algebra, we have $z^2 \neq 0$. Choose

$$j := \frac{z}{\sqrt{-z^2}}$$

and let $k := ij$. Then $1, i, j, k$ are linearly independent, and

$$j^2 = k^2 = ijk = -1.$$

□

Let A be an $\mathbf{R}$-algebra and let V be an A-module with $\dim_{\mathbf{R}} V < \infty$. By Schur's lemma and Proposition 10.12, the $\mathbf{R}$-algebra $\mathrm{End}_A(V)$ is isomorphic to $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. According to the isomorphism type of $\mathrm{End}_A(V)$, we call V a real representation, complex representation, quaternionic representation of A.

11. Finite dimensional representations

Let $\mathbf{k}$ be a field and let A be an associative $\mathbf{k}$-algebra. We say that an A-module V has *finite dimension* if V is a finite dimensional $\mathbf{k}$-vector space.

11.1. Endomorphisms of indecomposable representations. We could compare the following proposition with Schur's lemma.

Proposition 11.1. *Let V be a finite dimensional A-module. Suppose that V is indecomposable. Then every $f \in \mathrm{End}(V)$ is either an isomorphism or a nilpotent (i.e. $f^n = 0$ for some integer $n > 0$).*

Proof. Since $\dim_{\mathbf{k}} V < \infty$, there exists an integer $n > 0$ such that both

$$\ker(f^n) \subset \ker(f^{n+1}) \text{ and } \mathrm{Im}(f^{n+1}) \subset \mathrm{Im}(f^n)$$

are equalities. It follows that $\ker(f^n) \cap \mathrm{Im}(f^n) = 0$, so

$$V = \ker(f^n) \oplus \mathrm{Im}(f^n) \tag{11.1}$$

as a $\mathbf{k}$-vector space. Since both $\ker(f^n)$ and $\mathrm{Im}(f^n)$ are A-submodules, (11.1) is also a decomposition of A-modules. Thus either $\ker(f^n) = 0$ or $\mathrm{Im}(f^n) = 0$. In the former (resp. latter) case f is an isomorphism (resp. nilpotent). □

11.2. The Krull–Schmidt theorem.

Theorem 11.2 (Krull–Schmidt theorem)**.** *Every finite dimensional A-module V can be decomposed into a finite direct sum of indecomposable representations:*

$$V = \bigoplus_{j=1}^{m} V_j.$$

Moreover, the decomposition is unique, up to permutation of the index j and up to isomorphism of each factor V_j.

Proof. The existence follows from $\dim_{\mathbf{k}} V < \infty$.

Suppose that

$$V = \bigoplus_{j=1}^{m} V_j = \bigoplus_{j=1}^{n} V'_j$$

are decompositions of V into indecomposable A-modules. We prove by induction on m that $m = n$ and $V_j \simeq V'_j$ up to permutation of the indices j. The case $m = 1$ is clear.

For every index j, let

$$\imath_j : V_j \hookrightarrow V, \quad \imath'_j : V'_j \hookrightarrow V, \quad p_j : V \twoheadrightarrow V_j, \quad p'_j : V \twoheadrightarrow V'_j$$

be the natural inclusions and projections. We have

$$\sum_{j=1}^{n} \imath'_j p'_j = \mathrm{Id}_V,$$

so

$$\sum_{j=1}^{n} p_1 \imath'_j p'_j \imath_1 = \mathrm{Id}_{V_1}.$$

The following lemma implies that $p_1 \imath'_j p'_j \imath_1 : V_1 \to V_1$ is an isomorphism for some j.

Lemma 11.3. *Let W be an indecomposable A-module and let $f_1, \ldots, f_k \in \mathrm{End}_A(W)$. Suppose that $f = \sum_{i=1}^{k} f_i$ is an isomorphism. Then one of f_i is an isomorphism.*

Proof. By induction, it suffices to prove for the case $k = 2$. Up to replacing f_i by $f_i \circ f^{-1}$, we can assume that $f_1 + f_2 = \mathrm{Id}$, so f_1 commutes with f_2. Assume that both f_1 and f_2 are not isomorphisms. Then they are nilpotent by Proposition 11.1. But then $\mathrm{Id} = (f_1 + f_2)^N = 0$ for large N, which is a contradiction. □

Up to permuting the indices of V'_j, we can assume that $p_1 \imath'_1 p'_1 \imath_1$ is an isomorphism. It follows from Exercise 10.3 that $p'_1 \imath_1 : V_1 \xrightarrow{\sim} V'_1$. Thus $V_1 \cap \ker(p'_1) = 0$. As

$$\dim \ker p_1 = \dim V - \dim V_1 = \dim V - \dim V'_1 = \dim \ker p'_1,$$

we have

$$V = V_1 \oplus \ker(p_1) = V_1 \oplus \ker(p'_1).$$

It follows that $\ker(p_1) \simeq \ker(p'_1)$. We conclude by the induction hypothesis. □

For any quiver Q, thanks to the equivalence of categories $\mathrm{Rep}(Q, \mathbf{k}) \simeq \mathbf{k}Q$-Mod, Theorem 4.1 is a particular case of Theorem 11.2.

11.3. The Jordan–Hölder theorem.

Theorem 11.4 (Jordan–Hölder theorem). *Let V be a finite dimensional A-module. Then there exists a finite chain*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V \tag{11.2}$$

of A-submodules such that each V_{i+1}/V_i is simple. Moreover, if

$$0 = V'_0 \subsetneq V'_1 \subsetneq \cdots \subsetneq V'_n = V$$

is another chain of A-submodules satisfying the same property, then there exists a permutation of indices σ such that

$$V_{i+1}/V_i \simeq V'_{\sigma(i)+1}/V'_{\sigma(i)}$$

for each i.

The chain of submodules (11.2) is called a *Jordan–Hölder filtration* of V. The quotients V_{i+1}/V_i are called the *Jordan–Hölder factors* of V.

Proof. We construct (11.2) by induction on $\dim V$. Since $\dim_{\mathbf{k}} V < \infty$, V has an irreducible submodule V_1. The induction hypothesis implies that the Jordan–Hölder filtration exists for V/V_1. Together with V_1, the latter filtration lifts to a Jordan–Hölder filtration of V.

For the uniqueness (up to permutations) of the Jordan–Hölder factors, we argue again by induction on $\dim_{\mathbf{k}} V$. If $V_1 = V'_1$ as subspaces in V, then the Jordan–Hölder factors of $V/V_1 = V/V'_1$ are unique by the induction hypothesis, so the statement follows. Suppose that $V_1 \neq V'_1$. Since V_1 and V'_1 are irreducible, we have $V_1 \cap V'_1 = 0$. Consider $W := V/(V_1 \oplus V'_1)$ and let $W_1, \ldots, W_k$ be its Jordan–Hölder factors. Then both

$$V_2/V_1, \ldots, V_n/V_{n-1} \quad \text{and} \quad V'_1, W_1, \ldots, W_k$$

are the Jordan–Hölder factors of V/V_1, so these two series of factors are isomorphic by induction hypothesis. Likewise, the two series of factors

$$V'_2/V'_1, \ldots, V'_n/V'_{n-1} \quad \text{and} \quad V_1, W_1, \ldots, W_k$$

are also isomorphic, which finishes the proof. □

Exercise 11.5. Construct a nonzero A-module V which does not have any irreducible submodule. (For instance, the regular representation of $\mathbf{C}[X]$.)

Exercise 11.6. Let V be a finite dimensional A-module and let $W \subset V$ be a submodule. Show that the multiset of isomorphism classes of the Jordan–Hölder factors of V is the union of that of W and V/W.

Remark 11.7. Suppose that A is a $\mathbf{k}$-algebra admitting a morphism $D \to A$ from a division $\mathbf{k}$-algebra, so that any A-module V has an induced D-module structure. Recall from Exercise 10.6 that D-modules are free. All the results in § 11 hold more generally with the same proof for any A-module V which has finite rank as a D-module.

12. Semisimple modules

12.1. Definition. An A-module V is called *semisimple* if every submodule of V is a direct summand.

Proposition 12.1. *Every submodule and quotient of a semisimple A-module V is also semisimple.*

Proof. We prove the statement for submodules; the argument for quotients is similar.

Let $W \subset V$ be a submodule of V. For every submodule $Z \subset W$, we have $V = Z \oplus Z'$ for some submodule Z'. The projection $\pi : V \to Z$ satisfies $(\pi_{|W})_{|Z} = \pi_{|Z} = \mathrm{Id}_Z$, so Z is a direct summand of W. □

Exercise 12.2. Let V be a finite dimensional semisimple A-module.

(1) Show that V is isomorphic to the direct sum $\bigoplus_{i=1}^n V_i$ of its Jordan–Hölder factors.
(2) Show that any A-submodule of V is isomorphic to

$$\bigoplus_{i \in S} V_i$$

for some subset $S \in \{1, \dots, n\}$.

12.2. The existence of maximal ideals and irreducible submodules. A left-ideal I of A is called *maximal* if the only left-ideals of A containing I are I and A.

Proposition 12.3. *Every left-ideal I of A is contained in a maximal left-ideal of A.*

Proof. Let Σ be the set of ideals of A containing I which is not A. The inclusion defines a partial order on Σ. For every totally ordered subset $S \subset \Sigma$, the union

$$\widetilde{J} := \bigcup_{J \in S} J$$

is a left-ideal of A and does not contain 1 (otherwise, one of $J \in S$ would be A). So $\widetilde{J} \in \Sigma$, and it is an upper bound of S. We conclude by Zorn's lemma. □

Corollary 12.4. *Every nonzero semisimple A-module V has an irreducible submodule.*

Proof. Let $v \in V$ be a nonzero element. Then $\mathrm{Ann}(v) \neq A$ and we have $A \cdot v \simeq A/\mathrm{Ann}(v)$. By Proposition 12.3, there exists a maximal left-ideal $\mathfrak{m}$ containing $\mathrm{Ann}(v)$, so we have a quotient map $A \cdot v \to A/\mathfrak{m}$ onto a simple A-module. By Proposition 12.1, $A/\mathfrak{m}$ is a direct summand of $A \cdot v$, in particular it is isomorphic to a submodule of V. □

12.3. Semisimple modules are completely reducible.

Proposition 12.5. *Let V be an A-module. The following assertions are equivalent.*

(1) V is semisimple.
(2) $V = \sum_{i \in S} V_i$ for a collection of simple A-submodules $V_i \subset V$.
(3) $V = \bigoplus_{i \in S} V_i$ for a collection of simple A-submodules $V_i \subset V$.

Proof. For we prove (1) $\implies$ (3). Let $\mathscr{S}$ be the set of all simple submodules of V and let

$$\Sigma := \left\{ \mathscr{S}' \subset \mathscr{S} \;\middle|\; \sum_{W \in \mathscr{S}'} W = \bigoplus_{W \in \mathscr{S}'} W \right\}.$$

Note that for every totally ordered subset $\{\mathscr{S}_i\}_{i \in I}$ of Σ, we have

$$\sum_{W \in \cup_{i \in I} \mathscr{S}_i} W = \bigoplus_{W \in \cup_{i \in I} \mathscr{S}_i} W,$$

so Σ has a maximal element $\mathscr{S}_0$ by Zorn's lemma. Let

$$V' := \bigoplus_{W \in \mathscr{S}_0} W.$$

Suppose that $V' \neq V$. Then $V = V' \oplus V''$ for some submodule V''. Since V'' contains a simple submodule V_1, we have $V' \cap V_1 = 0$, which contradicts the maximality of $\mathscr{S}_0$. Hence $V = V'$, which implies (3).

(3) $\implies$ (2) is obvious. Now we prove (2) $\implies$ (1). Let $W \subset V$ be a submodule. By Zorn's lemma, there exists a maximal subset $S_0 \subset S$ such that

$$W \cap \sum_{i \in S_0} V_i = 0.$$

Let $W' := \sum_{i\in S_0} V_i$. If $W \oplus W' \neq V$, then there exists $j \in S$ such that $V_j \not\subset W \oplus W'$ (so $j \notin S_0$). As V_j is simple, we have $V_j \cap (W \oplus W') = 0$, so $W \cap (W' + V_j) = 0$, which contradicts the maximality of S_0. □

12.4. Semisimple k-algebras.

Corollary 12.6. *An A-module V is semisimple if and only if every* finitely generated *A-submodule of V is semisimple.*

Proof. The "only if" part follows from Proposition 12.1.

Assume that every finitely generated A-submodule of V is semisimple. Since V is a sum of finitely generated A-submodules V_i, which are semisimple, it follows from Proposition 12.5 that V is a sum of simple A-submodules of V. Thus V is semisimple by Proposition 12.5. □

A $\mathbf{k}$-algebra A is called *semisimple* if every A-module V is semisimple.

Corollary 12.7. *A* $\mathbf{k}$*-algebra A is semisimple if and only if the regular A-representation is semisimple.*

Proof. The "only if" part is immediate.

Assume that the regular A-representation is semisimple. Then by Proposition 12.5, any free A-module is semisimple. Since any A-module V is a quotient of a free A-module, it follows from Proposition 12.1 that V is semisimple. □

Corollary 12.8. *Let A and B be semisimple* $\mathbf{k}$*-algebra. Then $A \times B$ is also semisimple.*

Proof. Since A and B are semisimple, by Proposition 12.5 we have

$$A = \bigoplus_{i\in S} V_i \quad \text{and} \quad B = \bigoplus_{j\in S'} W_j$$

for a collection of simple A-submodules $V_i \subset A$ and simple B-submodules $W_j \subset A$. Then $V_i \times 0$ and $0 \times W_j$ are simple $(A \times B)$-modules and

$$A \times B = \bigoplus_{i\in S}(V_i \times 0) \oplus \bigoplus_{j\in S'}(0 \times W_j).$$

Hence $A \times B$ is semisimple by Corollary 12.7. □

12.5. Example: representations of finite groups. Let G be a finite group.

Theorem 12.9 (Maschke)**.** *Assume that* $\mathrm{char}(\mathbf{k})$ *does not divide $|G|$ (so $|G|$ is invertible in* $\mathbf{k}$*). Any finite dimensional representation V of G over* $\mathbf{k}$ *is semisimple. As a consequence, the group algebra* $\mathbf{k}[G]$ *is semisimple.*

Proof. Let $W \subset V$ be a subrepresentation. Choose a $\mathbf{k}$-linear map $P : V \to W$ such that $P_{|W} = \mathrm{Id}_W$. Let

$$\overline{P} := \frac{1}{|G|}\sum_{g\in G} gPg^{-1} : V \to W.$$

We have $\overline{P}_{|W} = \mathrm{Id}_W$, so $V = W \oplus \ker(\overline{P})$.

For every $v \in \ker(\overline{P})$ and every $h \in G$, since $g \mapsto h^{-1}g$ is a permutation of G, we have the first equality of

$$\sum_{g\in G} gPg^{-1}h(v) = \sum_{g\in G} hgPg^{-1}(v) = h\left(\sum_{g\in G} gPg^{-1}(v)\right) = 0.$$

Thus $\ker(\overline{P})$ is G-stable, and hence a subrepresentation of G.

For the final statement, note that since G is finite, we have $\dim_{\mathbf{k}} \mathbf{k}[G] < \infty$, so every finitely generated $\mathbf{k}[G]$-module V has finite $\dim_{\mathbf{k}} V$. We conclude by Corollary 12.6. □

12.6. Example: the ring of matrices. Let D be a division $\mathbf{k}$-algebra (e.g. $D = \mathbf{k}$). Let n be a positive integer and let $A = \mathrm{Mat}_n(D)$ be the $\mathbf{k}$-algebra of $n \times n$ matrices with coefficients in D.

Proposition 12.10.

(1) The only irreducible A-module is $V := D^n$, with $\mathrm{Mat}_n(D)$ acting on the space D^n of column matrices by left-multiplication.

(2) $\mathrm{Mat}_n(D)$ is semisimple and as A-modules,

$$\mathrm{Mat}_n(D) \simeq V^n.$$

(3) Every finitely generated A-module is of the form V^m for some $m \in \mathbf{Z}_{\geqslant 0}$.

Proof. For any nonzero $v \in V$ and $w \in V$, there exists $M \in \mathrm{Mat}_n(D)$ such that $Mv = w$. Thus V is irreducible. Regarding $\mathrm{Mat}_n(D)$ as a direct sum of n spaces of column matrices, we have $A \simeq V^n$. It follows from Corollary 12.7 that A is semisimple.

Now let W be a finitely generated A-module. Then W is a quotient of A^N and as a D-module, W has finite rank. Since V is the unique Jordan–Hölder factor (where we apply Remark 11.7 to obtain the existence of the Jordan–Hölder filtration and the uniqueness of the Jordan–Hölder factors) of A, by Exercise 11.6 it is also the unique Jordan–Hölder factor of W. Since W is semisimple, we have $W \simeq V^m$ for some $m \in \mathbf{Z}_{\geqslant 0}$. □

Together with Corollary 12.8, we obtain the following.

Corollary 12.11. *Let $D_1, \dots, D_n$ be division $\mathbf{k}$-algebra and let $m_1, \dots, m_n$ be positive integers. The product*

$$\mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n)$$

is a semisimple $\mathbf{k}$-algebra.

12.7. The endomorphism ring of finite direct sums.

Exercise 12.12. Let V, W_1, W_2 be A-modules. Show that

$$\mathrm{Hom}_A(V, W_1 \oplus W_2) \simeq \mathrm{Hom}_A(V, W_1) \oplus \mathrm{Hom}_A(V, W_2)$$

and

$$\mathrm{Hom}_A(W_1 \oplus W_2, V) \simeq \mathrm{Hom}_A(W_1, V) \oplus \mathrm{Hom}_A(W_2, V)$$

as $\mathbf{k}$-vector spaces.

Lemma 12.13. *Let V be an A-module and let $R := \mathrm{End}_A(V)$. Then*

$$\mathrm{End}_A(V^n) \simeq \mathrm{Mat}_n(R)$$

as $\mathbf{k}$-algebras.

Proof. Write $V^n = \oplus_{i=1}^n V_i$ with $V_i = V$. Then

$$\mathrm{End}_A(V^n) = \bigoplus_{i,j=1}^{n} \mathrm{Hom}_A(V_i, V_j) \simeq \bigoplus_{i,j=1}^{n} R$$

as $\mathbf{k}$-vector spaces. Regarding elements of the (i, j)-summand R as the (i, j) coefficient of a matrix, the composition in $\mathrm{End}_A(V^n)$ becomes matrix product. □

Exercise 12.14. Let V and W be A-modules such that $\mathrm{Hom}_A(V, W) = 0$ and $\mathrm{Hom}_A(W, V) = 0$. Show that

$$\mathrm{End}_A(V \oplus W) = \mathrm{End}_A(V) \times \mathrm{End}_A(W)$$

as $\mathbf{k}$-algebras.

Proposition 12.15. *Let $V_1, \dots, V_n$ be irreducible A-modules which are pairwise non isomorphic. Let $D_i := \mathrm{End}_A(V_i)$. Given n positive integers $m_1, \dots, m_n$, we have*

$$\mathrm{End}_A\left(\bigoplus_{i=1}^{n} V_i^{m_i}\right) \simeq \mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n)$$

as $\mathbf{k}$-algebras.

Proof. For every pair of indices $i \neq j$, since V_i and V_j are irreducible and $V_i \not\simeq V_j$, we have $\mathrm{Hom}(V_i, V_j) = 0$ by Schur's lemma. Hence Proposition 12.15 follows from Lemma 12.13 and Exercise 12.14. □

12.8. The Wedderburn–Artin theorem. The following structural theorem shows that up to isomorphisms, the **k**-algebras in Corollary 12.11 are all the semisimple **k**-algebras.

Theorem 12.16 (Wedderburn–Artin)**.** *Let A be a semisimple* **k***-algebra. There exists division* **k***-algebra $D_1, \dots, D_n$ such that*

$$A \simeq \mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n)$$

as **k***-algebras.*

Proof. Since A is semisimple, its regular representation is semisimple. So

$$A \simeq \bigoplus_{i \in S} V_i$$

for some simple modules V_i. We have $1 = \sum_{i \in S'} v_i$ for some finite subset $S' \subset S$ and $v_i \in V_i$, which implies $S = S'$. So

$$A \simeq \bigoplus_{i=1}^{n} V_i$$

for finitely many simple A-modules.

Let $D_i := \mathrm{End}_A(V_i)$, which is a division ring because V_i is simple. By Exercise 9.1 and Proposition 12.15, we have

$$A^{\mathrm{op}} \simeq \mathrm{End}_A(A) \simeq \mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n).$$

For any division **k**-algebra D, the opposite algebra D^{op} is also a division ring and for any positive integer n, we have an isomorphism

$$\begin{aligned} \mathrm{Mat}_n(D)^{\mathrm{op}} &\to \mathrm{Mat}_n(D^{\mathrm{op}}) \\ (c_{ij}) &\mapsto {}^t(c_{ij}), \end{aligned} \tag{12.1}$$

as **k**-algebras. This concludes the proof. □

12.9. The structure of regular representation of a semisimple ring.

Corollary 12.17. *Let A be a semisimple* **k***-algebra.*

(1) There exist only finitely many isomorphism classes of irreducible A-modules $V_1, \dots, V_n$.

(2) We have

$$A = \mathrm{End}_{D_1}(V_1) \oplus \cdots \oplus \mathrm{End}_{D_n}(V_n),$$

where $D_i := \mathrm{End}_A(V_i)$.

Proof. We have

$$A \simeq \mathrm{Mat}_{m_1}(D_1^{\mathrm{op}}) \times \cdots \times \mathrm{Mat}_{m_n}(D_n^{\mathrm{op}})$$

as in the Wedderburn–Artin theorem. Let $V_i := D_i^{m_i}$. By Proposition 12.10, V_i is a simple $\mathrm{Mat}_{m_i}(D_i)$-module, so it is also a simple A-module (through the projection $A \to \mathrm{Mat}_{m_i}(D_i)$). As $\mathrm{Mat}_{m_i}(D_i^{\mathrm{op}}) \simeq \mathrm{End}_{D_i}(V_i)$ as **k**-algebras, we have

$$A \simeq \mathrm{End}_{D_1}(V_1) \times \cdots \times \mathrm{End}_{D_n}(V_n).$$

Note that $V_i \not\simeq V_j$ as A-modules whenever $i \neq j$, because $\mathrm{Mat}_{m_i}(D_i)$ acts non trivially on V_i but trivially on V_j. It remains to show that $V_1, \dots, V_n$ are all the simple A-modules up to isomorphisms.

Let V be an irreducible A-module. Let $v \in V$ be a nonzero element. Then $A \cdot v = V$ and we have a surjective morphism

$$A \to A \cdot v = V$$

of A-modules. Since

$$A = V_1^{m_1} \oplus \cdots \oplus V_n^{m_n} \tag{12.2}$$

the restriction of $A \twoheadrightarrow V$ to one of V_i is nonzero. As both V and V_i are irreducible, we have $V_i \simeq V$. □

12.10. A consequence on finite group representations.

Corollary 12.18. *Let G be a finite group. Suppose that* $\mathrm{char}(\mathbf{k})$ *does not divide $|G|$ and that $\mathbf{k}$ is algebraically closed.*

(1) G has only finitely many isomorphism classes of irreducible representations $V_1, \dots, V_n$ over $\mathbf{k}$.
(2) We have

$$\mathbf{k}[G] \simeq \mathrm{End}_{\mathbf{k}}(V_1) \oplus \cdots \oplus \mathrm{End}_{\mathbf{k}}(V_n)$$

as G-representations. As a consequence,

$$|G| = \sum_{i=i}^{n} (\dim V_i)^2.$$

Proof. The first statement follows from Corollary 12.17, by noticing that since $\mathbf{k}$ is algebraically closed, we have $\mathrm{End}_A(V_i) \simeq \mathbf{k}$ by Schur's lemma. The isomorphism in (2) is also an isomorphism of $\mathbf{k}$-vector spaces, which implies $|G| = \dim_{\mathbf{k}} \mathbf{k}[G] = \sum_{i=i}^{n} (\dim V_i)^2$. □

12.11. Radicals. Let A be a $\mathbf{k}$-algebra with $\dim_{\mathbf{k}} A < \infty$. We define the *radical* of A to be

$$\mathrm{Rad}(A) := \{ a \in A \mid a \cdot V = 0 \text{ for every irreducible } A\text{-modules } V \}.$$

Proposition 12.19.

(1) We have

$$\mathrm{Rad}(A) = \{ a \in A \mid a \textit{ nilpotent } \}.$$

(2) $\mathrm{Rad}(A)$ *is a nilpotent two-sided ideal, and every nilpotent two-sided ideal of A is contained in* $\mathrm{Rad}(A)$.

Here, an element $a \subset A$ is called *nilpotent* if there exists a positive integer n such that $a^n = 0$. A two-sided ideal $I \subset A$ is called *nilpotent* if there exists a positive integer n such that $a_1 \cdots a_n = 0$ for all $a_1, \dots a_n \in I$.

Proof. It is clear that $\mathrm{Rad}(A)$ is a two-sided ideal. Since A is finite dimensional, as an A-module it admits a Jordan–Hölder filtration. Let $V_1, \dots, V_n$ be the Jordan factors of A. Since $a \cdot V_i = 0$ for every $a \in \mathrm{Rad}(A)$ and every V_i, it follows that $a_1 \cdots a_n = 0$ for all $a_1, \dots a_n \in \mathrm{Rad}(A)$. Hence $\mathrm{Rad}(A)$ is nilpotent.

Note that if $a \in A$ is a nilpotent (namely, $a^N = 0$ for some positive integer N), then $a \in \mathrm{Rad}(A)$. Indeed, for every $x \in A \backslash \mathrm{Rad}(A)$, there exists an irreducible A-module V such that $x \cdot V \neq 0$. By Schur's lemma, $V \xrightarrow{x \times} V$ is injective, so $x^N \neq 0$ for every positive integer N. This proves (1). As every element of a nilpotent two-sided ideal is nilpotent, (1) implies the second statement of (2). □

Proposition 12.20. *A is semisimple if and only if* $\mathrm{Rad}(A) = 0$.

Proof. Suppose that A is semisimple. Then $A = \bigoplus_i V_i$ where V_i are irreducible A-modules. It follows that $\mathrm{Rad}(A) \cdot A = 0$, so $\mathrm{Rad}(A) = \mathrm{Rad}(A) \cdot 1 = 0$.

Suppose that A is not semisimple. Then by the Krull–Schmidt theorem, some indecomposable A-submodule $B \subset A$ is not semisimple. Let $V \subset B$ be an irreducible B-submodule. Then $V \simeq B/\mathfrak{m}$ for some maximal left-ideal $\mathfrak{m}$ of B. Let $x \in \mathfrak{m}$ be a nonzero element. We have $x \cdot V = 0$, so $x \cdot : B \to B$ is not an isomorphism. As B is indecomposable, x is nilpotent by Proposition 11.1. Hence $x \in \mathrm{Rad}(A)$ by Proposition 12.19, which shows that $\mathrm{Rad}(A) \neq 0$. □

Exercise 12.21. Show that $A/\mathrm{Rad}(A)$ is semisimple.

13. Grothendieck groups and characters

13.1. Grothendieck groups. Let A-Mod_f be the category of finite dimensional A-modules. The *Grothendieck group* of A-Mod_f is the $\mathbf{Z}$-module $K_0(A\text{-}\mathrm{Mod}_f)$ defined by generators and relations as follows.

- Generators: finite dimensional A-modules V.
- Relations: the submodule generated by $[V]-[U]-[W]$, whenever we have a short exact sequence of A-modules
$$0 \to U \to V \to W \to 0.$$

We can also define the *additive Grothendieck group* $K_0^{\mathrm{add}}(A\text{-Mod}_f)$ of A-Mod$_f$: the definition is the same as $K_0(A\text{-Mod}_f)$ except for the relations, which is the submodule generated by $[V]-[U]-[W]$ whenever we have
$$V \simeq U \oplus W.$$

Exercise 13.1. Show that $K_0(A\text{-Mod}_f)$ (resp. $K_0^{\mathrm{add}}(A\text{-Mod}_f)$) is isomorphic to the free abelian group generated by the isomorphism classes of irreducible (resp. indecomposable) A-modules.

13.2. Characters. Let $\rho : A \to \mathrm{End}_{\mathbf{k}}(V)$ be a finite dimensional representation of A. The *character* of ρ is defined as

$$\begin{aligned} \chi_V : A &\to \mathbf{k} \\ a &\mapsto \mathrm{Tr}(\rho(a)). \end{aligned} \tag{13.1}$$

Note that χ_V is $\mathbf{k}$-linear, and since $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$ for any $f, g \in \mathrm{End}_{\mathbf{k}}(V)$, the character χ_V descends to a $\mathbf{k}$-linear map
$$\chi_V : A/[A,A] \to \mathbf{k},$$
where $[A,A]$ is the $\mathbf{k}$-linear subspace generated by
$$[a,b] := ab - ba \in A$$
for all $a, b \in A$.

Exercise 13.2. Let G be a group and let $A = \mathbf{k}[G]$. Show that
$$\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k}) \simeq \{ \text{ maps } C(G) \to \mathbf{k} \},$$
where $C(G)$ is the set of conjugacy classes of G.

Exercise 13.3. Let V be a finite dimensional A-module and let $W \subset V$ be a submodule. Show that
$$\chi_V = \chi_W + \chi_{V/W}.$$

We have group homomorphisms
$$\begin{aligned} K_0^{\mathrm{add}}(A\text{-Mod}_f) \to K_0(A\text{-Mod}_f) &\to \mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k}) \\ V &\mapsto \chi_V. \end{aligned} \tag{13.2}$$

We will prove the following, as a consequence of the Jacobson density theorem.

Corollary 13.4. *Assume that* $\mathbf{k}$ *is either algebraically closed or* $\mathrm{char}\mathbf{k} = 0$*. Let* $V_1, \ldots, V_n$ *be irreducible* A*-modules which are pairwise non isomorphic. Then*
$$\chi_{V_1}, \ldots, \chi_{V_n} \in \mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$$
are linearly independent.

As a consequence, it follows from Exercise 13.1 that if $\mathbf{k}$ is as in Corollary 13.5 and $\mathrm{char}\mathbf{k} = p \geqslant 0$, then character map
$$K_0(A\text{-Mod}_f) \otimes \mathbf{Z}/p\mathbf{Z} \to \mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$$
is injective.

Corollary 13.5. *Assume that* $\mathrm{char}\mathbf{k} = 0$*. Up to isomorphisms, a finite dimensional semisimple representation* V *of* A *is uniquely determined by its character* χ_V*.*

13.3. The Jacobson density theorem. Let A be *any* $\mathbf{k}$-algebra and let $\rho : A \to \mathrm{End}_{\mathbf{k}}(V)$ be an irreducible representation of A. Let $D := \mathrm{End}_A(V)$. Then for every $a \in A$, $f \in D$ and $v \in V$, we have

$$f(av) = af(v).$$

It follows that multiplication by a is D-linear, so the image of $\rho : A \to \mathrm{End}_{\mathbf{k}}(V)$ lies in $\mathrm{End}_D(V)$.

The Jacobson density theorem asserts that when V is finite dimensional the image of ρ is exactly $\mathrm{End}_D(V)$. Here we prove a slightly more general statement.

Theorem 13.6 (Jacobson density theorem). *Let A be any $\mathbf{k}$-algebra and let $V_1, \ldots, V_n$ be finitely many finite dimensional irreducible A-modules such that $V_i \not\simeq V_j$ whenever $i \neq j$. Let $D_i := \mathrm{End}_A(V_i)$ and let $\mu_i : A \to \mathrm{End}_D(V_i)$ be the map sending a to $v_i \mapsto av_i$. Then*

$$\mu : (\mu_1, \ldots, \mu_n) : A \to \mathrm{End}_{D_1}(V_1) \times \cdots \times \mathrm{End}_{D_n}(V_n)$$

is surjective.

Proof. Since each V_i satisfies $\dim_{\mathbf{k}} V_i < \infty$ and D_i is a $\mathbf{k}$-algebra, we have $m_i := \mathrm{rank}_{D_i} V_i < \infty$. As A-modules, we have

$$\mathrm{End}_{D_1}(V_1) \times \cdots \times \mathrm{End}_{D_n}(V_n) \simeq V_1^{m_1} \oplus \cdots \oplus V_n^{m_n},$$

which is a direct sum decomposition into irreducible A-modules. Thus by Exercise 12.2, we have

$$\overline{A} := \mu(A) \simeq V_1^{\ell_1} \oplus \cdots \oplus V_n^{\ell_n}$$

as A-modules for some integers $\ell_j \leqslant m_j$. It also follows that

$$\overline{A}^{\mathrm{op}} \simeq \bigoplus_{i=1}^{n} \mathrm{End}_{\overline{A}}(V_i^{\ell_i}, V_i^{\ell_i}) \simeq \bigoplus_{i=1}^{n} \mathrm{Mat}_{\ell_i}(D_i)$$

as A-modules. Since

$$\sum_{i=1}^{n} \dim_{\mathbf{k}} V_i^{\ell_i} = \dim_{\mathbf{k}} \overline{A} = \dim_{\mathbf{k}} \overline{A}^{\mathrm{op}} = \sum_{i=1}^{n} \dim_{\mathbf{k}} \mathrm{Mat}_{\ell_i}(D_i),$$

we have

$$\sum_{i=1}^{n} \ell_i m_i \cdot \dim_{\mathbf{k}} D_i = \sum_{i=1}^{n} \ell_i^2 \cdot \dim_{\mathbf{k}} D_i,$$

which shows that $\ell_i = m_i$ for all i. Hence $\overline{A} = \mathrm{End}_D(V)$. □

Exercise 13.7. Deduce the following consequence of the Jacobson density theorem. Let V be a finite dimensional semisimple A-module and let $R := \mathrm{End}_A(V)$. For every $v_1, \ldots, v_n$ and $s \in \mathrm{End}_R(V)$, there exists $a \in A$ such that

$$a \cdot v_i = s(v_i)$$

for all i.

13.4. Independence of characters.

Proof of Corollary 13.5. Suppose that

$$\sum_{i=1}^{n} \lambda_i \cdot \chi_{V_i} = 0$$

for some $\lambda_i \in \mathbf{k}$. Let $D_i := \mathrm{End}_A(V_i)$ and let

$$\mu : (\mu_1, \ldots, \mu_n) : A \to \mathrm{End}_{D_1}(V_1) \times \cdots \times \mathrm{End}_{D_n}(V_n) \tag{13.3}$$

be the surjective map as in Theorem 13.6.

First we assume that char$\mathbf{k} = 0$. Since (13.3) is surjective, for every i there exists $a_i \in A$ such that $\mu_i(a_i) = \mathrm{Id}_{V_i}$ and $\mu_j(a_i) = 0$ for every $j \neq i$. Thus

$$0 = \sum_{i=1}^{n} \lambda_i \cdot \chi_{V_i}(a) = \lambda_i \cdot \dim V_i,$$

showing that $\lambda_i = 0$.

Now assume that $\mathbf{k}$ is algebraically closed. Then $D_i \simeq \mathbf{k}$ for all $\mathbf{k}$, so the surjectivity of (13.3) provides again $a_i \in A$ for each i such that $\mathrm{Tr}(\mu_i(a_i)) = 1$ and $\mu_j(a_i) = 0$ for all j, showing that $\lambda_i = 0$. □

Remark 13.8. Without assuming that $\mathbf{k}$ is algebraically closed or char$\mathbf{k} = 0$, there exist irreducible representations V with $\chi_V = 0$. For instance, there exist division algebras D over some field $\mathbf{k}$ of characteristic $p > 0$ such that $\dim_{\mathbf{k}} D$ is a power of p^2 [**4**, Theorem 4.7.3], and we have $\chi_D = 0$ for the regular representation of D (see e.g. [**2**, Proposition 2.6.3]).

13.5. The character map for semisimple rings.

Theorem 13.9. *Assume that* $\mathbf{k}$ *is algebraically closed. Let* A *be a semisimple* $\mathbf{k}$*-algebra with* $\dim_{\mathbf{k}} A < \infty$*, and let* $V_1, \dots, V_n$ *be the finite dimensional irreducible* A*-modules (up to isomorphisms). Then* $\chi_{V_1}, \dots, \chi_{V_n}$ *form a basis of* $\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$.

In other words, for $\mathbf{k}$ and A as in the theorem, the character map

$$K_0(A\text{-Mod}_f) \otimes \mathbf{Z}/p\mathbf{Z} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$$

is an isomorphism by Exercise 13.1.

Proof. By Corollary 13.5, it remains to show that $\chi_{V_1}, \dots, \chi_{V_n}$ generate $\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$. Since $\mathbf{k}$ is algebraically closed, we have $\mathrm{Hom}_A(V_i, V_i) \simeq \mathbf{k}$ be Schur's lemma. Thus

$$A \simeq \mathrm{Mat}_{m_1}(\mathbf{k}) \times \cdots \times \mathrm{Mat}_{m_n}(\mathbf{k})$$

as $\mathbf{k}$-algebras for some positive integers $m_1, \dots, m_n$. It follows from Exercise 13.10 that $A/[A,A] \simeq \mathbf{k}^n$ as $\mathbf{k}$-vector spaces. As $\chi_{V_1}, \dots, \chi_{V_n}$ are $\mathbf{k}$-linearly independent, necessarily that generate $\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$. □

Exercise 13.10. Let n be a positive integer.

(1) E_{ij} be the $n \times n$ elementary matrix whose (i,j) entry is 1. Compute $[E_{i,j}, E_{k,\ell}]$ for all indices i, j, k, ℓ.

(2) Show that

$$[\mathrm{Mat}_n(\mathbf{k}), \mathrm{Mat}_n(\mathbf{k})] = \mathfrak{sl}_n(\mathbf{k}) := \ker(\mathrm{Tr} : \mathrm{Mat}_n(\mathbf{k}) \to \mathbf{k}).$$

13.6. A symmetric bilinear form on the space of characters. Assume that $\mathbf{k}$ is algebraically closed of characteristic zero. Let A be a finite dimensional semisimple $\mathbf{k}$-algebra, and let $V_1, \dots, V_n$ be the irreducible A-modules. The character of these representations $\chi_i := \chi_{V_i}$ form a basis of the $\mathbf{k}$-vector space $\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$ by Theorem 13.9. Let $(\bullet, \bullet)$ be the symmetric bilinear form on $\mathrm{Hom}_{\mathbf{k}}(A/[A,A], \mathbf{k})$ having $\chi_1, \dots, \chi_n$ as an orthonormal basis. This bilinear form is useful to organize some information about finite dimensional A-modules, as we now explain.

Every finite dimensional A-module V is uniquely isomorphic to

$$V \simeq V_1^{\oplus m_1} \oplus \cdots \oplus V_n^{\oplus m_n}.$$

We call m_i the *multiplicity* of V_i in V.

Exercise 13.11. Show that $m_i = (\chi_i, \chi_V)$.

Proposition 13.12. *Assume that* $\mathbf{k}$ *is algebraically closed. Let* V *and* W *be finite dimensional* A*-modules. We have*

$$(\chi_V, \chi_W) = \dim_{\mathbf{k}} \mathrm{Hom}_A(V, W).$$

Proof. Write
$$V \simeq V_1^{\oplus m_1} \oplus \cdots \oplus V_n^{\oplus m_n} \text{ and } W \simeq V_1^{\oplus m_1'} \oplus \cdots \oplus V_n^{\oplus m_n'}.$$
Then
$$\dim_{\mathbf{k}} \operatorname{Hom}_A(V, W) = \dim_{\mathbf{k}} \operatorname{Hom}_A(V_i^{\oplus m_i}, V_i^{\oplus m_i'}) = \sum_{i=1}^n m_i m_i' = (\chi_V, \chi_W),$$
where the second equality follows from Schur's lemma. □

Introducing the bilinear form $(\bullet, \bullet)$ does not add new pieces of information, unless we can explicitly compute it. We will provide an explicit formula for representations of finite groups.

13.7. Over nonclosed fields. We continue the setting in § 13.6, but without assuming that $\mathbf{k}$ is algebraically closed. We can generalize § 13.6 as follows.

The characters $\chi_i := \chi_{V_i}$ are still $\mathbf{k}$-linearly independent in $\operatorname{Hom}_{\mathbf{k}}(A/[A, A], \mathbf{k})$, and we let
$$\mathbf{X}(A) := \bigoplus_i^n \mathbf{k} \cdot \chi_i \subset \operatorname{Hom}_{\mathbf{k}}(A/[A, A], \mathbf{k}).$$
Let $(\bullet, \bullet)$ be the symmetric bilinear form on $\mathbf{X}(A)$ defined by
$$(\chi_i, \chi_j) = \dim_{\mathbf{k}} \operatorname{Hom}_A(V_i, V_j).$$

Exercise 13.13. Show that $\chi_1, \ldots, \chi_n$ are orthogonal. Prove Proposition 13.12 without assuming that $\mathbf{k}$ is algebraically closed.

Exercise 13.14. Let V be a finite dimensional A-module. Show that
$$m_i = \frac{(\chi_i, \chi_V)}{(\chi_i, \chi_i)}.$$

Example 13.15. Suppose that $\mathbf{k} = \mathbf{R}$ and V is a finite dimensional irreducible A-module. Then V is real, complex, quaternionic, if and only if
$$(\chi_V, \chi_V) = 1, 2, 4$$
respectively.

LECTURE 5

Tensor products of group representations

Let $\mathbf{k}$ be a field. Let G be a group.

14. Tensor products of group representations

We first recall how tensor product $\otimes$ is defined for vector spaces. Here are some guiding principles:

- For finite dimensional vector spaces, $\oplus$ linearizes addition, $\otimes$ linearizes multiplication (both through dim). If $e_1, \ldots, e_m$ is a basis of V and $e'_1, \ldots, e'_n$ is a basis of W, then $\{e_i \otimes e'_j\}_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ is a basis of $V \otimes_{\mathbf{k}} W$.
- The tensor product $V \otimes_{\mathbf{k}} W$ of two $\mathbf{k}$-vector spaces V and W is the universal target of bilinear maps from $V \times W$.

Here is the precise statement for the second point.

Theorem-Definition 14.1 (Universal property of tensor products)**.** *Let V and W be two $\mathbf{k}$-vector spaces. There exists a $\mathbf{k}$-vector space $V \otimes_{\mathbf{k}} W$, together with a $\mathbf{k}$-bilinear map*

$$\phi : V \times W \to V \otimes_{\mathbf{k}} W,$$

For any $\mathbf{k}$-bilinear map $\psi : V \times W \to L$ to some $\mathbf{k}$-vector space L, there exists a unique $\mathbf{k}$-linear map $\tilde{\psi} : V \otimes_{\mathbf{k}} W \to L$ such that

$$\begin{array}{ccc} V \times W & \xrightarrow{\ \forall \psi\ } & L \\ & \searrow^{\phi} & \uparrow{\scriptstyle \exists ! \tilde{\psi}} \\ & & V \otimes_{\mathbf{k}} W \end{array}$$

commutes. Moreover, the pair $(V \otimes_{\mathbf{k}} W, \phi)$ is unique up to unique isomorphism. The $\mathbf{k}$-vector space $V \otimes_{\mathbf{k}} W$ is called the tensor product of V and W over $\mathbf{k}$.

14.1. Construction of tensor products. Let V and W be $\mathbf{k}$-vector spaces. The simplest way of constructing the tensor product $V \otimes_{\mathbf{k}} W$ is first by choosing a basis $\{e_i\}_{i \in I}$ of V and a basis $\{e'_j\}_{j \in J}$ of W, then define $V \otimes_{\mathbf{k}} W$ as the $\mathbf{k}$-vector space freely generated by

$$\{e_i \otimes e'_j\}_{i \in I, j \in J}.$$

We have a bilinear map

$$\begin{aligned} \phi : V \times W &\to V \otimes_{\mathbf{k}} W \\ (\sum_{i \in I} \lambda_i e_i, \sum_{j \in J} \lambda'_j e'_j) &\mapsto \sum_{i \in I, j \in J} (\lambda_i \lambda'_j) e_i \otimes e_j, \end{aligned} \tag{14.1}$$

where $\{\lambda_i\}_{i \in I}$ is a collection of elements of $\mathbf{k}$ which is zero for all but finitely many i; same for $\{\lambda'_j\}_{j \in J}$.

Exercise 14.2. Prove Theorem 14.1 for $\phi : V \times W \to V \otimes_{\mathbf{k}} W$.

The universal property also implies that up to a unique isomorphism, our previous construction of $\phi : V \times W \to V \otimes_{\mathbf{k}} W$ does not depend on the choices of bases $\{e_i\}$ and $\{e'_j\}$

14.2. Second construction. Here is another way to construct $V \otimes_{\mathbf{k}} W$, which does not rely on the choice of bases. Since we want $\phi : V \times W \to V \otimes_{\mathbf{k}} W$ to be $\mathbf{k}$-bilinear, we just define $V \otimes_{\mathbf{k}} W$ straightforwardly by generators and relations as follows:

- Generators: $v \otimes w$ for all $v \in V$ and $w \in W$.
- The **k**-subspace of relations $\mathscr{R}$ is generated by

$$(v + v') \otimes w = v \otimes w + v' \otimes w, \quad v \otimes (w + w') = v \otimes w + v \otimes w',$$

$$(\lambda v) \otimes w = v \otimes (\lambda w) = \lambda(v \otimes w),$$

for all $v, v' \in V$, $w, w' \in W$, and $\lambda \in \mathbf{k}$.

Namely,

$$V \otimes_{\mathbf{k}} W := \left(\bigoplus_{v \in V, w \in W} \mathbf{k} \cdot (v \otimes w) \right) / \mathscr{R}.$$

For every $v \in V$ and $w \in W$, we still use $v \otimes w$ to denote its image in $W \otimes_{\mathbf{k}} W$. Elements in $V \otimes_{\mathbf{k}} W$ of the form $v \otimes w$ are called *pure tensors*, or *simple tensors*.

Exercise 14.3. Prove Theorem 14.1 for the second construction of $V \otimes_{\mathbf{k}} W$. What is the unique isomorphism between the first construction and the second construction?

Given a finite number of **k**-vector space $V_1, \ldots, V_n$, we define the tensor product

$$V_1 \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} V_n$$

in a similar way. It satisfies a similar universal property, replacing bilinear maps with multilinear maps.

14.3. Interlude: Hilbert's third problem. Let P_1 and P_2 be two polytopes in $\mathbf{R}^n$. We say that P_1 and P_2 are *scissors-congruent* if they can be decomposed into finitely many polytopes, and these pieces can be reassembled in $\mathbf{R}^N$ into congruent polytopes.

Two scissors-congruent polytopes have same volume.

Exercise 14.4. The converse is true in dimension 2, which has been proven by e.g. Wallace (1807), Bolyai (1833), Gerwein (1835). Prove this.

Question 14.5 (Kretkowski 1882, Hilbert's third problem 1900)**.** *If polytopes (in dimension* 3*) have same volume, are they scissors-congruent?*

Let P be a polytope and let e be an edge of P. Let $\ell(e)$ be the length of the edge and let $\theta_0(e)$ denote the angle of the faces of P adjacent to e divided by 2π, viewed as an element of $\mathbf{R}/\mathbf{Z}$. Let $\theta(e)$ be its image in $\mathbf{R}/\mathbf{Q}$

Exercise 14.6. Let P be a polytope.

(1) Show that the Dehn invariant

$$D(P) := \sum_{e \text{ edges of } P} \ell(e) \otimes \theta(e) \in \mathbf{R} \otimes_{\mathbf{Q}} \mathbf{R}/\mathbf{Q}$$

is preserved under scissors-congruent.

(2) Compute the Dehn invariant of a regular tetrahedron and that of a cube with same volume, and show that they are not equal.

Hilbert's third problem thus has a negative answer. The Dehn invariant was considered by Dehn in 1901 to solve the Hilbert's third problem. Actually before Dehn, the problem had already been solved by Birkenmajer in 1882, even before Hilbert asked the question.

14.4. Tensor product of endomorphisms. Let V and W be **k**-vector spaces. For every $\phi \in \mathrm{End}_{\mathbf{k}}(V)$ and $\psi \in \mathrm{End}_{\mathbf{k}}(W)$, we define their *tensor product* to be the endomorphism $\phi \otimes \psi$ on $V \otimes_{\mathbf{k}} W$ defined by

$$(\phi \otimes \psi)(v \otimes w) = \phi(v) \otimes \psi(w)$$

for pure tensors, then extends linearly to the whole $V \otimes_{\mathbf{k}} W$.

Exercise 14.7. Give an equivalent definition of $\phi \otimes \psi$ using the universal property of tensor products.

Exercise 14.8. Suppose that V and W are finite dimensional, and let $e_1, \ldots, e_m$ and $e'_1, \ldots, e'_n$ be bases of V and W respectively. Show that if the matrix representing ϕ is $M = (m_{ij})_{1 \leqslant i,j \leqslant m}$ and the matrix representing ϕ is $M' = (m'_{i'j'})_{1 \leqslant i',j' \leqslant n}$, withe respect to the above bases, then the matrix representing $\phi \otimes \psi$ with respect to the basis $\{e_i \otimes e'_j\}_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ of $V \otimes_{\mathbf{k}} W$ is the Kronecker product of M and M', namely $(m_{ij} m'_{i'j'})_{1 \leqslant i,j \leqslant m, 1 \leqslant i',j' \leqslant n}$.

14.5. Tensor products of group representations. For every pair of G-representations $\rho_V : G \to \mathrm{End}_{\mathbf{k}}(V)$ and $\rho_W : G \to \mathrm{End}_{\mathbf{k}}(W)$, we define the *tensor product* of ρ_V and ρ_W to be the G-representation $\rho_{V \otimes W}$ on $V \otimes_{\mathbf{k}} W$ defined by

$$\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g).$$

Exercise 14.9. Let $G \circlearrowright S, S'$ be a pair of finite G-sets, and let $S \times S'$ be the product G-set. Show that we have

$$\mathbf{k}[S \times S'] \simeq \mathbf{k}[S] \otimes_{\mathbf{k}} \mathbf{k}[S']$$

as G-representations.

Exercise 14.10. Let U, V, W be G-representations. Show that we have canonical isomorphisms of G-representations

- $\mathbf{k} \otimes_{\mathbf{k}} V \xrightarrow{\sim} V$,
- $V \otimes_{\mathbf{k}} W \xrightarrow{\sim} W \otimes_{\mathbf{k}} W$,
- $(U \otimes_{\mathbf{k}} V) \otimes_{\mathbf{k}} W \xrightarrow{\sim} U \otimes_{\mathbf{k}} (V \otimes_{\mathbf{k}} W)$,
- $(U \oplus V) \otimes_{\mathbf{k}} W \xrightarrow{\sim} (U \otimes_{\mathbf{k}} W) \oplus (\mathbf{k} \otimes_{\mathbf{k}} W)$,

defined by $\lambda \otimes v \mapsto \lambda v$, $\; v \otimes w \mapsto w \otimes v$, etc.

14.6. $\mathrm{Hom}_{\mathbf{k}}(V, W)$ as G-representations. Let V and W be a pair of G-representations over $\mathbf{k}$. Then $\mathrm{Hom}_{\mathbf{k}}(V, W)$ is endowed with a natural linear G-action defined by

$$g \cdot \phi = g \circ \phi \circ g^{-1}$$

for every $g \in G$ and $\phi \in \mathrm{Hom}_{\mathbf{k}}(V, W)$. In these lectures, when we regard $\mathrm{Hom}_{\mathbf{k}}(V, W)$ as a G-representation, it is defined as above unless otherwise specified.

Exercise 14.11. Verify that the above construction indeed defines a G-representation on $\mathrm{Hom}_{\mathbf{k}}(V, W)$. Show that

$$\mathrm{Hom}_G(V, W) := \mathrm{Hom}_{\mathbf{k}[G]}(V, W) = \mathrm{Hom}(V, W)^G.$$

When W is the trivial G-representation of $\mathbf{k}$, we call

$$V^\vee := \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$$

the *dual representation* of V.

14.7. $\otimes$ and Hom form an adjoint pair.

Proposition-Exercise 14.12. *Let V be a G-representation over $\mathbf{k}$. As functors from $\mathrm{Rep}(G, \mathbf{k})$ to itself, $\bullet \otimes_{\mathbf{k}} V$ is left adjoint to $\mathrm{Hom}_{\mathbf{k}}(V, \bullet)$. Namely, for any G-representations U and W over $\mathbf{k}$, there exist isomorphisms of $\mathbf{k}$-vector spaces*

$$\phi_{UW} : \mathrm{Hom}_{\mathbf{k}}(U \otimes_{\mathbf{k}} V, W) \simeq \mathrm{Hom}_{\mathbf{k}}(U, \mathrm{Hom}_R(V, W))$$

which are functorial in U and W. This means that for every G-equivariant $\mathbf{k}$-linear map $f : W \to W'$, the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{k}}(U \otimes_{\mathbf{k}} V, W) & \xrightarrow{\phi_{UW}} & \mathrm{Hom}_{\mathbf{k}}(U, \mathrm{Hom}_{\mathbf{k}}(V, W)) \\
\Big\downarrow{\scriptstyle f \circ} & & \Big\downarrow{\scriptstyle f \circ} \\
\mathrm{Hom}_{\mathbf{k}}(U \otimes_{\mathbf{k}} V, W') & \xrightarrow{\phi_{UW'}} & \mathrm{Hom}_{\mathbf{k}}(U, \mathrm{Hom}_{\mathbf{k}}(V, W'))
\end{array}$$

commutes. Also the similar diagram for every G-equivariant $\mathbf{k}$-linear map $h : V \to V'$ commutes.

Exercise 14.13. Using the property that ϕ_{UW} is functorial in U and W, show that ϕ_{UW} are actually isomorphisms of G-representations.

In particular, when $W = \mathbf{k}$, we have

$$(U \otimes_{\mathbf{k}} V)^{\vee} \simeq \mathrm{Hom}_{\mathbf{k}}(U, V^{\vee}),$$

and these isomorphisms are functorial in U and V.

Exercise 14.14. Suppose that V and W are finite dimensional G-representations over $\mathbf{k}$. Show that

$$V^{\vee} \otimes_{\mathbf{k}} W \simeq \mathrm{Hom}_{\mathbf{k}}(V, W),$$

and these isomorphisms are functorial in V and W among finite dimensional G-representations over $\mathbf{k}$.

15. Tensor algebras

Let V be a $\mathbf{k}$-vector space.

15.1. Tensor algebras. For every $n \in \mathbf{Z}_{\geqslant 0}$, we define inductively

$$T^0(V) := \mathbf{k}, \quad T^n(V) := T^{n-1}(V) \otimes_{\mathbf{k}} V$$

and let

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V).$$

We define product on $T(V)$, first for pure tensors by

$$(x_1 \otimes \cdots \otimes x_i) \cdot (y_1 \otimes \cdots \otimes y_j) = (x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j),$$

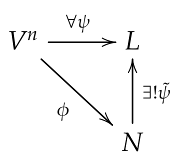
then extend by linearity. We can therefore consider $T(V)$ as a graded associative $\mathbf{k}$-algebra, and call it the tensor algebra associated to V.

15.2. Symmetric and exterior algebras.

Theorem-Definition 15.1 (Universal property of symmetric and exterior powers)**.** *Let n be a positive integer. There exists an $\mathbf{k}$-vector space N together with an $\mathbf{k}$-multilinear symmetric (resp. alternating) map*

$$\phi : V^n \to N$$

which satisfies the following universal property: for any symmetric (resp. alternating) $\mathbf{k}$-multilinear map $\psi : V^n \to L$ to some $\mathbf{k}$-vector space L, there exists a unique $\mathbf{k}$-linear map $\tilde{\psi} : N \to L$ such that



commutes. Moreover, the pair (N, ϕ) is unique up to unique isomorphism. The $\mathbf{k}$-vector space N is called the symmetric power *(resp. the* exterior power*) of V over $\mathbf{k}$, and is denoted* $\mathrm{Sym}^n M$ *(resp. $\bigwedge^n M$).*

The symmetric algebra associated to an $\mathbf{k}$-vector space V is defined as

$$\mathrm{Sym}(V) := \frac{T(V)}{<x \otimes y - y \otimes x \mid x, y \in V>},$$

where the denominator is the two-sided ideal generated by all the $x \otimes y - y \otimes x$.

Exercise 15.2. Show that the grading on $T(V)$ induces a grading $\oplus_i \mathrm{Sym}^i(V)$ on $\mathrm{Sym}(V)$, and that $\mathrm{Sym}(V)$ is a *commutative* graded $\mathbf{k}$-algebra. Show that the composition

$$\phi : V^n \to T^n(V) \to \mathrm{Sym}^n(V)$$

satisfies the universal property in Theorem 15.1.

The image of a pure tensor $x_1 \otimes \cdots \otimes x_n$ in $\mathrm{Sym}^n(V)$ is denoted by

$$x_1 \cdots x_n.$$

When $n!$ is invertible in $\mathbf{k}$, the quotient $q : T^n(V) \to \mathrm{Sym}^n(V)$ splits: the map defined by

$$x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

extends to a morphism of $\mathbf{k}$-vector space $\iota : \mathrm{Sym}^n(M) \to T^n(M)$ such that $q \circ \iota$ is the identity.

Exercise 15.3. Let V be a vector space over a field $\mathbf{k}$ of finite dimension n. Show that $\mathrm{Sym}(V^\vee)$ is identified with the ring of polynomials on V. You may show that explicitly, if $e_1, \ldots, e_n$ is a basis of V and $e_1^\vee, \ldots, e_n^\vee$ its dual basis, then there is an isomorphism

$$\mathrm{Sym}(V^\vee) = \mathbf{k}[e_1^\vee, \ldots, e_n^\vee]$$

as graded $\mathbf{k}$-algebras.

The exterior algebra associated to an $\mathbf{k}$-vector space V is defined as

$$\bigwedge V := \frac{T(V)}{<x \otimes x \mid x \in V>},$$

where the denominator is the two-sided ideal generated by all the $x \otimes x$.

Exercise 15.4. Likewise, show that the grading on $T(V)$ induces a grading $\oplus_i \bigwedge^i V$ on $\bigwedge V$, and that $\bigwedge V$ is a *graded-commutative* graded $\mathbf{k}$-algebra: namely, for every $a \in \bigwedge^i V$ and $b \in \bigwedge^j V$, we have

$$b \wedge a = (-1)^{ij} a \wedge b,$$

where $\wedge$ is the product on $\bigwedge V$. Show that the composition

$$\phi : V^n \to T^n(V) \to \bigwedge^n V$$

satisfies the universal property in Theorem 15.1.

The image of a pure tensor $x_1 \otimes \cdots \otimes x_n$ in $\bigwedge^n(V)$ is denoted by

$$x_1 \wedge \cdots \wedge x_n.$$

When $n!$ is invertible in $\mathbf{k}$, the map defined by

$$x_1 \wedge \cdots \wedge x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

extends to a morphism of $\mathbf{k}$-vector space $\bigwedge^n V \to T^n(V)$ and defines a splitting of the quotient $q : T^n(V) \to \bigwedge^n V$.

Exercise 15.5. Construct a natural $\mathbf{k}$-linear identification between $\mathrm{Sym}^n(V^\vee)$ (resp. $\bigwedge^n V^\vee$) and the space of symmetric (resp. alternating) multilinear forms on V.

15.3. Symmetric power and exterior power of G-representations. Let V be a $\mathbf{k}$-vector space and let $n \in \mathbf{Z}_{>0}$. For every $\phi \in \mathrm{End}_{\mathbf{k}}(V)$, we define the nth *symmetric power* ϕ to be the endomorphism $\mathrm{Sym}^n \phi$ on $\mathrm{Sym}^n V$ defined by

$$(\mathrm{Sym}^n \phi)(v_1 \cdots v_n) = \phi(v_1) \cdots \phi(v_n)$$

for pure tensors, then extends linearly to the whole $\mathrm{Sym}^n V$. Likewise, we define the nth *exterior power* of ϕ to be the endomorphism $\bigwedge^n \phi$ on $\bigwedge^n V$ defined by

$$(\bigwedge^n \phi)(v_1 \wedge \cdots \wedge v_n) = \phi(v_1) \wedge \cdots \wedge \phi(v_n)$$

for pure tensors, then extends linearly to the whole $\bigwedge^n V$.

Now assume that $\rho_V : G \to \mathrm{End}_{\mathbf{k}}(V)$ is a G-representation over $\mathbf{k}$. We define the nth symmetric power and the nth exterior power of ρ_V to be the G-representation on $\mathrm{Sym}^n V$ and $\bigwedge^n V$ defined by

$$\rho_{\mathrm{Sym}^n V}(g) = \mathrm{Sym}^n \rho_V(g) \quad \text{and} \quad \rho_{\bigwedge^n V}(g) = \bigwedge^n \rho_V(g)$$

respectively, for every $g \in G$.

Exercise 15.6. Show that

$$V \otimes_{\mathbf{k}} V \simeq \mathrm{Sym}^2(V) \oplus \bigwedge^2 V$$

as G-representations.

Exercise 15.7. Let V and W be G-representations over $\mathbf{k}$. Let $n \in \mathbf{Z}_{>0}$. Show that

$$\mathrm{Sym}^n(V \oplus W) \simeq \bigoplus_{i+j=n} \mathrm{Sym}^i V \otimes_{\mathbf{k}} \mathrm{Sym}^j V$$

and

$$\bigwedge^n (V \oplus W) \simeq \bigoplus_{i+j=n} \wedge^i V \otimes_{\mathbf{k}} \wedge^j V$$

as G-representations.

Exercise 15.8. The aim of this exercise is the prove the following proposition.

Proposition 15.9. *Let G be a finite group and let $\mathbf{k}$ be an algebraically closed field such that* $\mathrm{char}\mathbf{k} \nmid |G|$. *Let V be a finite dimensional faithful G-representation over $\mathbf{k}$. Every irreducible G-representation W over $\mathbf{k}$ is a G-subrepresentation of* $\mathrm{Sym}^n V$ *(in particular, a G-subrepresentation of $V^{\otimes n}$) for some integer n.*

(1) Show that there exists $u \in V^\vee$ such that $\mathrm{Stab}(u)$ is trivial for the induced G-action $G \circlearrowright V^\vee$.
(2) Show that the map

$$\mathrm{Sym}(V) \to \mathrm{Map}(G, \mathbf{k})$$

sending a polynomial f on $V^\vee$ to the map $g \mapsto f(g \cdot u)$ is a surjective maps of G-representations.
(3) Conclude.

Exercise 15.10. Let V_n be the standard representation of $\mathfrak{S}_n$. Show that $\bigwedge^d V_n$ is irreducible for all $d = 1, \ldots, n$.

15.4. Ring structures on the Grothendieck groups. Let $\mathrm{Rep}(G, \mathbf{k})_f = \mathbf{k}[G]\text{-}\mathrm{Mod}_f$ be the category of finite dimensional G-representations.

Exercise 15.11. Let V be a G-representation over $\mathbf{k}$ and let

$$0 \to W_1 \to W_2 \to W_3 \to 0$$

be a short exact sequence in $\mathrm{Rep}(G, \mathbf{k})$. Show that

$$0 \to W_1 \otimes_{\mathbf{k}} V \to W_2 \otimes_{\mathbf{k}} V \to W_3 \otimes_{\mathbf{k}} V \to 0$$

is a short exact sequence in $\mathrm{Rep}(G, \mathbf{k})$.

By Exercises 15.11 and 14.10, the product

$$[V] \cdot [W] := [V \otimes W]$$

is well-defined in $K_0(\mathrm{Rep}(G, \mathbf{k})_f)$ and $K_0^{\mathrm{add}}(\mathrm{Rep}(G, \mathbf{k})_f)$ for any finite dimensional G-representations V, W over $\mathbf{k}$, and we can extend by linearity to a product on $K_0(\mathrm{Rep}(G, \mathbf{k})_f)$ and $K_0^{\mathrm{add}}(\mathrm{Rep}(G, \mathbf{k})_f)$, making them into rings. We call them the Grothendieck ring of $\mathrm{Rep}(G, \mathbf{k})_f$, and the additive Grothendieck ring of $\mathrm{Rep}(G, \mathbf{k})_f$.

Exercise 15.12.

(1) Show that $K_0^{\mathrm{add}}(\mathrm{Rep}(G, \mathbf{k})_f) \to K_0(\mathrm{Rep}(G, \mathbf{k})_f)$ is a ring homomorphism.

(2) Show that we have a ring homomorphism

$$B(G) \to K_0^{\mathrm{add}}(\mathrm{Rep}(G,\mathbf{k})_f)$$

sending a G-set S to $\mathbf{k}[S]$.

Exercise 15.13. Show that the permutation representations associated to the G-sets

$$\mathrm{PGL}(3,\mathbf{F}_2) \circlearrowright \mathbf{P}(V) \quad \text{and} \quad \mathrm{PGL}(3,\mathbf{F}_2) \circlearrowright \mathbf{P}(V^\vee)$$

in Exercise 3.4 are isomorphic. Thus in general, $B(G) \to K_0^{\mathrm{add}}(\mathrm{Rep}(G,\mathbf{k})_f)$ is not injective.

15.5. Characters of tensor products. Let V and W be two finite dimensional G-representations over $\mathbf{k}$.

Exercise 15.14. Prove the following equalities.

(1) $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.
(2) $\chi_{V^\vee}(g) = \chi_V(g^{-1})$ for all $g \in G$.
(3) $\chi_{\mathrm{Hom}(V,W)} = \chi_{V^\vee} \cdot \chi_W$.

Thus the character map

$$\begin{aligned} K_0(\mathrm{Rep}(G,\mathbf{k})_f) &\to \mathrm{Map}(C(G),\mathbf{k}) \\ [V] &\mapsto \chi_V. \end{aligned} \tag{15.1}$$

is a ring homomorphism.

Exercise 15.15. Show that for every $g \in G$, we have

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2} \quad \text{and} \quad \chi_{\bigwedge^2(V)}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}.$$

Exercise 15.16. Let U, V, W be finite dimensional G-representations. Show that

$$(\chi_{U\otimes_{\mathbf{k}}V}, \chi_W) = (\chi_U, \chi_{\mathrm{Hom}_{\mathbf{k}}(V,W)}).$$

(Hint: use Exercise 14.13 and Proposition 13.12)

Exercise 15.17. Let V be a complex representation of a finite group G. Show that

$$\chi_{V^\vee} = \overline{\chi_V}.$$

16. Examples of representations of finite groups

Let G be a finite group and let $\mathbf{k}$ be a field of characteristic zero.

16.1. Character tables. Recall that by Maschke's theorem, $\mathbf{k}[G]$ is semisimple. So every G-representation is a direct sum of irreducible G-representations, and the isomorphism classes of irreducible representations V_i are completely their characters χ_{V_i}. Therefore the information of representations of G over $\mathbf{k}$ is essentially contained in the *character table* of G, which consists of

$$\chi_{V_i}(g_j) \in \mathbf{k}$$

with V_i runs through all irreducible G-representations, and g_j runs through all conjugacy classes of G. If we further assume that $\mathbf{k}$ algebraically closed, then these characters χ_{V_i} form a basis of $\mathrm{Map}(C(G),\mathbf{k})$, so in this case, the character table is a square table.

16.2. A trace formula. We could have proven the following statement together with Maschke's theorem

Proposition 16.1. *Let V be a G-representation over* $\mathbf{k}$. *Assume that* $\mathrm{char}(\mathbf{k}) = 0$. *We have*

$$\frac{1}{|G|}\sum_{g\in G} \chi_V(g) = \dim V^G.$$

Proof. Let

$$P := \frac{1}{|G|}\sum_{g\in G}\rho_V(g).$$

We have $\frac{1}{|G|}\sum_{g\in G}\chi_V(g) = \mathrm{Tr}P$. Since P is a projector onto V^G (namely $\mathrm{Im}P = V^G$ and $P_{|V^G} = V^G$), we have $\mathrm{Tr}P = \dim V^G$. □

16.3. Explicit description of the bilinear form for representations of finite groups. We assume that $\mathbf{k}$ is algebraically closed and $\mathrm{char}(\mathbf{k}) = 0$. Since $\mathbf{k}[G]$ is semisimple by Maschke's theorem, up to isomorphisms there exist only finitely many irreducible G-representations $V_1, \dots, V_n$. The character of these representations $\chi_i := \chi_{V_i}$ form a basis of the $\mathbf{k}$-vector space $\mathrm{Map}(C(G), \mathbf{k})$ by Theorem 13.9.

Let $(\bullet, \bullet)$ be the symmetric bilinear form on $\mathrm{Map}(C(G), \mathbf{k})$ having $\chi_1, \dots, \chi_n$ as an orthonormal basis.

Proposition 16.2. *For every pair of maps $f_1, f_2 : C(G) \to \mathbf{k}$, we have*

$$(f_1, f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1}).$$

Proof. Since the right hand side of the equality is also bilinear in f_1 and f_2, it suffices to prove for $f_1 = \chi_i$ and $f_2 = \chi_j$ for any i and j. By Proposition 16.1, we have

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\chi_{V_i\otimes V_j^\vee}(g) = \dim \mathrm{Hom}_{\mathbf{k}}(V_j, V_i)^G = \delta_{ij} = (\chi_i, \chi_j)$$

where the second last equality follows from Schur's lemma. □

Exercise 16.3. Without assuming that $\mathbf{k}$ is algebraically closed, show that for every pair of elements $f_1, f_2 : C(G) \to \mathbf{k}$ in $\mathbf{X}(G)_{\mathbf{k}} := \mathbf{X}(\mathbf{k}[G])$, we still have

$$(f_1, f_2) = \frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1}).$$

16.4. The standard representations of symmetric groups. Consider the natural action of $\mathfrak{S}_n$ on $S = \{1, \dots, n\}$, and the induced permutation representation $\mathbf{k}[S]$. Let $[1], \dots, [n]$ be the basis of $\mathbf{k}[S]$ corresponding to $1, \dots, n$. Note that $[1] + \cdots + [n]$ is $\mathfrak{S}_n$-invariant, and we have a decomposition

$$\mathbf{k}[S] \simeq V_n \oplus \mathbf{k}$$

of $\mathfrak{S}_n$-representations, where V_n is the linear subspace generated by $[i] - [j]$ for all i and j. We call $\mathfrak{S}_n \circlearrowright V_n$ the *standard representation* of $\mathfrak{S}_n$.

Proposition 16.4. *Assume that $\mathbf{k}$ is algebraically closed and $\mathrm{char}(\mathbf{k}) \nmid n!$. For every integer $n \geqslant 2$, the standard representation V_n of $\mathfrak{S}_n$ is irreducible.*

Proof. By Exercise 13.11, it suffices to show that $(\chi_{V_n}, \chi_{V_n}) = 1$. Let $\mathfrak{S}_n \circlearrowright S := \{1, \dots, n\}$ be the natural action.

We first show that $(\chi_{\mathbf{k}[S]}, \chi_{\mathbf{k}[S]}) = 2$. By Proposition 16.2, we have

$$(\chi_{\mathbf{k}[S]}, \chi_{\mathbf{k}[S]}) = \sum_{\sigma\in\mathfrak{S}_n}|\mathrm{Fix}(\sigma \circlearrowright S)|^2 = \sum_{\sigma\in\mathfrak{S}_n}\sum_{i,j=1}^{n}\delta_{i,\sigma(i)}\delta_{j,\sigma(j)} = \sum_{i,j=1}^{n}\sum_{\sigma\in\mathfrak{S}_n}\delta_{i,\sigma(i)}\delta_{j,\sigma(j)},$$

where δ is the Kronecker delta. We have

$$\sum_{\sigma\in\mathfrak{S}_n}\delta_{i,\sigma(i)}\delta_{j,\sigma(j)} = \begin{cases}(n-1)! & \text{if } i = j\\ (n-2)! & \text{if } i \neq j,\end{cases} \tag{16.1}$$

thus

$$\sum_{i,j=1}^{n}\sum_{\sigma\in\mathfrak{S}_n}\delta_{i,\sigma(i)}\delta_{j,\sigma(j)} = n\cdot(n-1)! + n\cdot(n-1)\cdot(n-2)! = 2n!.$$

Since

$$(\chi_{V_n}, \chi_{V_n}) = (\chi_{\mathbf{k}[S]}, \chi_{\mathbf{k}[S]}) - 2(\chi_{\mathbf{k}}, \chi_{V_n}) - (\chi_{\mathbf{k}}, \chi_{\mathbf{k}}) \leqslant (\chi_{\mathbf{k}[S]}, \chi_{\mathbf{k}[S]}) - (\chi_{\mathbf{k}}, \chi_{\mathbf{k}}) = 1,$$

we have $(\chi_{V_n}, \chi_{V_n}) = 1$.

□

16.5. Example: complex representations of cyclic groups. Assume that $\mathbf{k} = \mathbf{C}$. Let G be a cyclic group of order n. We know by Corollary 10.9 that the irreducible representations of G all have dimension 1. Also, since G is an abelian group, we have $C(G) = G$. So there are exactly $|G|$ isomorphism classes of irreducible G-representations.

Let $g \in G$ be a generator. Let $\zeta_n := e^{2\pi\imath/n}$. For every $i \in \mathbf{Z}$, consider the irreducible G-representation $\rho_i : G \to \mathrm{End}_{\mathbf{C}}(\mathbf{C})$ defined by $\rho_i(g)(v) = \zeta_n^i \cdot v$ for all $v \in \mathbf{C}$. Let $\chi_i := \chi_{\rho_i}$. We have

$$\chi_i(g^j) = \zeta_n^{ij}.$$

Exercise 16.5. Show that $\chi_0, \ldots, \chi_{n-1}$ are linearly independent in $\mathrm{Map}(G, \mathbf{C})$. Deduce that the irreducible G-representations are exactly $\mathrm{Id}_{\mathbf{k}}, \rho_1, \ldots, \rho_{n-1}$ up to isomorphisms.

16.6. Example: real representations of cyclic groups. Let G be a cyclic group of order n. Fix a generator $g \in G$. The representations ρ_j of G that we consider previously are also representations over $\mathbf{R}$, which are isomorphic to $\rho_j^{\mathbf{R}}$ defined by rotations of $\mathbf{R}^2$ of degree $\frac{2\pi}{n} j$. The representation $\rho_j^{\mathbf{R}}$ is irreducible if and only if rotation by $\frac{2\pi}{n} j$ doesn't have eigenspace, which is equivalent to $2j \neq n$. When $2j = n$, the representation $\rho_j^{\mathbf{R}}$ is decomposed as

$$\mathbf{R}^2 = \mathbf{R} \oplus \mathbf{R}_-$$

where $\mathbf{R}$ is the trivial representation of G, and $\mathbf{R}_-$ is the representation defined by $\sigma(g) = -\mathrm{Id}_{\mathbf{R}}$. Let $\chi_j^{\mathbf{R}} := \chi_{\rho_j^{\mathbf{R}}}$. We have

$$\chi_j^{\mathbf{R}}(g^k) = 2\cos(2\pi jk/n).$$

Note that $\chi_j^{\mathbf{R}} = \chi_{n-j}^{\mathbf{R}}$, so $\rho_j^{\mathbf{R}} \simeq \rho_{n-j}^{\mathbf{R}}$ by Corollary 13.5.

Exercise 16.6.

(1) Show that the characters $\chi_{\mathrm{Id}_{\mathbf{R}}}, \chi_\sigma, \chi_1^{\mathbf{R}}, \ldots, \chi_{\lceil n/2 \rceil - 1}^{\mathbf{R}}$ are linearly independent.
(2) Deduce from Corollary 12.17 the classification of irreducible real representations of G. Which of them are real, complex, quaternionic?

16.7. Product of groups. Let G and H be finite groups.

Exercise 16.7.

(1) Show that the set of conjugacy classes $C(G \times H)$ is in bijection with $C(G) \times C(H)$.
(2) Let $V_1, \ldots, V_m$ (resp. $W_1, \ldots, W_n$) be the isomorphism classes of irreducible representations of G (resp. H). Show that the the isomorphism classes of irreducible representations of $G \times H$ is

$$V_i \otimes_{\mathbf{k}} W_j \quad (i = 1, \ldots, m, \quad j = 1, \ldots, n),$$

where we regard V_i as the $(G \times H)$-representation defined by the composition $G \times H \to G \to \mathrm{End}_{\mathbf{k}}(V_i)$, and same for W_j.
(3) What is the character table over $\mathbf{C}$ of a finite abelian group?

16.8. Example: $\mathfrak{S}_3$. We work with $\mathbf{k} = \mathbf{C}$. Recall that each element of $\mathfrak{S}_n$ can be decomposed into a composition of cyclic permutations with disjoint cycles, and the conjugacy classes of $\mathfrak{S}_n$ is in bijection with the partitions of n, which correspond to the lengths of the cyclic permutations in the decomposition. Thus $\mathfrak{S}_3$ has three conjugacy classes, represented by the neutral element e, the transposition (12), and the 3-cycle $(1,2,3)$. So up to isomorphisms there are exactly three irreducible representations ρ_1, ρ_2, ρ_3 of $\mathfrak{S}_3$. One of them ρ_1 is the trivial representation $G \circlearrowright \mathbf{k}$. What are the others?

Let V_i be the underlying $\mathbf{k}$-vector space of ρ_i. First we notice that by Corollary 12.18, we have

$$6 = |G| = 1 + (\dim V_2)^2 + (\dim V_3)^2.$$

So necessarily $\dim V_2 = 1$ and $\dim V_3 = 2$ (up to permutations).

For ρ_2, we can consider the signature $\sigma : \mathfrak{S}_3 \to \{\pm 1\}$. The group $\{\pm 1\}$ acts on $\mathbf{k}$ by $\pm\mathrm{Id}$. Composing it with σ gives a representation ρ_2 of $\mathfrak{S}_3$ on $\mathbf{k}$, which is non-trivial (e.g. because $\chi_{\rho_1} \neq \chi_{\rho_2}$). As $\mathbf{k}$ is one-dimensional, ρ_2 is irreducible. The remaining irreducible representation ρ_3 is the standard representation of $\mathfrak{S}_3$.

	(1)	(12)	(123)
ρ_1	1	1	1
ρ_2	1	−1	1
ρ_3	2	0	−1

TABLE 1. Character table of $\mathfrak{S}_3$

16.9. Example: $\mathfrak{S}_4$. We work with $\mathbf{k} = \mathbf{C}$. The conjugacy classes of $\mathfrak{S}_4$ are represented by

$$e, (12), (123), (12)(34), (1234).$$

As in the case of $\mathfrak{S}_3$, the trivial representation ρ_1, the signature representation ρ_2, and the standard representation ρ_3 are irreducible representations of $\mathfrak{S}_4$ on $\mathbf{k}$.

Exercise 16.8. Show that ρ_3 is also the representation of $\mathbf{C}^3$ defined by the rotations of a cube.

Considering the $\mathfrak{S}_4$-action on the pairs of skew-edges in a tetrahedron, we have a surjective homomorphism $\mathfrak{S}_4 \to \mathfrak{S}_3$. Thus the 2-dimensional irreducible representation V of $\mathfrak{S}_3$ induces an irreducible representation ρ_4 of $\mathfrak{S}_4$.

Exercise 16.9. Let $\rho : G \to \mathrm{End}_{\mathbf{k}}(V)$ be an irreducible representation of G and let χ be a one-dimensional representation of G. Show that $\rho \otimes \chi$ is irreducible.

The remaining irreducible $\mathfrak{S}_4$-representation is $\rho_5 = \rho_3 \otimes \rho_4$.

Exercise 16.10. What is the character table of $\mathfrak{S}_4$ over $\mathbf{C}$?

16.10. Example: $\mathfrak{A}_4$. Let $P(\sigma)$ be the partition of n which corresponds to an element of $\sigma \in \mathfrak{S}_n$. Recall that $\sigma, \sigma' \in \mathfrak{A}_n$ are conjugate then $P(\sigma) \neq P(\sigma')$. Conversely, there exist exactly two (resp. one) conjugacy classes with the same partition P if P consists of distinct odd numbers (resp. otherwise). If $P(\sigma) = P(\sigma')$ consists of distinct odd numbers, then σ and σ' are conjugate if and only if they are conjugate by an *even* permutation in $\mathfrak{S}_n$. The conjugacy classes of $\mathfrak{A}_4$ are thus represented by

$$e, (123), (132), (12)(34).$$

Note that the set of irreducible representations of a group G contains those of G/H for every normal subgroup $H \trianglelefteq G$. To find the irreducible representations of $\mathfrak{A}_4$, we can first consider the quotient $\mathfrak{A}_4 \to \mathbf{Z}/3\mathbf{Z}$, defined by the action of $\mathfrak{A}_4$ on the pairs of skew-edges in a tetrahedron (which consists of even permutations). The cyclic group $\mathbf{Z}/3\mathbf{Z}$ has three irreducible complex representations, which lifts to irreducible complex representations ρ_1, ρ_2, ρ_3 of $\mathfrak{A}_4$. By the orthogonality of the character, the character table of $\mathfrak{A}_4$ is

	e	(123)	(132)	(12)(34)
ρ_1	1	1	1	1
ρ_2	1	ζ_3	ζ_3^2	1
ρ_3	1	ζ_3^2	ζ_3	1
ρ_4	3	0	0	−1

where $\zeta_3 = e^{2\pi\imath/3}$. Computing the character of the restriction of the standard representation $\mathfrak{S}_4 \circlearrowright \mathbf{C}^3$ to $\mathfrak{A}_4$, we see that it coincides with the character of ρ_4. Hence ρ_4 is the standard representation of $\mathfrak{A}_4$.

16.11. Example: $\mathfrak{A}_5$. The conjugacy classes of $\mathfrak{A}_5$ are represented by

$$e, (123), (12)(34), (12345), (12354).$$

Exercise 16.11. Show that the five classes of irreducible representation of $\mathfrak{A}_5$ over $\mathbf{C}$ are:

(1) The trivial representation ρ_1.
(2) The standard representation of $\mathfrak{S}_5$ restricted to $\mathfrak{A}_5$
(3) The representation ρ_3 of $\mathbf{C}^3$ defined by the rotations of a regular dodecahedron.
(4) The composition $\rho_3 \circ \alpha$ where $\alpha \in \mathrm{Aut}(\mathfrak{A}_5)$ is the conjugation of $\mathfrak{A}_5$ in $\mathfrak{S}_5$ by (45).
(5) The representation of $\mathfrak{A}_5$ on

$$\left\{ f : S \to \mathbf{C} \,\middle|\, \sum_{s \in S} f(s) = 0 \right\}$$

induced by the permutation of $\mathfrak{A}_5$ on the set S of pairs of opposite faces of a regular dodecahedron.

Describe the character table of $\mathfrak{A}_5$ over $\mathbf{C}$.

Exercise 16.12. Classify the irreducible representation of $\mathfrak{G}_5$ over $\mathbf{C}$, and describe the character table.

16.12. Example: complex representations of Q_8. Let

$$Q_8 := \{ \pm 1, \pm i, \pm j, \pm k \}$$

be the multiplicative subgroup of the quaternion algebra $\mathbf{H}$ over $\mathbf{R}$. The conjugacy classes are

$$1, -1, \pm i, \pm j, \pm k,$$

so Q_8 has five isomorphism classes of complex irreducible representations, which have dimension $1, 1, 1, 1, 2$ by Corollary 12.18.

To find the irreducible representations of Q_8, we can first consider the quotient

$$Q_8/Z(Q_8) \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$$

by the center $Z(Q_8) = \{\pm 1\}$, There are four irreducible representations of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, which give the one-dimensional representations of Q_8. Explicitly, these are the trivial representation ρ_1, the representation ρ_2 defined by $\rho_2(i) = \mathrm{Id}_{\mathbf{C}}$, $\rho_2(j) = \rho_2(k) = -\mathrm{Id}_{\mathbf{C}}$, and the other two ρ_3, ρ_4 defined similarly under permutations of i, j, k. By the orthogonality of the characters of irreducible representations and Proposition 16.2, we can deduce the character of the remaining irreducible representation ρ_5 of dimension 2, and the character table of Q_8 is

	1	-1	$\pm i$	$\pm j$	$\pm k$
ρ_1	1	1	1	1	1
ρ_2	1	1	1	-1	-1
ρ_3	1	1	-1	1	-1
ρ_4	1	1	-1	-1	1
ρ_5	2	-2	0	0	0

Consider the regular representation of $\mathbf{H}$ on $\mathbf{H}$. Regarding $\mathbf{H}$ as a $\mathbf{C}$-vector space defined by left-multiplication, then $1, j \in \mathbf{H}$ form a basis of $\mathbf{H}$. Let Q_8 act on $\mathbf{H}$ by *right*-multiplication, which is a 2-dimensional representation ρ_5 over $\mathbf{C}$. In terms of matrices with respect to the basis $1, j$, we have $\rho_5(-1) = -\mathrm{Id}$ and

$$\rho_5(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \rho_5(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_5(k) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Hence ρ_5 is the remaining irreducible representation of Q_8.

Remark 16.13. The matrices

$$\rho_5(i)/\sqrt{-1}, \quad \rho_5(j)/\sqrt{-1}, \quad \rho_5(k)/\sqrt{-1}$$

are call Pauli matrices in quantum mechanics.

16.13. Multiplication table of tensor product. For simplicity, assume that $\mathbf{k} = \mathbf{C}$. Recall that if V and W are finite dimensional G-representations, then $\chi_{V\otimes W} = \chi_V \chi_W$. The character $\chi_{V\otimes W}$ determines the G-representation $V \otimes W$, and we can compute $\chi_V \chi_W$ using the character table. For instance, if $G = \mathfrak{S}_3$, then we obtain the following Multiplication table of tensor products of irreducible $\mathfrak{S}_3$-representations.

	ρ_1	ρ_2	ρ_3
ρ_1	ρ_1	ρ_2	ρ_3
ρ_2		ρ_1	ρ_3
ρ_3			$\rho_1 \oplus \rho_2 \oplus \rho_3$

Exercise 16.14. Compute the multiplication table of tensor products of complex irreducible representations of $\mathfrak{S}_4$.

16.14. McKay quiver. The *McKay quiver* of a complex representation V of of G is a weighted quiver Q described as follows.

- The vertices i of Q are the irreducible representations V_i of G, and each vertex i is assigned with the number (weight) $\dim_{\mathbf{C}} V_i$.
- The number of arrows n_{ij} from i to j is equal to the multiplicity of V_j in $V \otimes_{\mathbf{k}} V_i$.

Exercise 16.15. Prove the following statements

(1) If $G \circlearrowright V$ is faithful, then the McKay graph of V is connected. (Hint: use Proposition 15.9.)
(2) If $V \simeq V^\vee$ as G-representations, then $n_{ij} = n_{ji}$.

16.15. Finite subgroups of SU(2) **and affine Dynkin diagram.** On the quaternion algebra $\mathbf{H}$ over $\mathbf{R}$, we have a norm defined by

$$N(a + bi + cj + dk) := \sqrt{a^2 + b^2 + c^2 + d^2},$$

with $a, b, c, d \in \mathbf{R}$. Alternatively, if $z = a + bi + cj + dk$ and $\bar{z} := a - bi - cj - dk$ denotes its conjugate, then $N(z) = z\bar{z}$. For every $z_1, z_2 \in \mathbf{H}$, since $\overline{z_1 z_2} = \overline{z_2} \cdot \overline{z_1}$, we have $N(z_1 z_2) = N(z_1)N(z_2)$. Thus the elements of norm 1 of $\mathbf{H}$ form a group, denoted by S^3.

Exercise 16.16. Show that S^3 is isomorphic to the group of special unitary matrices

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \middle| \, \alpha, \beta \in \mathbf{C}, \ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Exercise 16.17. Let ρ be the real representation of S^3 on $\mathbf{H}$ defined by conjugation

$$\rho(g) : z \mapsto gzg^{-1}.$$

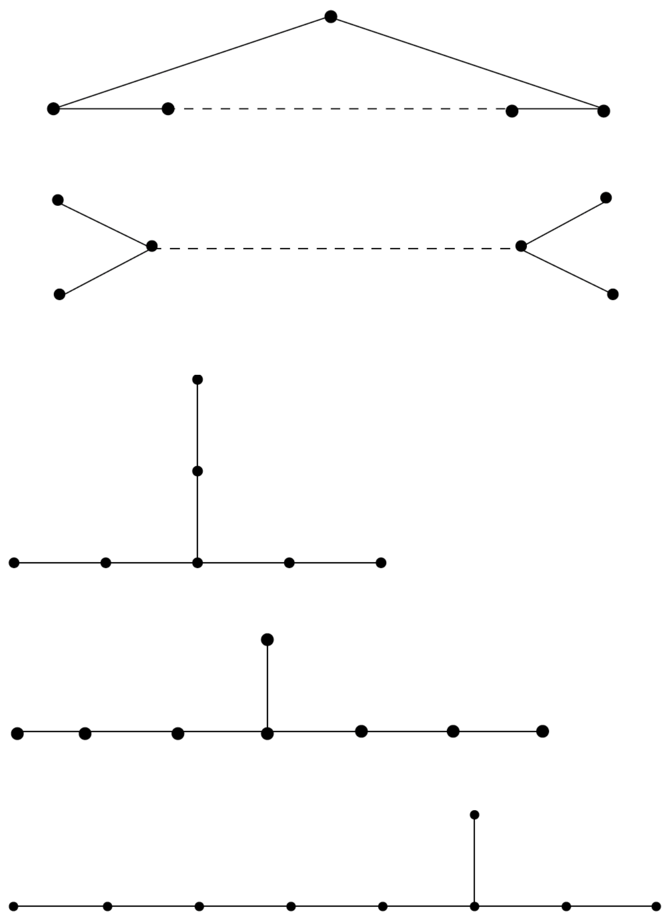
(1) Show that it preserves the norm N, and it restricts to a subrepresentation on the $\mathbf{R}$-linear subspace $W \subset \mathbf{H}$ spanned by i, j, k.
(2) Deduce that there is a surjective group homomorphism $S^3 \to \mathrm{SO}(3, \mathbf{R})$, whose kernel is $\{\pm 1\}$.
(3) Classifies finite subgroups which are isomorphic to some subgroup of $S^3 \simeq \mathrm{SU}(2)$.

Now let $G \leqslant \mathrm{SU}(2)$ be a nontrivial finite subgroup and let $\rho : G \to \mathrm{End}_{\mathbf{C}}(V)$ be the restriction of the standard representation of SU(2) on $\mathbf{C}^2$. Since $\mathrm{SU}(2) \circlearrowright V$ is faithful and self-dual, so is $G \circlearrowright V$. Thus the McKay quiver of $G \circlearrowright V$ is connected and satisfies $n_{ij} = n_{ji}$. Let $M(\rho)$ be the undirected graph, having the same vertices as the McKay quiver with n_{ij} edges between i and j.

Exercise 16.18. The aim of this exercise is to classify $M(\rho)$.

(1) Show that $2 - \chi_\rho$ is the character of a complex representation of G. Deduce that the quadratic form
$$f \mapsto ((2 - \chi_\rho)f, f)$$
on $\mathrm{Map}(C(G), \mathbf{R})$ is positive semidefinite, and not definite.

(2) Let $A(\rho)$ be the adjacent matrix of $M(\rho)$ and let
$$C(\rho) = 2 \cdot \mathrm{Id} - A(\rho).$$
Show that $C(\rho)$ is positive semidefinite and not definite if and only if $M(\rho)$ is one of the following.



They are the *affine Dynkin diagrams* of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$

(3) Describe the McKay quiver of G (including the weights).

Intermezzo

17. Frobenius divisibility

17.1. The statement.

Theorem 17.1. *Let G be a finite group and let $\rho : G \to \mathrm{End}_{\mathbf{C}}(V)$ be a finite dimensional irreducible complex representation of G. Then $\dim V$ divides $|G|$.*

17.2. The proof. For every $g \in G$, let

$$C(g) := \{ \gamma g \gamma^{-1} \mid \gamma \in G \}.$$

Define

$$\Phi_g := \sum_{h \in C(g)} \rho(h) \in \mathrm{End}_{\mathbf{C}}(V).$$

Note that $\Phi_g : V \to V$ is G-equivariant, so by Schur lemma, we have

$$\Phi_g = \lambda_g \mathrm{Id}_V$$

with

$$\lambda_g = \frac{|C(g)|}{d} \chi_V(g)$$

where $d = \dim V$.

We have

$$\frac{|G|}{d} = \sum_{g \in C(G)} \frac{\lambda_g}{\chi_V(g)} = \sum_{g \in C(G)} \lambda_g \overline{\chi_V(g)}.$$

Since $\mathbf{Z}$ is integrally closed and algebraic integers form a subring of $\mathbf{C}$, it suffices to show prove the following two lemmas.

Lemma 17.2. *$\chi_V(g)$ are algebraic integers.*

Proof. Since $\rho(g)$ has finite order, the eigenvalues of $\rho(g)$ are roots of unity. Thus $\chi_V(g) = \mathrm{Tr}\rho(g)$ is an algebraic integer. □

Lemma 17.3. *λ_g are algebraic integers.*

Proof. Let $C_g := \sum_{h \in C(g)} h \in \mathbf{Z}[G]$ and let $\mathbf{Z}[C_g]$ be the (commutative) $\mathbf{Z}$-subalgebra generated by C_g.

We have a surjective ring homomorphism $\mathbf{Z}[C_g] \twoheadrightarrow \mathbf{Z}[\Phi_g] \subset \mathrm{End}_{\mathbf{C}}(V)$ and $\mathbf{Z}[\Phi_g] \simeq \mathbf{Z}[\lambda_g] \subset \mathbf{C}$ as rings. Since $\mathbf{Z}[G]$ is a finitely generated $\mathbf{Z}$-module, so are $\mathbf{Z}[C_g]$ and $\mathbf{Z}[\lambda_g]$. Hence λ_g is integral over $\mathbf{Z}$ (by Cayley–Hamilton, see Modern Algebra II). □

17.3. An improvement.

Corollary 17.4. *Let G be a finite group and let $\rho : G \to \mathrm{End}_{\mathbf{C}}(V)$ be a finite dimensional irreducible complex representation of G. Then $\dim V$ divides $|G/Z|$, where Z is the center of G.*

Exercise 17.5. We shall prove Corollary 17.4.

(1) G_1 and G_2 be two finite groups. Let V_1 be a complex G_1-representation and let V_2 be a complex G_2-representation. Let $V_1 \boxtimes V_2$ be the $(G_1 \times G_2)$-representation on $V_1 \otimes_{\mathbf{k}} V_2$ defined by

$$(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 v_1) \otimes g_2 v_2.$$

Show that if V_1 and V_2 are irreducible representations, then so is $V_1 \boxtimes V_2$.

(2) Let $m \in \mathbf{Z}_{>0}$. Let $e \in G$ be the neutral element. Show that

$$K = \{ (z_1, \dots, z_m) \in Z^m \mid z_1 \cdots z_m = e \}$$

lies in the kernel of the G^m-representation $V^{\boxtimes m}$.

(3) Conclude.

LECTURE 6

Induction and restriction

18. Induced modules and restricted modules

18.1. First example: extension of scalars. Let $\mathbf{k}$ be a field and let V be a vector space of dimension d over a field $\mathbf{k}$. If we choose a basis $e_1, \ldots, e_d$ of V, then every element of V is a linear combination of the e_i's with coefficients in $\mathbf{k}$, which gives a $\mathbf{k}$-linear isomorphism

$$V \simeq \oplus_{i=1}^{d} \mathbf{k} \cdot e_i.$$

Now let $L/\mathbf{k}$ be a field extension. The tensor product

$$V_L := V \otimes_{\mathbf{k}} L$$

that we will define *as an L-vector space* can be understood as the extension of scalars. With the above chosen basis $e_1, \ldots, e_d$, there exists a canonical isomorphism

$$V_L \simeq \oplus_{i=1}^{d} L \cdot e_i,$$

through which V_L can be described as an L-vector space having the same basis elements as V, but replacing the coefficient field with L. If we have a $\mathbf{k}$-linear map $\phi : U \to V$ between $\mathbf{k}$-vector spaces, it also extends to an L-linear map

$$\phi_L : U_L \to V_L$$

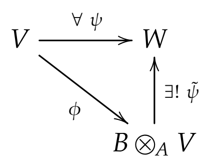
defined by the same matrix.

18.2. Universal property of induced modules. We also notice that if V is a $\mathbf{k}$-vector space and W is an L-vector space, then any $\mathbf{k}$-linear map $\psi : V \to W$ has a unique L-linear extension $\tilde{\psi} : V_L \to W$. This motivates the following general definition.

Theorem-Definition 18.1 (Universal property of induced modules)**.** *Let $r : A \to B$ be a morphism of $\mathbf{k}$-algebras. Let V be an A-module. There exists a B-module $B \otimes_A V$, together with an A-linear map*

$$\phi : V \to B \otimes_A V,$$

satisfying the following universal property: for any A-linear map $\psi : V \to W$ to some B-module W, there exists a unique B-linear map $\tilde{\psi} : B \otimes_A V \to W$ such that



commutes. Moreover, the pair $(B \otimes_A V, \phi)$ is unique up to unique isomorphism. We call $B \otimes_A V$ the induced B-module. *It is also denoted by $\mathrm{Ind}_A^B V$.*

18.3. Construction. The construction of $B \otimes_A V$ is similar to the construction of tensor products we've seen previously. We define the B-module $B \otimes_A V$ by generators and relations as follows:

- Generators: $b \otimes v$ for all $b \in B$ and $v \in V$.
- The R-submodule of relations $\mathscr{R}$ is generated by

$$(b + b') \otimes v = b \otimes v + b' \otimes v, \quad b \otimes (v + v') = b \otimes v + b \otimes v',$$

$$(ba) \otimes v = b \otimes (av) = b(a \otimes v), \quad \text{and} \quad b(b' \otimes v) = (bb') \otimes v$$

for all $b, b' \in B$, $v, v' \in V$, and $a \in A$. Here, the image of a in B is still denoted by a by abuse of notation.

In other words,,

$$B \otimes_A V := \left(\bigoplus_{b \in B, v \in V} B \cdot (b \otimes v) \right) / \mathscr{R}.$$

Exercise 18.2. Prove Theorem 18.1.

Exercise 18.3. Let $A \to B \to C$ be morphisms of $\mathbf{k}$-algebras. Show that we have the following canonical isomorphisms.

- $B \otimes_A (U \oplus V) \xrightarrow{\sim} (B \otimes_A U) \oplus (B \otimes_A V)$;
- $A \otimes_B (B \otimes_C V) \xrightarrow{\sim} A \otimes_C V$.

18.4. Restricted modules. Let A be a $\mathbf{k}$-algebra and let $\phi : A \to B$ be an A-algebra. Any B-module V has an induced A-module structure, defined by

$$a \cdot v := \phi(a) \cdot v$$

for every $a \in A$ and $v \in V$. As a morphism of B-modules is naturally a morphism of A-modules, we thus have a functor

$$\mathrm{Res}_A : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

from the category of B-modules to the category of A-modules, called the restriction functor. For every B-module V, the A-module $\mathrm{Res}_A V$ is called the restricted module.

18.5. Frobenius reciprocity. Restricted modules and induced modules are related as follows.

Exercise 18.4. Show that Ind_A^B and Res_A form an adjoint pair: there exist natural $\mathbf{k}$-linear isomorphisms

$$\mathrm{Hom}_B(\mathrm{Ind}_A^B V, W) \simeq \mathrm{Hom}_A(V, \mathrm{Res}_A W)$$

for any A-module V and B-module W. (Hint: use the universal property.)

19. Example: extension and restriction of scalars

Let $\mathbf{k}$ be a field and let $L/\mathbf{k}$ be a field extension. Let G be a finite group.

19.1. Endomorphisms under extension of scalars. Let V be a finite dimensional $\mathbf{k}$-vector space and let $f \in \mathrm{End}_{\mathbf{k}}(V)$. Let $V_L := L \otimes_{\mathbf{k}} V$ and

$$f_L := \mathrm{Id}_L \otimes f \in \mathrm{End}_L(V_L).$$

Then

$$\mathrm{Tr}(f) = \mathrm{Tr}(f_L).$$

In particular, if $G \circlearrowright V$ is a finite dimensional representation over $\mathbf{k}$, then for the induced a G-representation on V_L over L (namely $V_L = \mathrm{Ind}_{\mathbf{k}[G]}^{L[G]} V$), we have

$$\chi_V = \chi_{V_L}.$$

Exercise 19.1. Let $\rho : G \to \mathrm{End}_W$ be a finite dimensional G-representation over L. Show that $W \simeq L \otimes_{\mathbf{k}} V$ for some G-representation V over $\mathbf{k}$ if and only if there exists a basis $\mathscr{B}$ on W such that for every $g \in G$, with respect to $\mathscr{B}$ the map $\rho(g)$ is a matrix with coefficients in $\mathbf{k}$.

19.2. Trace map.

Exercise 19.2. Let $L/\mathbf{k}$ be a finite Galois extension. Let $\alpha \in L$. Let $P_\alpha \in \mathbf{k}[X]$ be the minimal polynomial of α and let $c_\alpha \in \mathbf{k}[X]$ be the characteristic polynomial of the $\mathbf{k}$-linear map

$$\begin{aligned} \mu_\alpha : L &\to L \\ x &\mapsto \alpha x. \end{aligned}$$

Show that

$$c_\alpha = P_\alpha^{[L:\mathbf{k}(\alpha)]}.$$

Deduce that

$$\mathrm{Tr}_{L/\mathbf{k}}(\alpha) := \mathrm{Tr}\mu_\alpha = \sum_{\sigma\in \mathrm{Gal}(L/K)} \sigma(\alpha).$$

We call $\mathrm{Tr}_{L/\mathbf{k}} : L \to \mathbf{k}$ the *trace* map.

Example 19.3. For every $z \in \mathbf{C}$, $\mathrm{Tr}_{\mathbf{C}/\mathbf{R}}(z) = z + \bar{z}$.

19.3. Endomorphisms under restriction of scalars. Now let V be a finite dimensional L-vector space and let $f \in \mathrm{End}_L(V)$. Let $V|_\mathbf{k}$ be the underlying $\mathbf{k}$-vector space of V and let $f|_\mathbf{k} \in \mathrm{End}_\mathbf{k}(V|_\mathbf{k})$ be the endomorphism f, viewed as a $\mathbf{k}$-linear map.

Suppose that $L/\mathbf{k}$ is a finite Galois extension. It follows from Exercise 19.2 that

$$\mathrm{Tr}(f|_\mathbf{k}) = \mathrm{Tr}_{L/\mathbf{k}}(\mathrm{Tr}(f)).$$

We have the following more precise statement.

Exercise 19.4. Let W be a finite dimensional $\mathbf{k}$-vector space such that $V \simeq L \otimes_\mathbf{k} W$ as L-vector spaces. Then the Galois action on L induces a $\mathrm{Gal}(L/\mathbf{k})$-action on V. For every $\sigma \in \mathrm{Gal}(L/\mathbf{k})$, let V_σ be the $L[f]$-module whose underlying L-vector space is V, such that f acts on V by

$$v \mapsto (\sigma \circ f \circ \sigma^{-1})(v)$$

Show that

$$L \otimes_\mathbf{k} V \simeq \bigoplus_{\sigma\in \mathrm{Gal}(L/\mathbf{k})} V_\sigma$$

as $L[f]$-modules, where f acts on $L \otimes_\mathbf{k} V$ by $\mathrm{Id}_L \otimes f$.

Remark 19.5. In Exercise 19.4, the $\mathbf{k}$-vector space W together with the isomorphism $V \simeq L \otimes_\mathbf{k} W$ is called a $\mathbf{k}$-*structure* of V. The $\mathrm{Gal}(L/\mathbf{k})$-action on V, depend on the choice of $\mathbf{k}$-structure. If W' is another $\mathbf{k}$-structure of V, then the $\mathrm{Gal}(L/\mathbf{k})$-action on V induced by W' is conjugate to the previous $\mathrm{Gal}(L/\mathbf{k})$-action by some L-linear automorphism of V.

In particular, if $\rho : G \to \mathrm{End}_L(V)$ is a finite dimensional representation over L, then for the restricted a G-representation on $V|_\mathbf{k}$ over $\mathbf{k}$ (namely $V|_\mathbf{k} = \mathrm{Res}_{\mathbf{k}[G]} V$), we have

$$\chi_{V|_\mathbf{k}} = \mathrm{Tr}_{L/\mathbf{k}} \circ \chi_V.$$

Exercise 19.6. Assume that $\mathrm{char}\mathbf{k} = 0$. Show that

$$(\chi_{V|_\mathbf{k}}, \chi_{V|_\mathbf{k}}) = [L : \mathbf{k}] \sum_{\sigma\in \mathrm{Gal}(L/\mathbf{k})} (\chi_V, \chi_{V_\sigma}),$$

where V_σ is the L-linear G-representation on V defined by

$$g \mapsto \sigma \circ \rho(g) \circ \sigma^{-1}.$$

Show that $\chi_{V_\sigma} = \sigma \circ \chi_V$.

19.4. Restriction of scalars: case C/R. Let's look at the case $\mathbf{C}/\mathbf{R}$.

Let V be an irreducible complex representation of G and let $V|_\mathbf{R}$ be the underlying real representation. Recall that

$$\overline{\chi_V} = \chi_{V^\vee}.$$

By Exercise 19.6, we then have

$$(\chi_{V|_\mathbf{R}}, \chi_{V|_\mathbf{R}}) = 2(\chi_V, \chi_V) + 2(\chi_V, \chi_{V^\vee}) = \begin{cases} 2 \text{ if } V \not\simeq V^\vee \\ 4 \text{ if } V \simeq V^\vee \end{cases}. \tag{19.1}$$

Exercise 19.7. Show that
$$\mathbf{C} \otimes_{\mathbf{R}} V \simeq V \oplus V^{\vee}$$
as complex G-representations.

Lemma-Definition 19.8 (Trichotomy of complex representations)**.** *Let V be an irreducible complex representation of G. Exactly one of the following happens.*

- *(real type) $V|_{\mathbf{R}}$ is not irreducible. In this case,*
$$V|_{\mathbf{R}} \simeq W \oplus W.$$
for some irreducible real representation W of G of real type.
- *(complex type) $V|_{\mathbf{R}}$ is irreducible of complex type.*
- *(quaternionic type) if $V|_{\mathbf{R}}$ is irreducible of quaternionic type.*

We have $(\chi_{V|_{\mathbf{R}}}, \chi_{V|_{\mathbf{R}}}) = 2$ (resp. $(\chi_{V|_{\mathbf{R}}}, \chi_{V|_{\mathbf{R}}}) = 4$) if and only if $G \circlearrowright V$ is of complex type (resp. real or quaternionic type).

Proof. Suppose that $V|_{\mathbf{R}}$ is irreducible. Then $V|_{\mathbf{R}}$ is not of real type because $(\chi_{V|_{\mathbf{R}}}, \chi_{V|_{\mathbf{R}}}) \neq 1$.

Assume that $V|_{\mathbf{R}}$ is not irreducible. Let $W \subset V|_{\mathbf{R}}$ be an irreducible $\mathbf{R}[G]$-submodule of multiplicity m. Since V is an irreducible $\mathbf{C}[G]$-module, we have $\imath \cdot W \neq W$. As $\imath \cdot W$ is also an irreducible $\mathbf{R}[G]$-submodule of $V|_{\mathbf{R}}$ and $W \simeq \imath \cdot W$ as $\mathbf{R}[G]$-modules, we have $m \geqslant 2$. Thus
$$4 \geqslant (\chi_{V|_{\mathbf{R}}}, \chi_{V|_{\mathbf{R}}}) \geqslant m^2(\chi_W, \chi_W) \geqslant 4,$$
which implies that $m = 2$ and $(\chi_W, \chi_W) = 1$. □

19.5. Extension of scalars: case $\mathbf{C}/\mathbf{R}$.

Corollary 19.9. *Let V be an irreducible real representation of G.*

1. *V is of real type if and only if $\mathbf{C} \otimes_{\mathbf{R}} V$ is an irreducible complex representation of G of real type.*
2. *V is of complex type if and only if $V \simeq W|_{\mathbf{R}}$ for some irreducible complex representation W of G of complex type; in this case $\mathbf{C} \otimes_{\mathbf{R}} V \simeq W \oplus W^{\vee}$ with $W \not\simeq W^{\vee}$.*
3. *V is of quaternionic type if and only if $V \simeq W|_{\mathbf{R}}$ for some irreducible complex representation W of G of quaternionic type; in this case $\mathbf{C} \otimes_{\mathbf{R}} V \simeq W \oplus W$ and $W \simeq W^{\vee}$.*

Proof. Let $V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} V$, regarded as a complex representation of G. We have $\chi_V = \chi_{V_{\mathbf{C}}}$.

If V is of real type, then $(\chi_{V_{\mathbf{C}}}, \chi_{V_{\mathbf{C}}}) = (\chi_V, \chi_V) = 1$, so the complex representation $V_{\mathbf{C}}$ is irreducible, which is necessarily of real type.

If V is of complex type, then $(\chi_{V_{\mathbf{C}}}, \chi_{V_{\mathbf{C}}}) = (\chi_V, \chi_V) = 2$. So $V_{\mathbf{C}} = W \oplus W'$ for some irreducible complex G-representations W and W' with $W \not\simeq W'$. Since
$$W|_{\mathbf{R}} \oplus W'|_{\mathbf{R}} = (V_{\mathbf{C}})|_{\mathbf{R}} \simeq V \oplus V$$
as real G-representations and V is irreducible, necessarily $W|_{\mathbf{R}} \simeq V$. Finally, since
$$W \oplus W' = V_{\mathbf{C}} \simeq \mathbf{C} \otimes_{\mathbf{R}} W \simeq W \oplus W^{\vee},$$
we have $W^{\vee} \simeq W' \not\simeq W$. Hence W is of complex type.

If V is of quaternionic type, then $(\chi_{V_{\mathbf{C}}}, \chi_{V_{\mathbf{C}}}) = (\chi_V, \chi_V) = 4$. So either $V_{\mathbf{C}} = W \oplus W$ or $V_{\mathbf{C}} \simeq W_1 \oplus W_2 \oplus W_3 \oplus W_4$ for some irreducible complex G-representations W, W_1, W_2, W_3, W_4. Since
$$(V_{\mathbf{C}})|_{\mathbf{R}} \simeq V \oplus V$$
as G-representations and V is irreducible, necessarily we are in the former case, so $W|_{\mathbf{R}} \simeq V$. Finally, since
$$W \oplus W = V_{\mathbf{C}} \simeq \mathbf{C} \otimes_{\mathbf{R}} W \simeq W \oplus W^{\vee},$$
we have $W^{\vee} \simeq W$. Hence W is of quaternionic type.

The "if" part of the statements follow from the uniqueness of the Krull-Schmidt decomposition. □

19.6. Counting irreducible real representations.

Proposition 19.10. *The following numbers are equal:*

(1) *The number of isomorphism classes of irreducible real representations of G.*
(2) $r + \frac{c}{2} + q$*, where r,c,q are the number of isomorphism classes of irreducible complex representations of G of real, complex, quaternionic type respectively.*
(3) *The dimension of the invariant subspace* $\mathrm{Map}(C(G), \mathbf{C})^\theta$ *under the involution*

$$\begin{aligned} \theta : \mathrm{Map}(C(G), \mathbf{C}) &\to \mathrm{Map}(C(G), \mathbf{C}) \\ f &\mapsto (g \mapsto f(g^{-1})). \end{aligned} \tag{19.2}$$

(4) $\frac{n+\#C(G)}{2}$*, where n be the number of conjugacy classes which is invariant under* $g \mapsto g^{-1}$.

Proof. The equality (1) = (2) follows from Corollary 19.9. Then the number of isomorphism classes of irreducible real representations of G is $r + \frac{c}{2} + q$.

Since the characters $\chi_1, \ldots, \chi_k$ of the irreducible complex representations of G form a basis of $\mathrm{Map}(C(G), \mathbf{C})$, and since $\theta(\chi_V) = \chi_{V^\vee}$ for any irreducible complex representation V, we have (3) = (4). Finally, since $\chi_{V^\vee} = \theta(\chi_V) = \chi_V$ if and only if V is not of complex type, we have (2) = (3). □

19.7. Invariant forms.

Lemma 19.11. *Let V be a real (resp. complex) finite dimensional G-representation. Then V admits a G-invariant positive definite scalar product (resp. positive definite Hermitian product).*

Proof. For real representation, start with any positive definite scalar product $(\bullet, \bullet)$. Then

$$(x, y) \mapsto \sum_{g \in G} (gx, gy)$$

is a G-invariant positive definite scalar product. For complex representation the proof is similar. □

Corollary 19.12. *Let V be an irreducible complex representation of G.*

(1) *V is of real type if and only if V has a G-invariant nondegenerate symmetric bilinear form.*
(2) *V is of complex type if and only if* $V \not\simeq V^\vee$ *as G-representations.*
(3) *V is of quaternionic type if and only if V has a G-invariant nondegenerate alternating bilinear form.*

In (1) and (3), the bilinear form is unique up to scalar.

Proof. By Lemma 19.9, V is of complex type if and only if $(\chi_{V|_\mathbf{R}}, \chi_{V|_\mathbf{R}}) = 2$. which is equivalent to $(\chi_V, \chi_{V^\vee}) = 0$ by (19.1), this proves (2).

Suppose that V is not of complex type. Then

$$(1, \chi_{V \otimes V}) = (\chi_V, \chi_{V^\vee}) = 1$$

by (19.1) and Lemma 19.9. Since $V \otimes V = S^2 V \oplus \bigwedge^2 V$, exactly one of $S^2 V$ and $\bigwedge^2 V$ has a nonzero G-invariant element. Thus it remains to construct the bilinear forms in (1) and (3) for V of real type and of quaternionic type respectively.

Suppose that V is of real type. Then $V \simeq \mathbf{C} \otimes_\mathbf{R} W$ for some real G-representation W. By Lemma 19.11, W has a G-invariant positive definite scalar product B. The complexification of B is a G-invariant nondegenerate symmetric bilinear form on V.

Suppose that V is of quaternionic type. By Lemma 19.11, V has a G-invariant positive definite Hermitian product H. Let $j \in \mathbf{H} \simeq \mathrm{End}(V|_\mathbf{R})$ and define

$$\alpha(x, y) = H(x, jy)$$

for every $x, y \in V$. For any complex number c, we have $jc = \bar{c} j$, so α is a complex bilinear form. As H is positive definite, α is nondegenerate. Finally, by the uniqueness of G-invariant bilinear form up to scalar, there exists $\varepsilon \in \mathbf{C}$ such that $\alpha(y,x) = \varepsilon\alpha(x,y)$ for all $x, y \in V$. Choose a nonzero element $x \in V$. Since

$$\alpha(x, jx) = -H(x,x) < 0 \quad \text{and} \quad \alpha(jx, x) = H(jx, jx) > 0,$$

we have $\varepsilon < 0$. As $\alpha(x, jx) = \varepsilon^2\alpha(x, jx)$, we have $\varepsilon^2 = 1$. Thus $\varepsilon = -1$, showing that α is alternating. □

Exercise 19.13. Determine the type of each irreducible complex representations of S_4. Same for D_4 and Q_8.

19.8. Frobenius–Schur indicator and 2-torsions of G. For every irreducible complex G-representation V, define the Frobenius–Schur indicator of V to be

$$\mathrm{FS}(V) := \dim(S^2V)^G - \dim(\wedge^2V)^G = \begin{cases} 1 & \text{if } V \text{ is of real type} \\ 0 & \text{if } V \text{ is of complex type} \\ -1 & \text{if } V \text{ is of quaternionic type} \end{cases} \tag{19.3}$$

Let

$$G[2] := \{ g \in G \mid g^2 = e \},$$

where $e \in G$ is the neutral element.

Proposition 19.14. *Let Σ be the set of isomorphism classes of irreducible complex G-representations. We have*

$$\#G[2] = \sum_{V\in\Sigma} \mathrm{FS}(V) \cdot \dim V.$$

Proof. For every $g \in G$ we have

$$\mathrm{Tr}(g \circlearrowright \mathbf{k}[G]) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{otherwise}. \end{cases} \tag{19.4}$$

Since

$$\mathbf{C}[G] \simeq \bigoplus_{V\in\Sigma} V^{\dim V},$$

we have

$$\#G[2] = \frac{1}{|G|}\sum_{g\in G} \chi_{\mathbf{k}[G]}(g^2) = \frac{1}{|G|}\sum_{V} \dim V \sum_{g\in G} \chi_V(g^2).$$

As $\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\wedge^2V}(g)$ by Exercise 15.15, it follows from the trace formula that $\#G[2] = \sum_{V\in\Sigma} \mathrm{FS}(V) \cdot \dim V$. □

20. Examples: induced representations

Let $\mathbf{k}$ be a field. For simplicity, we assume $\mathrm{char}\,\mathbf{k} = 0$. Let G be a finite group and let $H \leqslant G$ be a subgroup. For every H-representation V over $\mathbf{k}$, set

$$\mathrm{Ind}_H^G V := \mathrm{Ind}_{\mathbf{k}[H]}^{\mathbf{k}[G]} V = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V.$$

We call $\mathrm{Ind}_H^G V$ the induced G-representation of V.

20.1. An explicit description of $\mathrm{Ind}_H^G V$.

Proposition-Exercise 20.1. *Let $S \subset G$ be a subset of representatives of G/H. For every $s \in S$, let*

$$V_s := s \otimes V \subset \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V.$$

Then

$$\bigoplus_{s\in S} V_s = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$$

as $\mathbf{k}$-vector spaces.

Through the decomposition in Proposition 20.1, the G-action on $\mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$ is described as follows. For every $s \in S$ and $g \in G$, let $s' \in S$ be the unique element such that

$$gs \in s'H.$$

Then $s'^{-1}gs \in H$, so for every $s \otimes v \in V_s$, we have

$$g \cdot (s \otimes v) = s' \otimes (s'^{-1}gs \cdot v) \in V_{s'}.$$

Note that if $sH = s'H$, then $V_s = V_{s'}$ as $\mathbf{k}$-vector subspaces of $\mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$. Thus if s is referred to an element of G/H, we may write $V_s := V_{\tilde{s}}$ where $\tilde{s} \in G$ is any lifting of s.

Exercise 20.2. Let $\mathbf{k}$ be the trivial H-representation. Show that $\mathrm{Ind}_H^G \mathbf{k}$ is the permutation G-representation of G/H by left-multiplication.

20.2. When is a representation induced from a subgroup representation? Let V be a finite dimensional G-representation. By construction of induced representations, one necessary condition for V to be isomorphic to some $\mathrm{Ind}_H^G W$ is the existence of

(1) a decomposition
$$V = \bigoplus_{i \in I} V_i$$
as $\mathbf{k}$-vector spaces

(2) a transitive G-action $G \circlearrowright I$ such that
$$g \cdot V_i = V_{g \cdot i}$$
for all $g \in G$ and $i \in I$.

Exercise 20.3. Suppose that conversely, V is a G-representation satisfying (1) and (2) above. Let $i_0 \in I$ and let $H = \mathrm{Stab}(i_0)$ for $G \circlearrowright I$. Show that

$$V \simeq \mathrm{Ind}_H^G V_{i_0}.$$

20.3. An application of induced representations. The following statement improves Corollary 17.4.

Corollary 20.4. *Let G be a finite group and let $A \trianglelefteq G$ be a normal abelian subgroup. Every finite dimensional irreducible complex G-representation $\rho : G \to \mathrm{End}_{\mathbf{C}}(V)$ satisfies*

$$\dim V \mid (G : A).$$

Proof. Let

$$V|_A := \mathrm{Res}_A V = \bigoplus_{i \in I} W_i^{\oplus m_i}$$

be the Krull–Schmidt decomposition of $V|_A$: each W_i is an irreducible A-representation and $W_i \not\simeq W_j$ whenever $i \neq j$. Write $V_i = W_i^{\oplus m_i}$. Since A is normal in G, there exists a group action $G \circlearrowright I$ such that

$$g \cdot V_i = V_{g \cdot i}$$

for all $g \in G$ and $i \in I$. As V is an irreducible G-representation, $G \circlearrowright I$ is transitive.

Let $i_0 \in I$ and let $H := \mathrm{Stab}(i_0) \leqslant G$. By Exercise 20.3, we have $V \simeq \mathrm{Ind}_H^G V_{i_0}$, so

$$\dim V = (G : H) \dim V_{i_0}.$$

Note that A is also a normal subgroup of H, so $(\dim V_{i_0}) | (H : A)$ implies $(\dim V) | (G : A)$. Thus by induction on $(G : H)$, it suffices to prove the statement for the case $G = H$.

Suppose that $G = H$. Then $V|_A \simeq W^m$ for some irreducible A-representation W. As A is abelian, this implies that for all $a \in A$, $\rho(a) = \lambda \mathrm{Id}$ for some $\lambda \in \mathbf{C}$. Thus $\rho(A)$ is in the center of $\rho(G)$. It follows that

$$\dim V \mid (\rho(G) : \rho(A)) \mid (G : A).$$

□

20.4. Frobenius formula for the character of an induced representation. We compute the character $\mathrm{Ind}_H^G\chi_V$ of $\mathrm{Ind}_H^G V$. By the description of the G-action on $\bigoplus_{s\in S} V_s$ through the isomorphism Proposition 20.1, V_s is G-stable if and only if $s^{-1}gs \in H$. Thus

$$\mathrm{Ind}_H^G\chi_V(g) = \sum_{s\in S, s^{-1}gs\in H} \chi_V(s^{-1}gs) \tag{20.1}$$

Exercise 20.5. Show that

$$\mathrm{Ind}_H^G\chi_V(g) = \frac{1}{|H|}\sum_{\gamma\in G,\gamma g\gamma^{-1}\in H} \chi_V(\gamma g\gamma^{-1}).$$

For instance, if H is a normal subgroup of G and $g \notin H$, then $\mathrm{Ind}_H^G\chi_V(g) = 0$.

20.5. Frobenius reciprocity for characters. Define

$$\mathrm{Ind}_H^G : \mathbf{X}(H)_\mathbf{k} \to \mathbf{X}(G)_\mathbf{k}$$

by (20.1) and

$$\mathrm{Res}_H : \mathbf{X}(G)_\mathbf{k} \to \mathbf{X}(H)_\mathbf{k}$$

by restriction.

Exercise 20.6. Show that

$$(\mathrm{Ind}_H^G\phi, \psi) = (\phi, \mathrm{Res}_H\psi)$$

for every $\phi \in \mathbf{X}(H)_\mathbf{k}$ and $\psi \in \mathbf{X}(G)_\mathbf{k}$.

20.6. Example: $\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}$. We identify $\mathfrak{S}_3$ with the permutation group of $\{1,2,3\}$, and $\mathfrak{S}_2 \leqslant \mathfrak{S}_3$ with the stabilizer of 3. Let $\mathbf{C}_-$ be the trivial representation of $\mathfrak{S}_2 = \{\mathrm{Id}, \sigma\}$, and let $\mathbf{C}_-$ be the "sign representation" of $\mathfrak{S}_2$ on $\mathbf{C}$ What is $\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}\mathbf{C}_-$?

Let ρ_1, ρ_2, ρ_3 be the three irreducible complex representations of $\mathfrak{S}_3$ we've studied previously. We have $\mathrm{Res}_{\mathfrak{S}_2}\rho_1 = \mathbf{C}$, $\mathrm{Res}_{\mathfrak{S}_2}\rho_2 = \mathbf{C}_-$, and $\mathrm{Res}_{\mathfrak{S}_2}\rho_3 = \mathbf{C}\oplus\mathbf{C}_-$

Using the character table of $\mathfrak{S}_3$ together with Frobenius reciprocity, we have $\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}\chi_{\mathbf{C}_-} = \chi_2 \oplus \chi_3$, thus

$$\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_3}\mathbf{C}_- \simeq \rho_2 \oplus \rho_3.$$

Exercise 20.7. Let $\mathbf{k} = \mathbf{C}$. We regard $\mathfrak{S}_3 \leqslant \mathfrak{S}_4$ as the stabilizer of 4 for the natural $\mathfrak{S}_4$-action on $\{1,2,3,4\}$. For every irreducible complex $\mathfrak{S}_3$-representation V, decompose $\mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_4}V$ into a direct sum of irreducible $\mathfrak{S}_4$-representations.

20.7. Mackey's decomposition. When is $\mathrm{Ind}_H^G V$ irreducible? To answer this question, we need to compute $(\mathrm{Ind}_H^G\chi_V, \mathrm{Ind}_H^G\chi_V) = 1$, and by Frobenius reciprocity, it would be helpful if we know what $\mathrm{Res}_H\mathrm{Ind}_H^G V$ is.

The following statement provides a decomposition of $\mathrm{Res}_H\mathrm{Ind}_H^G V$, in a more general setting.

Theorem 20.8 (Mackey's decomposition)**.** *Let $K, H \leqslant G$ be subgroups of G. Let $\rho : H \to \mathrm{End}_\mathbf{k}(V)$ be a representation of H over $\mathbf{k}$. For each $i \in K\backslash G/H$, choose a representative $s_i \in G$. We have an isomorphism of K-representations*

$$\mathrm{Res}_K\mathrm{Ind}_H^G V \simeq \bigoplus_{i\in K\backslash G/H} \mathrm{Ind}_{(s_iHs_i^{-1})\cap K}^K V^{s_i}$$

where V^{s_i} is the representation $\rho_{s_i} : s_iHs_i^{-1} \to \mathrm{End}_\mathbf{k}(V)$ defined by

$$\rho_{s_i}(g) = \rho(s_i^{-1}gs_i).$$

Proof. Let $S \subset G$ be a subset of representatives of G/H. Recall from Exercise 1.9 that

$$K\backslash G/H \xrightarrow{\sim} \mathrm{Orb}\,(K \circlearrowright (G/H))$$

sending KgH to $K\cdot(gH)$ is a bijection. Decomposing G/H into K-orbits $G/H = \bigsqcup_{i\in K\backslash G/H} S_i$ such that the image of s_i lies in S_i, we have

$$\bigoplus_{i\in K\backslash G/H}\bigoplus_{s\in S_i} V_s = \mathbf{k}[G]\otimes_{\mathbf{k}[H]} V$$

as **k**-vector spaces. Note that for each $i \in K\backslash G/H$, $W_i := \bigoplus_{s \in S_i} V_s$ is K-stable, so we can regard W_i as a K-representation.

The stabilizer of s_i for $K \circlearrowright S_i$ is $(s_i H s_i^{-1}) \cap K$, which induces a bijection

$$\frac{K}{(s_i H s_i^{-1}) \cap K} \simeq S_i.$$

If $k, k' \in K$ has the same image in $K/(s_i H s_i^{-1} \cap K)$, then $V_{ks_i} = V_{k's_i}$ as **k**-linear subspaces of W_i. It follows that

$$W_i = \bigoplus_{k \in K/(s_i H s_i^{-1} \cap K)} V_{ks_i} \simeq \mathrm{Ind}^K_{(s_i H s_i^{-1}) \cap K} V_{s_i}$$

as K-representations. Finally, since $V^{s_i} \to V_{s_i}$ sending $s_i \otimes v$ to v is a K-equivariant isomorphism, we have

$$W_i \simeq \mathrm{Ind}^K_{(s_i H s_i^{-1}) \cap K} V^{s_i}$$

as K-representations, which finishes the proof. □

20.8. Mackey's irreducibility criterion. As a consequence,

$$(\mathrm{Ind}^G_H \chi_V, \mathrm{Ind}^G_H \chi_V) = (\chi_V, \mathrm{Res}_H \mathrm{Ind}^G_H \chi_V) = \sum_{i \in H\backslash G/H} \left(\chi_V, \mathrm{Ind}^H_{(s_i H s_i^{-1}) \cap H} \chi_V^{s_i}\right)$$

where $\chi_V^{s_i}$ is the character of $V^{s_i}|_{(s_i H s_i^{-1}) \cap H}$. Since

$$\left(\mathrm{Ind}^H_{(s_i H s_i^{-1}) \cap H} \chi^{s_i}, \chi_V\right) = (\chi^{s_i}, \mathrm{Res}_{(s_i H s_i^{-1}) \cap H} \chi_V),$$

we have

$$(\mathrm{Ind}^G_H \chi_V, \mathrm{Ind}^G_H \chi_V) = (\chi_V, \chi_V) + \sum_{i \in H\backslash G/H - \{e\}} (\chi^{s_i}, \mathrm{Res}_{(s_i H s_i^{-1}) \cap H} \chi_V),$$

where e is the class of the neutral element of G.

Corollary 20.9. *The induced representation* $\mathrm{Ind}^G_H V$ *is irreducible if and only if the following properties hold:*

- *V is irreducible.*
- *For all $g \notin H$, V^g and V viewed as representations of $(gHg^{-1}) \cap H$ don't have common irreducible factors.*

20.9. An application: irreducible representations of some semidirect products. Let G be a finite group and let $A \trianglelefteq G$ be a normal abelian subgroup. Assume that

$$G = A \rtimes H$$

for some subgroup H. Recall that if $(a, h), (a', h') \in A \rtimes H$, then

$$(a, h) \cdot (a', h') = (a(ha'h^{-1}), hh')).$$

We want to describe the complex irreducible representations of G.

Here is a construction of irreducible representations of G. Since A is abelian, any complex A-representation χ is one-dimensional, and is identified with its character $\chi : A \to \mathbf{C}^\times$. Consider the G-action on the character space $G \circlearrowright \mathbf{X}(A)_{\mathbf{C}}$ defined by

$$(g \cdot \chi)(a) = \chi(gag^{-1})$$

for all $g \in G$, $\chi \in \mathbf{X}(A)_{\mathbf{C}}$, $a \in A$.

Exercise 20.10. For every $\chi \in \mathbf{X}(A)_{\mathbf{C}}$, show that

$$G_\chi := \mathrm{Stab}(\chi) = A \rtimes H_\chi \leqslant G$$

for some subgroup $H_\chi \leqslant H$.

Let W be a finite dimensional irreducible complex representation of H_χ, which we regard as a representation of G_χ through $G_\chi \twoheadrightarrow H_\chi$. The character χ is identified with an A-representation on $\mathbf{C}$; regarded χ as a G_χ-representation, let $\widetilde{W} := \chi \otimes W$ Finally, let

$$V_{\chi,W} := \mathrm{Ind}_{G_\chi}^{G} \widetilde{W}.$$

Exercise 20.11.

(1) Show that the G-representation $V_{\chi,W}$ is irreducible. (Hint: use the Mackey irreducibility criterion.)
(2) Show that $V_{\chi,W} \simeq V_{\chi',W'}$ if and only if χ is in the same orbit as χ' (so H_χ is conjugates to $H_{\chi'}$, which yields an isomorphism $G_\chi \simeq G_{\chi'}$), and that $W \simeq W'$ as G_χ representations.
(3) Show that every finite dimensional irreducible complex G-representation is isomorphic to some $V_{\chi,W}$.

LECTURE 7

Representations of symmetric groups

21. Irreducible complex representations of symmetric groups

Let n be a positive integer. What are the irreducible complex representations of $\mathfrak{S}_n$?

21.1. Young diagrams. The number of isomorphism classes of irreducible complex representations of $\mathfrak{S}_n$ is equal to the number of conjugacy classes of $\mathfrak{S}_n$, which are in bijection with the partitions of n.

A partition of n is a sequence of positive integers

$$\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k),$$

such that $\sum_{i=1}^k \lambda_i = n$. We represent λ by a Young diagram, namely k rows of boxes of lengths $\lambda_1, \ldots, \lambda_k$ from top to bottom, with each row starting at the same horizontal position. We call k the length of λ.

21.2. Young symmetrizers. Let λ be a partition. A *Young tableau* $T(\lambda)$ associated to λ is a filling of every integer from 1 to n into the Young diagram λ, one in each box. Let $S_i \subset [1,n] \cap \mathbf{Z}$ (resp. $S_i' \subset [1,n] \cap \mathbf{Z}$) be the set of integers in the ith row (resp. column) of T_λ. We have

$$S_1 \sqcup \cdots \sqcup S_k = S_1' \sqcup \cdots \sqcup S_\ell' = [1,n] \cap \mathbf{Z}$$

where ℓ is the number of columns of λ. Let

$$P_{T(\lambda)} := \mathrm{Bij}(S_1) \times \cdots \times \mathrm{Bij}(S_k) \subset \mathfrak{S}_n,$$
$$Q_{T(\lambda)} := \mathrm{Bij}(S_1') \times \cdots \times \mathrm{Bij}(S_\ell') \subset \mathfrak{S}_n,$$

and let

$$a_{T(\lambda)} := \frac{1}{|P_{T(\lambda)}|} \sum_{g \in P_{T(\lambda)}} g \in \mathbf{C}[\mathfrak{S}_n],$$
$$b_{T(\lambda)} := \frac{1}{|Q_{T(\lambda)}|} \sum_{g \in Q_{T(\lambda)}} (-1)^{\sigma(g)} g \in \mathbf{C}[\mathfrak{S}_n],$$

where $\sigma(g)$ is the signature of g. Finally, let

$$c_{T(\lambda)} = a_{T(\lambda)} b_{T(\lambda)}.$$

We call $c_{T(\lambda)}$ a Young symmetrizer.

21.3. Specht modules. For every Young tableau $T(\lambda)$, define the $\mathfrak{S}_n$-representations

$$V_{T(\lambda)} := \mathbf{C}[\mathfrak{S}_n] c_{T(\lambda)}.$$

Theorem 21.1.

(1) *The isomorphism class V_λ of the $\mathfrak{S}_n$-representation $V_{T(\lambda)}$ only depends on the Young diagram λ.*
(2) *For every pair of Young diagrams λ and λ', $V_\lambda \simeq V_{\lambda'}$ if and only if $\lambda = \lambda'$.*
(3) *Each V_λ is irreducible, and every finite dimensional irreducible complex $\mathfrak{S}_n$-representation is isomorphic to one of V_λ.*

We call $V_{T(\lambda)}$ the Specht modules.

Remark 21.2. Theorem 21.1 implies that the same statements hold with $\mathbf{C}$ replaced by $\mathbf{Q}$, and if we define $V_{T(\lambda)} = \mathbf{Q}[\mathfrak{S}_n] c_{T(\lambda)}$ instead.

By construction, every statement in §21 still holds if we replace $\mathbf{C}$ be $\mathbf{Q}$.

21.4. The dependence of $V_{T(\lambda)}$ on the Young tableau. First we prove Theorem 21.1.(1).

Let λ be a Young diagram. The group action

$$\mathfrak{S}_n \circlearrowright \{ \text{ Young tableaux associated to } \lambda \}$$

sending $T(\lambda)$ to the permutation of the entries by $\mathfrak{S}_n$ is transitive. For every $g \in \mathfrak{S}_n$, we have

$$P_{g\cdot T(\lambda)} = gP_{T(\lambda)}g^{-1} \quad \text{and} \quad Q_{g\cdot T(\lambda)} = gQ_{T(\lambda)}g^{-1}.$$

Thus

$$V_{g\cdot T(\lambda)} \simeq V_{T(\lambda)} \cdot g^{-1}$$

as $\mathfrak{S}_n$-representations.

21.5. First examples.

Exercise 21.3.

(1) Suppose that λ has only one row. Show that V_λ is the trivial representation.
(2) Suppose that λ has only one column. Show that V_λ is the sign representation $\mathbf{C}_{\mathrm{sgn}}$.
(3) Let $T(\lambda)$ be a Young tableau. Show that

$$V_{{}^tT(\lambda)} \simeq \mathbf{C}_{\mathrm{sgn}} \otimes_{\mathbf{C}} V_\lambda,$$

where ${}^tT(\lambda)$ is the transpose of λ (i.e., the rows of ${}^tT(\lambda)$ are the columns of $T(\lambda)$).

21.6. A key property of $a_{T(\lambda)}$ and $b_{T(\lambda)}$. The elements $a_{T(\lambda)}$ and $b_{T(\lambda)}$ satisfy the following key property.

Proposition 21.4. *We have*

$$a_{T(\lambda)}\mathbf{C}[\mathfrak{S}_n]b_{T(\lambda)} = \mathbf{C} \cdot c_{T(\lambda)}$$

We will use the following lemma to prove Proposition 21.4.

Lemma 21.5. *Let $T(\lambda)$ and $T(\mu)$ be two Young tableaux. Let $g \in \mathfrak{S}_n$. Suppose that $P_{T(\lambda)} \cap (gQ_{T(\mu)}g^{-1})$ contains a permutation $\sigma \in \Sigma_n$ of signature -1. Then*

$$a_{T(\lambda)}gb_{T(\mu)} = 0.$$

Proof. We have

$$a_{T(\lambda)}gb_{T(\mu)} = a_{T(\lambda)}\sigma gb_{T(\mu)} = a_{T(\lambda)}g(g^{-1}\sigma g)b_{T(\mu)} = -a_{T(\lambda)}gb_{T(\mu)},$$

so $a_{T(\lambda)}gb_{T(\mu)} = 0$. □

Proof of Proposition 21.4. Note that if $g = pq$ for $h \in P_{T(\lambda)}$ and $q \in Q_{T(\lambda)}$, then

$$a_{T(\lambda)}gb_{T(\lambda)} = (-1)^{\mathrm{sgn}(q)}c_{T(\lambda)}.$$

Thus (1) follows from Lemma 21.5 and the following.

Lemma 21.6. *Suppose that $g \notin P_{T(\lambda)}Q_{T(\lambda)}$, then there exists a transposition σ in $P_{T(\lambda)} \cap (gQ_{T(\lambda)}g^{-1})$.*

Proof. Let $g \in \mathfrak{S}_n$ such that $P_{T(\lambda)} \cap (gQ_{T(\lambda)}g^{-1})$ contains no transposition. Note that $gQ_{T(\lambda)}g^{-1} = V_{g\cdot T(\lambda)}$. So any pair of elements of $[1,n] \cap \mathbf{Z}$ in the same row of $T(\lambda)$ are not in the same column of $g \cdot T(\lambda)$. Thus there exists $q' \in Q_{g\cdot T(\lambda)}$ and $p \in P_{T(\lambda)}$ such that

$$q'g \cdot T(\lambda) = p \cdot T(\lambda).$$

Thus $g = pq$ for $q := g^{-1}q'^{-1}g \in Q_{T(\lambda)}$. □

□

Let λ and μ be two Young diagrams. We write $\lambda > \mu$ if $\lambda_i > \mu_i$ for the smallest i such that $\lambda_i \neq \mu_i$.

Proposition 21.7. *Let $T(\lambda)$ and $T(\mu)$ be two Young tableaux. If $\lambda > \mu$, then*

$$a_{T(\lambda)}\mathbf{C}[\mathfrak{S}_n]b_{T(\mu)} = 0.$$

Proof. Let $g \in \mathfrak{S}_n$ By Lemma 21.5, it suffices to show that there exists $P_{T(\lambda)} \cap (gQ_{T(\mu)}g^{-1})$ contains a transposition.

Let i be the smallest index such that $\lambda_i > \mu_i$. We can assume that there is no pair of integers x, y lying in the jth row in $T(\lambda)$ with $j < i$ and in the same column of $g \cdot T(\mu)$. By the same argument as in the proof of Lemma 21.6, there exist $p \in P_{T(\lambda)}$ and $q' \in Q_{g \cdot T(\mu)}$ such that the first $i - 1$ rows of $p \cdot T(\lambda)$ and $q'g \cdot T(\mu)$ are equal. It follows from the pigeonhole principle that there exists two integers x, y lying in the ith row of $p \cdot T(\lambda)$ and in the same column of $q'g \cdot T(\mu)$. Hence

$$P_{p \cdot T(\lambda)} \cap Q_{q'g \cdot T(\mu)} = (pP_{T(\lambda)}p^{-1}) \cap (q'gQ_{T(\mu)}g^{-1}q'^{-1}) = P_{T(\lambda)} \cap (gQ_{T(\mu)}g^{-1})$$

contains a transposition. □

21.7. Idempotents. Let $\mathbf{k}$ be a field and let A be a $\mathbf{k}$-algebra. An element $e \in A$ is called an *idempotent* if $e^2 = e$. If $e \in A$ is idempotent, then so is $1 - e \in A$, and we have

$$A = Ae \oplus A(1 - e)$$

as A-modules

Lemma 21.8. *Let $e \in A$ be an idempotent. For every A-module V, we have*

$$\mathrm{Hom}_A(Ae, V) \simeq eV$$

as $\mathbf{k}$-vector spaces.

Proof. We have an isomorphism $\Phi : \mathrm{Hom}_A(A, V) \xrightarrow{\sim} V$ sending $f \in \mathrm{Hom}_A(A, V)$ to $f(1)$. Consider the decomposition

$$\mathrm{Hom}_A(A, V) = \mathrm{Hom}_A(Ae, V) \oplus \mathrm{Hom}_A(Ae, V).$$

If $f \in \mathrm{Hom}_A(A, V)$ lies in $\mathrm{Hom}_A(Ae, V)$, then

$$f(1) = f(e) = ef(1) \in eV.$$

So

$$\Phi(\mathrm{Hom}_A(Ae, V)) \subset eV \quad \text{and} \quad \Phi(\mathrm{Hom}_A(A(1 - e), V)) \subset (1 - e)V.$$

As

$$V = eV \oplus (1 - e)V,$$

necessarily $\mathrm{Hom}_A(Ae, V) \simeq eV$. □

Exercise 21.9. Let $T(\lambda)$ and $T(\mu)$ be two Young tableaux.

(1) Show that

$$c_{T(\lambda)}^2 = \frac{n!}{|P_{T(\lambda)}||Q_{T(\lambda)}| \dim V_{T(\lambda)}} c_{T(\lambda)}.$$

(2) Show that

$$\mathrm{Hom}_{\mathfrak{S}_n}(V_{T(\lambda)}, V_{T(\mu)}) \simeq c_{T(\lambda)}\mathbf{C}[\mathfrak{S}_n]c_{T(\mu)}$$

as $\mathbf{C}$-vector spaces.

21.8. Irreducible representations of $\mathfrak{S}_n$. Now we finish the prove of Theorem 21.1. We first prove Theorem 21.1.(2) and the first statement of Theorem 21.1.(3).

Proposition 21.10.

(1) Each V_λ is irreducible.

(2) If λ and μ are distinct Young diagrams, then $V_\lambda \not\simeq V_\mu$.

Proof. We can assume that $\lambda > \mu$ without loss of generality. Let $T(\lambda)$ and $T(\mu)$ be Young tableaux whose underlying Young diagrams are λ and μ respectively. By Exercise 21.9, we have

$$\mathrm{Hom}_{\mathfrak{S}_n}(V_{T(\lambda)}, V_{T(\mu)}) \simeq c_{T(\lambda)}\mathbf{C}[\mathfrak{S}_n]c_{T(\mu)}.$$

Hence $V_{T(\lambda)}$ is irreducible by Proposition 21.4 and Exercise 21.9.(1), and $V_\lambda \not\simeq V_\mu$ by Proposition 21.7. □

Finally, since there is a bijection between the conjugacy classes of $\mathfrak{S}_n$ and the partitions of n, the second statement of Theorem 21.1.(3) follows.

Exercise 21.11. Identify the Young diagram for each finite dimensional irreducible complex representations of $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

22. Frobenius character formula

This part is handwritten.

LECTURE 8

Lie algebras

23. Algebraic groups and Lie algebras

This part is handwritten.

24. The category of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{k}$. We have defined Lie algebras and morphism of Lie algebras in the previous lecture.

24.1. Ideals. An *ideal* of $\mathfrak{g}$ is a $\mathbf{k}$-linear subspace $I \subset \mathfrak{g}$ such that $[I, \mathfrak{g}] \subset I$. (Note that since $[I, \mathfrak{g}] = [\mathfrak{g}, I]$, we don't need to define left of right ideals.) An ideal is in particular a Lie subalgebra, but the converse is false in general.

Note that ideals of $\mathfrak{g}$ are nothing but subrepresentations of the adjoint representation of $\mathfrak{g}$.

Exercise 24.1. Let $I, J \subset \mathfrak{g}$ be two ideals. Show that $I + J, I \cap J$, and $[I, J]$ are also ideals of $\mathfrak{g}$.

Exercise 24.2.

(1) Let $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. Show that $\ker(f)$ is an ideal of $\mathfrak{g}$.
(2) Let $I \subset \mathfrak{g}$ be an ideal. Show that the Lie bracket on $\mathfrak{g}$ descends to a Lie bracket on the quotient vector space $\mathfrak{g}/I$. Thus $\mathfrak{g}/I$ is a Lie algebra.

24.2. Universal enveloping algebra. Every associative $\mathbf{k}$-algebra A is a Lie algebra, with the Lie bracket defined by

$$[x, y] = xy - yx$$

for all $x, y \in A$. The forgetful functor from the category of associative $\mathbf{k}$-algebras to the category of Lie algebras over $\mathbf{k}$ has a left adjoint.

Theorem-Definition 24.3. *For every Lie algebra $\mathfrak{g}$, define the associative $\mathbf{k}$-algebra*

$$U(\mathfrak{g}) := \frac{T^\bullet(\mathfrak{g})}{<v \otimes w - w \otimes v - [v, w] | v, w \in \mathfrak{g}>}.$$

The functor U is left adjoint to the forgetful functor: for every associative $\mathbf{k}$-algebras, we have natural isomorphisms

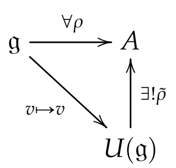
$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A) \simeq \mathrm{Hom}_{\mathbf{k}\text{-}\mathrm{Alg}}(U(\mathfrak{g}), A).$$

The $\mathbf{k}$-algebra $U(\mathfrak{g})$ is called the universal enveloping algebra *of* $\mathfrak{g}$.

In particular, the universal enveloping algebra satisfies the following universal property. For any associative $\mathbf{k}$-algebra A and any morphism of Lie algebras $\rho : \mathfrak{g} \to A$ satisfying

$$\rho([x, y]) = xy - yx$$

for all $x, y \in \mathfrak{g}$, there exists a unique morphism $\tilde{\rho} : U(\mathfrak{g}) \to A$ of $\mathbf{k}$-algebras such that



commutes.

Exercise 24.4. Prove Theorem 24.3.

24.3. Poincaré-Birkhoff-Witt theorem.

Theorem 24.5. *Let $X_{i i\in I}$ be a basis of $\mathfrak{g}$ as a $\mathbf{k}$-vector space. Fix a total order $<$ on I. Then*

$$\{ X_{i_1} \cdots X_{i_n} \mid i_1 < \cdots < i_n \}$$

is a basis of the $\mathbf{k}$-vector space $U(\mathfrak{g})$.

We refer to [**3**, Theorem 9.10] for a proof of the PBW theorem for finite dimensional $\mathfrak{g}$. The general statement follows from the finite dimensional case.

Corollary 24.6. *The canonical map $\mathfrak{g} \to U(\mathfrak{g})$ is injective.*

24.4. The category of Lie algebra representations. As a corollary of Theorem 24.3, the category of Lie algebra representations of $\mathfrak{g}$ is equivalent to the category of the representations of $U(\mathfrak{g})$.

Corollary 24.7. *We have an equivalence of categories*

$$\mathrm{Rep}(\mathfrak{g}, \mathbf{k}) \simeq \mathrm{Rep}(U(\mathfrak{g}), \mathbf{k}).$$

We can therefore transfer every notion and theorem about the representations of $U(\mathfrak{g})$ to the representations of $\mathfrak{g}$ (subrepresentations, quotients, irreducibility, semisimplicity, etc.).

24.5. Centers. The center of $\mathfrak{g}$ is defined as

$$Z(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g} \}.$$

We say that the Lie algebra $\mathfrak{g}$ is *abelian* if $\mathfrak{g} = Z(\mathfrak{g})$.

Exercise 24.8. Show that $\mathfrak{g}$ is abelian if and only if $U(\mathfrak{g})$ is commutative.

24.6. Hom **and** $\otimes$**.** Let V and W be $\mathbf{k}$-vector spaces. Let $\rho_V : \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(V)$ and $\rho_W : \mathfrak{g} \to \mathrm{End}_{\mathbf{k}}(W)$ be two representations of $\mathfrak{g}$. Then we have a representation of $\mathfrak{g}$ on the tensor product $V \otimes_{\mathbf{k}} W$, defined as

$$(\rho_V \otimes \rho_W)(x) = \rho_V(x) \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \rho_W(x).$$

Similarly, we have a representation of $\mathfrak{g}$ on $\mathrm{Hom}_{\mathbf{k}}(V, W)$ defined by

$$x \cdot f = \rho_W(x) \circ f - f \circ \rho_V(x)$$

for any $x \in \mathfrak{g}$ and $f \in \mathrm{Hom}_{\mathbf{k}}(V, W)$.

Exercise 24.9. Show that the constructions $\otimes$ and Hom for Lie algebra representations coincide with the constructions $\otimes$ and Hom for representations of associative algebras under the equivalence in Corollary 24.7.

In particular, the dual representation of $\mathfrak{g} \to \mathrm{End}(V)$ is

$$\rho_{V^\vee}(x) = -\rho_V(x)^\vee$$

for every $x \in \mathfrak{g}$.

Exercise 24.10. Let V be a Lie algebra representation of $\mathfrak{g}$. Show that $\bullet \otimes_{\mathbf{k}} V$ is left adjoint to $\mathrm{Hom}_{\mathbf{k}}(V, \bullet)$ in the category of $\mathfrak{g}$-representations.

24.7. Invariant elements and an example: the Killing form. Let V be a representation of $\mathfrak{g}$. An element $v \in V$ is called invariant if $\mathfrak{g} \cdot v = 0$.

Example 24.11. By definition A bilinear form $B \in V^\vee \otimes_{\mathbf{k}} V^\vee$ on V is invariant if

$$B(x \cdot v, w) + B(v, x \cdot w) = 0$$

for all $v, w \in V$.

Exercise 24.12. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbf{k}$. For every $x, y \in \mathfrak{g}$, define

$$K(x, y) := \operatorname{Tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)) \in \mathbf{k}.$$

Show that K is an invariant symmetric bilinear form on $\mathfrak{g}$.

We call K the *Killing form* of $\mathfrak{g}$.

25. $\mathfrak{sl}_2$

This part is handwritten.

26. On the structure of Lie algebras

This part is handwritten.

27. The Schur–Weyl duality

The Schur–Weyl duality exhibits some parallels between the irreducible representations of $\mathfrak{S}_n$ and those of $\operatorname{GL}(W)$ for a vector space W. We start with an example.

27.1. $S^m W$ and $\bigwedge^m W$ as irreducible $\operatorname{GL}(W)$ representations. Let W be a $\mathbf{C}$-vector space of dimension n.

Exercise 27.1. Let $m \in \mathbf{Z}_{>0}$. Let $\rho : \operatorname{GL}(W) \to \operatorname{End}_{\mathbf{k}}(S^m W)$ or $\rho : \operatorname{GL}(W) \to \operatorname{End}_{\mathbf{k}}(\bigwedge^m W)$ be the induced representations on $E := S^m W$ or $E := \bigwedge^m W$. The aim of this exercise is to show that $S^m W$ and $\bigwedge^m W$ (for $m \leqslant n$) are irreducible $\operatorname{GL}(W)$-representations.

(1) Find $h \in \operatorname{GL}(W)$ such that $\rho(h)$ is diagonalisable with distinct eigenvalues.
(2) Let $B = \{ e_1, \dots, e_\ell \}$ be a basis of E which diagonalize $\rho(h)$. Show that any $\operatorname{GL}(W)$-subrepresentation $U \subset E$ is generated by a subset of B.
(3) Choose a basis $\{ w_1, \dots, w_n \}$ of W, and let E_{ij} denote the elementary matrices on W. By considering the matrices $\operatorname{Id} + E_{ij}$ with $i \neq j$, show that E is an irreducible $\operatorname{GL}(W)$-representation.

We will see that the irreducible representations $S^m W$ and $\bigwedge^m W$ correspond to the Specht modules V_λ for $\lambda = (m)$ and $\lambda = (1, \dots, m)$ respectively.

27.2. Dual pairs. Let A and B be $\mathbf{k}$-algebras. Let V be an (A, B)-bimodule. The (A, B)-bimodule structure on V gives rise to morphisms of $\mathbf{k}$-algebras

$$B \to \operatorname{End}_A(V) \quad \text{and} \quad A \to \operatorname{End}_B(V).$$

We say that A and B form a *dual pair* with respect to the (A, B)-bimodule V if the above morphisms are isomorphisms.

Exercise 27.2. Show that A and A^{op} form a dual pair with respect to A viewed as a natural (A, A^{op})-bimodule.

Lemma 27.3. *Let A and B be a dual pair with respect to an (A, B)-bimodule V.*

(1) The left A-module and the right B-module structures of V are faithful.
(2) Suppose that V is finite dimensional. Then A is semisimple if and only if B is semisimple.

Proof. (1) is clear. The first statement of (2) follows from Proposition 12.15 and Corollary 12.11. □

27.3. The double centralizer theorem.

Theorem 27.4. *Let A be a semisimple* **k***-algebra and let V be a finite dimensional faithful A-module. Then A and $B := \mathrm{End}_A(V)$ form a dual pair with respect to the (A, B)-bimodule V. Moreover, we have*

$$V \simeq \bigoplus_{i=1}^{n} V_i \otimes_{\mathbf{k}} W_i$$

as (A, B)-bimodules, for some irreducible left A-modules V_i and irreducible right B-modules W_i. The left A-modules V_i are pairwise non-isomorphic, and so are the right B-modules W_i.

Proof. Let $V_1, \dots, V_n$ be the irreducible A-submodules of V which are pairwise non-isomorphic with multiplicities $m_1, \dots, m_n$. By Proposition 12.15, we have

$$B \simeq \mathrm{Mat}_{m_1}(D_1) \times \cdots \times \mathrm{Mat}_{m_n}(D_n),$$

where $D_i := \mathrm{End}_A(V_i)$. If we define $W_i := \mathrm{Hom}_A(V_i, V)$ as a right B-module, then

$$V \simeq \bigoplus_{i=1}^{n} V_i \otimes_{\mathbf{k}} W_i$$

as (A, B)-bi modules. Since $\mathrm{Hom}_A(V_i, V) \simeq D_i^{m_i}$, the right B-modules W_i are irreducible by Proposition 12.10.(1). Finally, for every i, let $w_i \in W_i$ be a nonzero element and let $\phi_i \in \mathrm{Mat}_{m_i}(D_i) \subset B$ be the identity matrix. We have $w_j \cdot \phi_i = \delta_{ij} w_i$, thus all W_i are non-isomorphic as B-modules. □

27.4. Symmetric groups and general linear groups. Let W be a $\mathbf{C}$-vector space. For every $n \in \mathbf{Z}_{>0}$, let $\mathfrak{S}_n$ act on $W^{\otimes n}$ by

$$\sigma \cdot (w_1 \otimes \cdots \otimes w_n) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$$

for pure tensors. We will apply Theorem 27.4 to the $\mathfrak{S}_n$-representation $W^{\otimes n}$, and see that $\mathfrak{S}_n$ and $\mathfrak{gl}(W)$ (or $\mathrm{GL}(W)$) form a dual pair.

Proposition 27.5. *We have*

$$\mathrm{End}_{\mathfrak{S}_n}(W^{\otimes n}) = \mathrm{Im}(U(\mathfrak{gl}(W)) \to \mathrm{End}_{\mathbf{k}}(W^{\otimes n})),$$

which is the subalgebra of $\mathrm{End}_{\mathbf{k}}(W^{\otimes n})$ generated by elements of the form

$$\Delta_n(f) := (f \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}) + (\mathrm{Id} \otimes f \otimes \cdots \otimes \mathrm{Id}) + \cdots + (\mathrm{Id} \otimes \mathrm{Id} \otimes \cdots \otimes f).$$

Proof. The last statement follows from the definition of tensor products of Lie algebra representations.

Since each $\Delta_n(f) : W^{\otimes n} \to W^{\otimes n}$ is $\mathfrak{S}_n$-equivariant, we have the inclusion $\supset$. For the other inclusion, we first observe the following.

Exercise 27.6. Show that $\mathrm{End}_{\mathfrak{S}_n}(W^{\otimes n}) \subset \mathrm{End}_{\mathbf{k}}(W^{\otimes n})$ is the image of the **k**-linear map

$$\begin{aligned} S^n\mathrm{End}_{\mathbf{k}}(W) &\to \mathrm{End}_{\mathbf{k}}(W^{\otimes n}) \\ f_1 \cdots f_n &\mapsto \sum_{\sigma \in \mathfrak{S}_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}. \end{aligned} \tag{27.1}$$

Let $U := \mathrm{End}_{\mathbf{k}}(W)$ as a **k**-vector space. Since the $\mathrm{GL}(U)$-representation on S^nU is irreducible by Exercise 27.1 and the subset $\{ u^n \in S^nU \mid u \in U \}$ is $\mathrm{GL}(U)$-stable, the whole space S^nU is generated by elements of the form u^n. For every positive integer i, let $H_i \in \mathbf{Q}[X_1, \dots, X_n]$ be the polynomial

$$H_i = X_1^i + \cdots + X_n^i.$$

By the Newton identities, there exists a polynomial $P \in \mathbf{Q}[X_1, \dots, X_n]$ such that

$$P(H_1, \dots, H_n) = X_1 \cdots X_n.$$

Let $u \in \mathrm{End}_{\mathbf{k}}(W)$ and consider the ring homomorphism

$$\mathbf{Q}[X_1, \dots, X_n] \to \mathrm{End}_{\mathbf{k}}(W^{\otimes n})$$

defined by

$$X_i \mapsto \mathrm{Id} \otimes \cdots \otimes u \otimes \cdots \otimes \mathrm{Id}$$

where u is in the ith place, we have

$$P(\Delta_n(u), \Delta_n(u^2), \ldots, \Delta_n(u^n)) = u \otimes \cdots \otimes u.$$

This shows the other inclusion $\subset$. □

27.5. The Schur–Weyl duality. Let $n \in \mathbf{Z}_{>0}$ and let λ be a partition of n. Let W be a $\mathbf{k}$-vector space. We define

$$S^\lambda W := \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, W^{\otimes n})$$

as a $\mathbf{k}$-vector space. Here, V_λ is the Specht module associated to λ, and $\mathfrak{S}_n$ acts on $W^{\otimes n}$ by

$$\sigma \cdot (w_1 \otimes \cdots \otimes w_n) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$$

for pure tensors. The space $S^\lambda W$ has a natural $\mathrm{GL}(W)$-representation and $\mathfrak{gl}(W)$-representation (on the right).

Exercise 27.7. Suppose that V has dimension $d \in \mathbf{Z}_{>0}$. Let $\Gamma_{d,n}$ be the set of partitions of n such that the number of rows of the corresponding Young diagram is less than or equal to d. Show that $S^\lambda W \neq 0$ if and only if $\lambda \in \Gamma_{d,n}$.

Corollary 27.8 (Schur–Weyl duality)**.** *Let V be a finite dimensional complex vector space of dimension d and let n be a positive integer. The $\mathbf{C}$-algebra $\mathbf{C}[\mathfrak{S}_n]$ and the image of $U(\mathfrak{gl}(W))$ form a dual pair with respect to $W^{\otimes n}$. In particular,*

$$W^{\otimes n} \simeq \bigoplus_{\lambda \in \Gamma_{d,n}}^{n} V_\lambda \otimes_{\mathbf{k}} S^\lambda W$$

as $(\mathbf{C}[\mathfrak{S}_n], U(\mathfrak{gl}(W)))$-bimodules, and also as $(\mathfrak{S}_n \times \mathrm{GL}(W))$-representations.

Proof. The isomorphism as $(\mathbf{C}[\mathfrak{S}_n], U(\mathfrak{gl}(W)))$-bimodules follows from Theorem 27.4 and Proposition 27.5. Since $\mathfrak{S}_n \times \mathrm{GL}(W) \subset \mathbf{C}[\mathfrak{S}_n] \times U(\mathfrak{gl}(W))$ and the $(\mathfrak{S}_n \times \mathrm{GL}(W))$-action is the restriction of the bimodule structure, the last statement follows. □

Corollary 27.9.

1. *$S^\lambda W$ is irreducible as $\mathfrak{gl}(W)$-representation and $\mathrm{GL}(W)$-representation.*
2. *For every $\lambda, \mu \in \Gamma_{d,n}$, we have $S^\lambda W \simeq S^\mu W$ if and only if $\lambda = \mu$ (as $\mathfrak{gl}(W)$-representations and $\mathrm{GL}(W)$-representations).*

Proof. The statements for $\mathfrak{gl}(W)$ follow from Corollary 27.8, Theorem 27.4, and Theorem 21.1. As $\mathrm{GL}(W)$ is Zariski dense in $\mathfrak{gl}(W)$ (i.e. if $\mathrm{GL}(W)$ is not contained in the zero locus $Z(P)$ of any polynomial function P on $\mathfrak{gl}(W)$ such that $Z(P) \subsetneq \mathfrak{gl}(W)$), we obtain the statements for $\mathrm{GL}(W)$. □

27.6. Schur functors. Schur functors S^λ generalize the constructions of symmetric and alternative powers of vector spaces.

Exercise 27.10.

1. Show that $S^{(n)} W \simeq \mathrm{Sym}^n W$ as $\mathrm{GL}(W)$-representations and $\lambda(W)$-representations.
2. Show that $S^{(1,\ldots,1)} W \simeq \bigwedge^n W$ as $\mathrm{GL}(W)$-representations and $\lambda(W)$-representations.

Here is an explicit description of $S^\lambda W$.

Exercise 27.11. Let λ be a partition of n and let

$$\mu_1 \geqslant \cdots \geqslant \mu_\ell$$

be the length of the columns of λ. We have

$$S^\lambda W \simeq (\bigwedge^{\mu_1} W \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \bigwedge^{\mu_\ell} W)/K,$$

where K is the subspace spanned by the elements of the form $w_{I,j}$ constructed as follows. Fix a Young tableau T associated to λ. Let I be a subset of a column C_i of T. Choose another column C_j of T. Let

$$\Sigma_{I,j} \subset \mathfrak{S}_n$$

be the subset such that $\sigma(I) \subset C_j$ and $\sigma_{|I}$ preserves the vertical order, and $\sigma(x) = x$ for every $x \notin I$. The elements $w_{I,j}$ are those of the form

$$w_{I,j} := w_1 \otimes \cdots \otimes w_n - \sum_{\sigma \in \Sigma_{I,j}} w_{\sigma(1)} \otimes \cdots w_{\sigma(n)}.$$

For instance, if $\lambda = (3,2)$ and we fill $1,\ldots,5$ from top to bottom, then left to right, then

$$\begin{aligned}&(w_1 \wedge w_2 \wedge w_3) \otimes (w_4 \wedge w_5)\\ =&(w_4 \wedge w_5 \wedge w_3) \otimes (w_1 \wedge w_2) + (w_4 \wedge w_2 \wedge w_5) \otimes (w_1 \wedge w_3) + (w_1 \wedge w_4 \wedge w_5) \otimes (w_2 \wedge w_3)\end{aligned} \tag{27.2}$$

in $S^\lambda V$.

Exercise 27.12.

(1) Show that the (comultiplication) map $c : \bigwedge^3 W \to (\bigwedge^2 W) \otimes W$ defined by

$$u \wedge v \wedge w \mapsto (u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v$$

on pure tensors is well defined.

(2) Show that we have an exact sequence

$$\bigwedge^3 W \xrightarrow{c} (\bigwedge^2 W) \otimes W \to S^{(2,1)} W \to 0.$$

27.7. Final remark: algebraic irreducible representations of $\mathrm{GL}(V)$**.** Let W be a finite dimensional complex vector space. What are the finite dimensional irreducible representations of $\mathrm{GL}(W)$? Apart from the Schur constructions $S^\lambda W$ are irreducible, the one-dimensional representations defined by

$$(\det)^k : \mathrm{GL}(W) \to \mathbf{C}^\times$$

for every $k \in \mathbf{Z}$ are also irreducible. The tensor products $S^\lambda W \otimes (\det)^k$ are also irreducible, and these are all the finite dimensional *algebraic* irreducible representations of $\mathrm{GL}(W)$. Here, a $\mathrm{GL}(W)$-representation V is called algebraic if

$$\mathrm{GL}(W) \to \mathrm{GL}(V)$$

is defined by rational functions.

Theorem 27.13 (See Fulton–Harris, Section 15.5)**.** *Every finite dimensional algebraic irreducible representations of* $\mathrm{GL}(W)$ *is isomorphic to* $S^\lambda W \otimes (\det)^k$ *for some* $k \in \mathbf{Z}$, $n \in \mathbf{Z}_{>0}$ *and* $\lambda \in \Gamma_{d,n}$.

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