CORRIGENDUM TO "ON THE CHOW GROUP OF ZERO-CYCLES OF A GENERALIZED KUMMER VARIETY"

by

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1 Error

Let A be an abelian surface and X the generalized Kummer variety of dimension $2n$ defined by A. The following theorem is proven in [**2**]. The reader is referred to *loc. cit.* for the definition of the induced Beauville decomposition.

***Theorem 1.1* ([2, Theorem 1.4]).** — *For any odd integer s, the s-th factor* $\mathrm{CH}_0(X)_s$ *of the induced Beauville decomposition vanishes.*

We gave two proofs of Theorem 1.1 in [**2**, Section 4] and [**2**, Section 5] (*cf.* [**2**, Remark 5.2]) respectively. However as Charles Vial pointed out to the author, the one presented in [**2**, Section 4] contains an error. More precisely, the author made the following incorrect claim in the proof of [**2**, Lemma 4.4]: Let A be a principally polarized abelian surface and let $\mu : A^n \to A$ and $p_j : A^n \to A$ denote the sum map and the j-th projection respectively. If $z \in \mathrm{CH}_0(A)_1$ and $L := \mathscr{F}(z) \in \mathrm{Pic}^0(A)$ where $\mathscr{F}$ denotes the Fourier-Mukai transform, then applying $\mathscr{F}$ to both sides of $\mu^*L = \sum_{j=1}^n p_j^*L$ gives $\mu^*z = \sum_{j=1}^n p_j^*z$.

The only places in [**2**] depending on the above false claim are Lemma 4.3, 4.4 and the proof of [**2**, Theorem 1.4] presented in [**2**, Section 4]. The rest of the article remains intact, especially Theorem 1.1 still holds true thanks to its second proof mentioned in [**2**, Remark 5.2].

The proof of Theorem 1.1 presented in [**2**, Remark 5.2] uses constant cycle subvarieties in X constructed and studied in [**2**, Section 3], whereas the (false) proof presented in [**2**, Section 4] is a formal computation of zero-cycles inside the Chow groups of abelian varieties. Despite the error, a proof of Theorem 1.1 by formal computations still exists, as we will see in the next section.

2 Alternative proof by formal computations of Theorem 1.1

As in [**2**], the class of a point $z \in A$ in $\mathrm{CH}_0(A)$ will be denoted by $\{z\}$.

Lemma 2.1. — *Let A be an abelian surface and let* $z_1, \ldots, z_n \in A$ *such that* $\sum_{i=1}^n z_i = 0$.

i) The equality $\sum_{i=1}^n \{z_i\} = \sum_{i=1}^n \{-z_i\}$ *holds in* $\mathrm{CH}_0(A)$.

ii) More generally for every integer $1 \leq k \leq n$,

$$\sum_{1\leq j_1,\dots,j_k\leq n} \left\{(z_{j_1},\dots,z_{j_k})\right\} = \sum_{1\leq j_1,\dots,j_k\leq n} \left\{(-z_{j_1},\dots,-z_{j_k})\right\} \in \mathrm{CH}_0(A^k).$$

Proof. — We prove the first statement of Lemma 2.1 by induction on n. When $n = 1$ or 2, Lemma 2.1 holds trivially. When $n = 3$, expanding Bloch's formula [**1**, Theorem 0.1]

$$(\{z_1\} - \{0\}) * (\{z_2\} - \{0\}) * (\{z_3\} - \{0\}) = 0$$

and taking $z_1 + z_2 + z_3 = 0$ into account yield

$$\{z_1\} + \{z_2\} + \{z_3\} = \{-z_1\} + \{-z_2\} + \{-z_3\}.$$

Now assume that Lemma 2.1 holds for $n \geq 3$. Let $z_1, \dots, z_{n+1} \in A$ such that $\sum_{i=1}^{n+1} z_i = 0$. On the one hand

$$\sum_{i=1}^{n-1}\{z_i\} + \{z_n + z_{n+1}\} = \sum_{i=1}^{n-1}\{-z_i\} + \{-z_n - z_{n+1}\} \tag{2.1}$$

by the induction hypothesis. On the other hand,

$$\left\{\sum_{i=1}^{n-1} z_i\right\} + \{z_n\} + \{z_{n+1}\} = \left\{-\sum_{i=1}^{n-1} z_i\right\} + \{-z_n\} + \{-z_{n+1}\} \tag{2.2}$$

by Bloch's formula. Adding equalities (2.1) and (2.2) then using the assumption $\sum_{i=1}^{n-1} z_i = -z_n - z_{n+1}$ give

$$\sum_{i=1}^{n+1}\{z_i\} = \sum_{i=1}^{n+1}\{-z_i\}.$$

The second statement of Lemma 2.1 follows immediately from the first one. Indeed, for any $c_1, \dots, c_k \in A$ and any integer $1 \leq l \leq k$, by pushing forward the equality $\sum_{i=1}^{n}\{z_i\} = \sum_{i=1}^{n}\{-z_i\}$ under the inclusion $c_1 \times \cdots \times c_{l-1} \times A \times c_{l+1} \times \cdots \times c_k \hookrightarrow A^k$ we have

$$\sum_{i=1}^{n} \{(c_1, \dots, c_{l-1}, z_i, c_{l+1}, \dots, c_k)\} = \sum_{i=1}^{n} \{(c_1, \dots, c_{l-1}, -z_i, c_{l+1}, \dots, c_k)\} \in \mathrm{CH}_0(A^k).$$

Therefore

$$\sum_{1\leq j_1,\dots,j_k\leq n} \left\{(z_{j_1},\dots,z_{j_k})\right\} = \sum_{1\leq j_1,\dots,j_k\leq n} \left\{(z_{j_1},\dots,z_{j_{k-1}},-z_{j_k})\right\} = \cdots = \sum_{1\leq j_1,\dots,j_k\leq n} \left\{(-z_{j_1},\dots,-z_{j_k})\right\}.$$

□

Lemma 2.2. — *Let A be an abelian surface and let $z_1, \dots, z_n \in A$ such that $\sum_{i=1}^{n} z_i = 0$. Then for every integer* $1 \leq k \leq n$,

$$\sum_{(j_1,\dots,j_k)\in\mathscr{P}} \left\{(z_{j_1},\dots,z_{j_k})\right\} = \sum_{(j_1,\dots,j_k)\in\mathscr{P}} \left\{(-z_{j_1},\dots,-z_{j_k})\right\} \in \mathrm{CH}_0(A^k)$$

where $\mathscr{P}$ is the set of k-tuples $(j_1, \dots, j_k)$ of integers in $[1, n]$ *with* mutually different entries.

Proof. — We prove Lemma 2.2 by induction on k. The base case $k = 1$ is the content of the first statement of Lemma 2.1.

Fix an integer $2 \leq k \leq n$ and assume that Lemma 2.2 holds for every integer between 1 and $k - 1$. For any integer p, set $[\![1, p]\!] := [1, p] \cap \mathbf{Z}$. We have the following obvious equalities

$$
\begin{aligned}
\sum_{1\le j_1,\dots,j_{k+1}\le n}\left\{(z_{j_1},\dots,z_{j_{k+1}})\right\} &= \sum_{p=1}^{k}\left(\sum_{I_1\sqcup\cdots\sqcup I_p=[\![1,k]\!]}\left(\sum_{(i_1,\dots,i_k)\in\mathscr{P}(I_1,\dots,I_p)}\{(z_{i_1},\dots,z_{i_k})\}\right)\right)\\
&= \sum_{(j_1,\dots,j_k)\in\mathscr{P}}\left\{(z_{j_1},\dots,z_{j_k})\right\} + \sum_{p=1}^{k-1}\left(\sum_{I_1\sqcup\cdots\sqcup I_p=[\![1,k]\!]}\left(\sum_{(i_1,\dots,i_k)\in\mathscr{P}(I_1,\dots,I_p)}\{(z_{i_1},\dots,z_{i_k})\}\right)\right)
\end{aligned}
\tag{2.3}
$$

where $\mathscr{P}(I_1,\dots,I_p)$ is the set of k-tuples $(i_1,\dots,i_k)\in[\![1,n]\!]^k$ such that $i_q=i_r$ if and only if $q,r\in I_s$ for some $s\in[\![1,p]\!]$. Here the sum $\sum_{I_1\sqcup\cdots\sqcup I_p=[\![1,k]\!]}$ is over all partition of $[\![1,k]\!]$ of length p such that $I_j\neq\emptyset$ for all $1\le j\le p$.

For any partition $I_1\sqcup\cdots\sqcup I_p=[\![1,k]\!]$ with $p\le k-1$, pushing forward the induction hypothesis on A^p *via* the embedding $A^p\hookrightarrow A^k$ defined by the partition $I_1\sqcup\cdots\sqcup I_p=[\![1,k]\!]$ yields

$$
\sum_{(i_1,\dots,i_k)\in\mathscr{P}(I_1,\dots,I_p)}\{(z_{i_1},\dots,z_{i_k})\} = \sum_{(i_1,\dots,i_k)\in\mathscr{P}(I_1,\dots,I_p)}\{(-z_{i_1},\dots,-z_{i_k})\}\,. \tag{2.4}
$$

As

$$
\sum_{1\le j_1,\dots,j_{k+1}\le n}\left\{(z_{j_1},\dots,z_{j_{k+1}})\right\} = \sum_{1\le j_1,\dots,j_{k+1}\le n}\left\{(-z_{j_1},\dots,-z_{j_{k+1}})\right\}
$$

by Lemma 2.1 *ii*), we conclude by (2.3) and (2.4) that

$$
\sum_{(j_1,\dots,j_k)\in\mathscr{P}}\left\{(z_{j_1},\dots,z_{j_k})\right\} = \sum_{(j_1,\dots,j_k)\in\mathscr{P}}\left\{(-z_{j_1},\dots,-z_{j_k})\right\}.
$$

□

Lemma 2.3. — *Let A be an abelian variety and $j: B\hookrightarrow A$ an inclusion of a sub-abelian variety B. The pushforward map*

$$
j_*:\mathrm{CH}(B)_{\mathbf{Q}}\to\mathrm{CH}(A)_{\mathbf{Q}}
$$

is injective.

Proof. — By Poincaré's reducibility theorem, the inclusion j factorizes through $B\xrightarrow{j'}B\times B'\xrightarrow{\pi}A$ where B' is an abelian variety, $j':B\hookrightarrow B\times B'$ is the standard inclusion with image $B\times\{0\}$, and π is an isogeny. Since $\pi_*:\mathrm{CH}(A)_{\mathbf{Q}}\xrightarrow{\sim}\mathrm{CH}(B\times B')_{\mathbf{Q}}$, it suffices to show that $j'_*:\mathrm{CH}(B)_{\mathbf{Q}}\to\mathrm{CH}(B\times B')_{\mathbf{Q}}$ is injective. As the composition

$$
\mathrm{CH}(B)_{\mathbf{Q}}\xrightarrow{j'_*}\mathrm{CH}(B\times B')_{\mathbf{Q}}\xrightarrow{p_*}\mathrm{CH}(B)_{\mathbf{Q}}
$$

is the identity map where $p:B\times B'\to B$ is the first projection, the map $j'_*:\mathrm{CH}(B)_{\mathbf{Q}}\to\mathrm{CH}(B\times B')_{\mathbf{Q}}$ is injective. □

Recall in [**2**] that for an abelian surface A, we defined $A^n_0:=\ker(\mathrm{alb}:A^n\to A)$, which is preserved by the action of the symmetry group $\mathfrak{S}_n$ on A^n permuting the factors.

Corollary 2.4. — *The involution $\iota:A\to A$ acts trivially on $\mathrm{CH}_0(A^n_0)^{\mathfrak{S}_n}$.*

Proof. — The $\mathfrak{S}_n$-invariant subgroup $\mathrm{CH}_0(A^n_0)^{\mathfrak{S}_n}$ is generated by elements of the form

$$
z:=\sum_{\sigma\in\mathfrak{S}_n}\{\sigma(z_1,\dots,z_n)\}
$$

with $\sum_{i=1}^{n} z_i = 0$. Applying Lemma 2.2 to z with $k = n$ yields $\iota^* z = z$ in $\mathrm{CH}_0(A^n)$. Hence $\iota^* z = z$ in $\mathrm{CH}_0(A_0^n)$ by Lemma 2.3. □

Proof of Theorem 1.1. — Since the involution acts trivially on $\mathrm{CH}_0(A_0^{n+1})^{\mathfrak{S}_{n+1}}$, the factor $\mathrm{CH}_0(X)_s \simeq \mathrm{CH}_0(A_0^{n+1})_s^{\mathfrak{S}_{n+1}}$ vanishes whenever s is odd. □

3 A counter-example to [2, Lemma 4.3 and 4.4]

If the statements of [2, Lemma 4.3] and [2, Lemma 4.4] still hold true despite their incorrect proof, then the original proof of [2, Theorem 1.4] in [2, Section 4] remains valid. However we will see in the following that it is not the case for both [2, Lemma 4.3] and [2, Lemma 4.4] already when $n = 2$.

If [2, Lemma 4.4] is true for $n = 2$, then

$$\mu^* z = p_1^* z + p_2^* z \tag{3.1}$$

where $z \in \mathrm{CH}_0(A)_1$. For any $a \in A$, restricting (3.1) to $p_1^{-1}(a)$ yields $z = t_a^* z$, which is equivalent to

$$z * (\{o\} - \{-a\}) = 0 \in \mathrm{CH}_0(A).$$

As $z * (\{o\} - \{-a\})$ generates $I^{*2} \subset \mathrm{CH}_0(A)$ when varying $z \in \mathrm{CH}_0(A)_1$ and $a \in A$ where

$$I := \ker(\deg : \mathrm{CH}_0(A) \to \mathbf{Z}),$$

we deduce that $I^{*2} = 0$, which is in contradiction with [1, Theorem 3.1].

Similarly if [2, Lemma 4.3] is true for $n = 2$, then restricting the equality in Lemma 4.3 to $p_1^{-1}(a)$ with $z_1 \in \mathrm{CH}_0(A)_1$ and $z_2 := \{a\}$ yields $z * (\{o\} - \{-a\}) = 0$ as well, which is not true in general for the same reason.

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References

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