Thermal Theory for Non-Boussinesq Gravity Currents Propagating on Inclined Boundaries

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Abstract: In this study the author derived the thermal theory for non-Boussinesq gravity currents produced from an instantaneous buoyancy source propagating on an inclined boundary. For Boussinesq gravity currents on a slope, it is known that the gravity current front location history follows an asymptotic relationship, \( x_f^{3/2} \sim t \), where \( x_f \) is the front location, and \( t \) is the time, when the gravity current is sufficiently far into the deceleration phase. For non-Boussinesq gravity currents, the distance for the acceleration phase is extended attributable to the non-Boussinesq effects. When the gravity current is sufficiently far into the deceleration phase, this paper shows that the non-Boussinesq gravity currents tend to approach similar asymptote in the deceleration phase as Boussinesq gravity currents, but the approach is less rapid in non-Boussinesq gravity currents. DOI: 10.1061/(ASCE)HY.1943-7900.0000949, © 2014 American Society of Civil Engineers.

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Introduction

Gravity currents, also known as density currents, are buoyancy-driven flows primarily in the horizontal direction (Huppert 2006). Gravity currents manifest in numerous situations, either as a current of heavy fluid running beneath light ambient fluid, or as a current of light fluid above heavy fluid. There are a number of factors which cause variations in the density of fluid, including temperature differentials, dissolved materials, and suspended sediments. Lock-exchange flows, in which gravity currents are produced from an instantaneous, finite buoyancy source and propagate on a horizontal boundary, have drawn much attention in the literature, e.g., Shin et al. (2004), Marino et al. (2005), Cantero et al. (2007), and La Rocca et al. (2008). Gravity currents on a slope have been considered less, but are also commonly encountered, such as powder-snow avalanches (Hopfinger 1983) and spillage of hazardous materials (Fannelop 1994). For more details about the diversity of gravity currents in geophysical environments and engineering applications, the readers are referred to Allen (1985) and Simpson (1997).

Gravity currents down an inclined boundary can be produced with a continuous inflow (Britten and Linden 1980; Garcia 1993, 1994) or with an instantaneous release of a finite volume of heavy fluid (Beghin et al. 1981; Dai et al. 2012; Dai 2013). For Boussinesq gravity currents produced from an instantaneous buoyancy source propagating on an inclined boundary, Beghin et al. (1981) reported that the gravity currents go through an acceleration phase followed by a deceleration phase. Thermal theory for Boussinesq gravity currents, which was developed therein following the famous Morton et al. (1956), has been implemented in related gravity current problems, e.g., Dade et al. (1994) for sediment deposition in gravity currents and Rastello and Hopfinger (2004) for particle-entraining snow avalanches. However, the non-Boussinesq effects were not included in these studies.

More recently, Maxworthy (2010) conducted a series of experiments aiming at Boussinesq gravity currents in the deceleration phase and proposed that the front history follows a power-relationship. For non-Boussinesq gravity currents, Lowe et al. (2005) and Birman et al. (2005) studied lock-exchange problems on a horizontal boundary with experiments and numerical simulations, respectively. But the geometric configuration makes their problem qualitatively different from that set forth in Beghin et al. (1981) in which thermal theory applies.

Non-Boussinesq gravity currents, in which the density contrasts are relatively larger, are important in quite a few situations. Dense gases are often stored as liquids at low temperatures, and these gases on release could have densities more than twice that of the ambient air. Powder-snow avalanches contain suspended snow grains, and the extra density of the suspended particles is large relative to that of air, even at low concentrations. Volcanic eruptions produce suspended ash and rocks which often take the form of gravity currents. Basically, the Boussinesq approximation does not hold for gravity currents driven by particles in the air when the particle concentration exceeds a few percent (Ancy 2004). Particulate gravity currents can also have considerable destructive potential, e.g., pyroclastic flows and snow avalanches, and understanding the flow dynamics is important for risk assessment in nearby regions (Gladstone et al. 2004). Unlike flows driven by particles in the air, turbidity currents with high sediment concentrations in the water owing to extreme precipitation events may still be considered in the Boussinesq regime (Jacobson and Testik 2013). Previous studies of gravity currents down an inclined boundary have neglected the non-Boussinesq effects. In this study, the author derived the thermal theory for non-Boussinesq gravity currents produced from an instantaneous buoyancy source propagating on an inclined boundary. The influence of non-Boussinesq effects is discussed for the first time with the help of analytical solutions.

Thermal Theory

The configuration of the problem is sketched in Fig. 1. Here the nomenclature primarily follows Beghin et al. (1981) for the reader’s convenience. The density of ambient fluid is taken as \( \rho_0 \), and the
density of heavy fluid in the lock region is \( \rho_1 \), where 
\( \epsilon = (\rho_1 - \rho_0)/\rho_0 \). The cross-sectional area of the lock, which 
represents the amount of heavy fluid in the lock, is \( A_0 = h_0l_0 \), where 
\( h_0 \) and \( l_0 \) are the height and length of the lock, respectively. 
After an instantaneous removal of the lock gate, the gravity current front 
develops, and the gravity current head approximately takes a 
semieliptical shape with a height-to-length aspect ratio, 
\( H/L \approx 1/2 \), where \( H \) is the height of the gravity current head, and 
\( L \) is the distance from the virtual origin to the mass-center of 
the gravity current head. The virtual origin is located \( x_0 \) beyond the 
initial mass-center location of the heavy fluid and can be identified by 
extrapolating the head height in the upslope direction. Please 
note that \( x_0 \) is another shape factor by 
Beghin et al. (1981), if the buoyancy could be released 
with a shape similar to the developed state and with appropriate 
vorticity, then \( x_0 \) would just be the distance from the virtual origin 
to the position of release. However, this is not possible in practice, 
and the initial state is situated somewhat downstream of the 
release gate.

The entrainment coefficient, \( \alpha \), is related to the angle of growth, 
\( \theta_0 \), through \( \alpha_0 = \theta_0/\theta_0 \approx \alpha_0 = \theta_0/\theta_0 \). Upon substitution 
of Eq. (4) into Eq. (1) and using \( U = dx/dt \), the momentum equation 
becomes

\[
U \frac{d}{dx} \left( x^2 U \right) + \epsilon \left( \frac{2}{\pi} \frac{k}{1 + 2k/\alpha_0} \right) \frac{dU^2}{dx} = C
\]

where

\[
C = \frac{4}{\pi} k \frac{1}{1 + 2k/\alpha_0} x B_0 \sin \theta \quad \text{with} \quad B_0 = \epsilon \theta_0
\]

is the driving force term. It should be pointed out that Eqs. (1) and 
(5) are consistent with Beghin et al. (1981) and related studies for the 
Boussinesq case except the second term on the left hand side of 
Eq. (5), which represents the influence of density difference on the 
inertia term and has been neglected in previous works.

**Boussinesq Case**

Eq. (5) may be rearranged in the following form:

\[
2xU^2 + \left[ \frac{1}{2} \frac{k}{1 + 2k/\alpha_0} \right] \frac{dU^2}{dx} = C
\]

after expanding the first term on the left hand side of Eq. (5). When the density difference is sufficiently small such that 
\( \epsilon \ll \theta_0/\alpha_0 \), the second term on the left hand side of 
Eq. (7) can be approximated as \( 0.5x^2 dU^2/dx \), although the right 
hand side of Eq. (7) remains unchanged. In other words, the influence 
of the density difference on the inertia term is neglected, although its influence on the driving force term is retained, namely, 
the Boussinesq case. In this case, the author derived an analytical 
solution of the form

\[
U^2 = U_0^2 \left( \frac{x_0}{x} \right)^4 + \frac{2}{3} C \left[ 1 - \left( \frac{x_0}{x} \right)^4 \right]
\]

which is identical to the solution given by Beghin et al. (1981) 
[cf. Eq. (5) therein].

It is worthy to note the condition under which the Boussinesq approximation can be applied, i.e., \( \epsilon \ll \theta_0/\alpha_0 \), and \( \pi(1 + 2k/\alpha_0) \approx \pi(1) \). The aforementioned condition for the Boussinesq approximation is 
equivalent to \( \epsilon \ll \theta_0/\alpha_0 \). However, a quantitative relationship 
between \( \epsilon \) and \( x_0 \) is not likely to be built simply based on theoretical 
arguments without laboratory experiments.

When the buoyancy is released with a quiescent initial condition, 
i.e., \( U_0 = 0 \), the gravity current reaches its maximum velocity

\[
H = \frac{1}{2} S_2 \left( 1 + \frac{2}{k} \right) \alpha_0 U
\]

and

\[
L = \frac{1}{2} S_1 \left( 1 + \frac{2}{k} \right) \alpha_0 U
\]
at \( x/x_0 = 4^{1/3} \). In other words, for \( 1 \leq x/x_0 \leq 4^{1/3} \), the gravity current is in the acceleration phase, and for \( x/x_0 \geq 4^{1/3} \), the gravity current is in the deceleration phase.

For sufficiently large values of \( x \) such that \( x/x_0 \gg 1 \) and with \( U_0 = 0 \), the solution [Eq. (7)] approaches the following asymptote:

\[
U = \frac{\sqrt{2}}{\sqrt[3]{3}} \sqrt{C} x^{-1/2} \tag{9}
\]

which can be integrated using \( U = dx/dt \) as

\[
x^{3/2} = \frac{\sqrt{3}}{\sqrt{2}} \sqrt{C(t + t_0)} \tag{10}
\]

where \( t_0 \) = integration constant.

Because the front location of the gravity current is a more readily measurable quantity, it is sometimes convenient and desired to write the solution in terms of the front location, \( x_f \). Using the geometric relation \( x_f = x + L/2 \), the front location, \( x_f \), is related to the mass-center location \( x \) through

\[
x_f = \left( 1 + \frac{\alpha_0}{2k} \right) x \tag{11}
\]

If the front location is used rather than the location of the mass-center, Eq. (9) becomes

\[
x_f^{3/2} = K_B^{3/2} B_0^{1/2} (t + t_0) \tag{12}
\]

where \( K_B \) is expressed in terms of \( k, \alpha_0, \) and \( \theta \) as

\[
K_B = 6 \left( \frac{1}{\pi} \right)^{1/3} \left( 1 + \frac{\alpha_0}{2k} \right) \left[ \frac{k \sin \theta}{(1 + 2k)\alpha_0} \right]^{1/3} \tag{13}
\]

Here, it is noted that the relationship between the front location and time [Eq. (12)] is equivalent to the asymptotic form of the velocity in the deceleration phase [Eq. (8)].

### Non-Boussinesq Case

Without the Boussinesq approximations, the influence of density variation on the inertia term, \( O(\epsilon) \), is retained, and the following closed-form solution is derived:

\[
U^2 = U_0^2 \left\{ \frac{1}{2} x^2 + \epsilon A_Q \right\}^2 \left\{ \frac{1}{2} x^2 + \epsilon A_Q \right\}^{-2} + C \left\{ \frac{1}{6} (x^3 - x_0^3) - \epsilon A_Q (x - x_0) \right\} \left\{ \frac{1}{2} x^2 + \epsilon A_Q \right\}^{-2} \tag{14}
\]

where \( U_0 \) = initial mass-center velocity and \( A_Q = 2^{\chi} k A_0 / (1 + 2k) \pi \alpha_0^2 \).

When the buoyancy is released with a quiescent initial condition, i.e., \( U_0 = 0 \)

\[
U^{3/2} = \left\{ \frac{1}{6} x^3 \right\} \left\{ 1 - \left( \frac{x_0}{x} \right)^3 \right\} + \delta \left( \frac{x}{x_0} \right) \left\{ 1 - \left( \frac{x_0}{x} \right) \right\} \left\{ \frac{1}{2} x^2 + \epsilon A_Q \right\}^{-2} \times \left( \frac{x_0}{x} \right) \left\{ \frac{1}{2} + \delta \left( \frac{x_0}{x} \right) \right\}^{-1} \tag{15}
\]

and the location at which a gravity current reaches its maximum velocity, \( (x_m/x_0) \), is found through \( dU/dx = 0, i.e., \)

\[
\left( \frac{x_m}{x_0} \right)^4 + 12 \delta \left( \frac{x_m}{x_0} \right)^2 - (4 + 24 \delta) \left( \frac{x_m}{x_0} \right) - 12 \delta^2 = 0 \tag{16}
\]

where \( \delta = \epsilon A_Q / \lambda_Q^2 \), named as effective density difference by the author, is a dimensionless parameter characterizing the non-Boussinesq effects. The Boussinesq case can be interpreted as \( \delta \to 0 \).

Here, the author gives an estimate for the typical values of \( \delta \) in different circumstances. Because \( x_0 / \alpha_0 \approx O(h_0) \) and \( A_Q / x^2_0 \approx O(1) \), it is estimated that \( \delta \approx O(\epsilon) \). Depending on how the gravity currents are produced, the parameter \( \delta \) assumes a wide range of values in different situations. For example, dense gases are typically stored as liquids with density in the range of 410–500 kg/m³ at low temperatures, and on release into the atmosphere, these gases could have densities more than twice that of the ambient air, i.e., \( \delta \geq 1 \). Powder-snow avalanches contain snow grains of which the particle concentration varies between 0.1 and 7%, and the effective density difference varies approximately in the range of \( 0.5 \leq \delta \leq 30 \). Volcanic eruptions often result in gravity currents with suspended ash and rocks, of which the density of particles ranges approximately from 700 to 3,200 kg/m³, and the resulting effective density difference is significantly higher, \( \delta \gg 5 \). Sediment-laden rivers often form gravity currents in a lake or reservoir, and the sediment concentrations can result in \( \delta \approx 0.4 \). In essence, the effective density difference gives a measure of the non-Boussinesq effects.

Fig. 2 shows \( (x_m/x_0) \) versus \( \delta \) and in the limit as \( \delta \to 0 \), \( (x_m/x_0) \to 4^{1/3} \). It is shown from the thermal theory that the acceleration phase distance is extended for non-Boussinesq gravity currents released with zero momentum.

For sufficiently large values of \( x \) such that \( x/x_0 \gg 1 \), Eq. (14) becomes

\[
U^{3/2} \approx \frac{U_0 \sqrt{\chi}}{\sqrt{C}} \frac{1}{6} \left( \frac{x}{x_0} \right)^3 + \delta \left( \frac{x}{x_0} \right) \frac{1}{2} \left( \frac{x_0}{x} \right) + \left( \frac{x_0}{x} \right) \left\{ \frac{1}{2} \right\}^{-1} \tag{17}
\]

and when \( x^2 / x_0^2 \gg 6 \delta \), the solution approaches

\[
U \approx \frac{U_0 \sqrt{\chi}}{\sqrt{C}} \frac{1}{6} \left( \frac{x_0}{x} \right)^{1/2} \tag{18}
\]

which is the same asymptote as Eq. (8).

It is apparent that for non-Boussinesq gravity currents, the asymptote [Eq. (8)] and consequently Eq. (9) are approached for sufficiently large values of \( x \) such that \( x/x_0 \gg 1 \) and \( x^2 / x_0^2 \gg 6 \delta \). To show the non-Boussinesq effects, Fig. 3 shows \( \tilde{U} = U^{3/2} / C^{1/2} \) versus \( x/x_0 \), and Fig. 4 shows \( (x/x_0)^{3/2} \) versus \( t = C t^{1/2} / x_0^{3/2} \) for \( \delta = 0, 0.5, 1, 2, 3 \). There is no question that \( \tilde{U} \sim (x/x_0)^{-1/2} \), and the relationship in Eq. (12) are approached for sufficiently large values of \( x \). However, it is interesting to

**Fig. 2.** \( (x_m/x_0)^{3/2} \) versus \( \delta \); the gravity current reaches its maximum velocity at \( (x_m/x_0) \); in the limit as \( \delta \to 0 \), \( (x_m/x_0) \to 4^{1/3} \)
observe that the non-Boussinesq gravity currents tend to approach the asymptotes less rapidly than the Boussinesq gravity currents ($\delta \to 0$). As clearly shown in Fig. 4, the relationship [Eq. (12)] is approached in the deceleration phase more rapidly when $\delta \to 0$ than in other cases when $\delta = 1, 2, 3$.

**Summary**

In this study the author derived the thermal theory for non-Boussinesq gravity currents produced from an instantaneous buoyancy source propagating on an inclined boundary. For Boussinesq gravity currents propagating on a slope, the front location follows a relationship [Eq. (12)] when the gravity current is sufficiently far into the deceleration phase. For non-Boussinesq gravity currents produced from an instantaneous buoyancy source, the acceleration phase distance is extended. When the gravity current is sufficiently far into the deceleration phase, the author showed that the non-Boussinesq gravity currents approach similar asymptotes in the deceleration phase as Boussinesq gravity currents, but the approach is less rapid.

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**References**


