從信號與系統到控制

單元：Z轉換-4

Z轉換範例 - 三角函數

授課老師： 連　豐　力

單元學習目標與大綱

- 根據 Z轉換 的公式與關係式
- 計算 三角函數 的 Z轉換
- 介紹 Z轉換 後的 收斂區間 特性

傅立葉轉換 與 Z轉換

$$x[n] \xleftrightarrow{FT} \boxed{X(e^{jw})} \qquad x[n] \xleftrightarrow{ZT} \boxed{X(z)}$$

$$\boxed{r}\,\boxed{e^{jw}} = \boxed{z}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]\,\boxed{e^{-jwn}} \qquad X(z) = \sum_{n=-\infty}^{+\infty} x[n]\,\boxed{z^{-n}}$$

$$X(e^{jw}) = \mathcal{F}\{\, x(t) \,\} \qquad X(z) = \mathcal{Z}\{\, x[n] \,\}$$

$$x[n] = \mathcal{F}^{-1}\{\, X(e^{jw}) \,\} \qquad x[n] = \mathcal{Z}^{-1}\{\, X(z) \,\}$$

三角函數 的 Z轉換

$$\sin(s) = \frac{1}{2j}\left(e^{js} - e^{-js}\right)$$

$$x[n] = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}\left[e^{j\frac{\pi}{4}\boxed{n}} - e^{-j\frac{\pi}{4}\boxed{n}}\right]$$

$$= \frac{1}{2j}\left[\left(e^{j\frac{\pi}{4}}\right)^n - \left(e^{-j\frac{\pi}{4}}\right)^n\right]$$

$$x[n] = \left(\frac{1}{3}\right)^n \frac{1}{2j}\left[\left(e^{j\frac{\pi}{4}}\right)^n - \left(e^{-j\frac{\pi}{4}}\right)^n\right] u[n]$$

三角函數 的 Z轉換

$$x[n] = \boxed{\left(\frac{1}{3}\right)^n} \frac{1}{2j}\left[\boxed{\left(e^{j\frac{\pi}{4}}\right)^n} - \boxed{\left(e^{-j\frac{\pi}{4}}\right)^n}\right] u[n]$$

$$= \frac{1}{2j}\left[\boxed{\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n} - \boxed{\left(\frac{1}{3}e^{-j\frac{\pi}{4}}\right)^n}\right] u[n]$$

$$x[n] = a^n u[n] \overset{ZT}{\longleftrightarrow} X(z) = \frac{z}{z-a} \qquad |z| > |a| \quad 1$$

$$\left(\boxed{\frac{1}{3}e^{j\frac{\pi}{4}}}\right)^n u[n] \overset{ZT}{\longleftrightarrow} \frac{z}{z - \boxed{\frac{1}{3}e^{j\frac{\pi}{4}}}} \qquad |z| > \left|\frac{1}{3}e^{j\frac{\pi}{4}}\right|$$

$$\frac{1}{3}$$

三角函數 的 Z轉換

$$x[n] = \frac{1}{2j}\left[\left(\boxed{\frac{1}{3}e^{j\frac{\pi}{4}}}\right)^n - \left(\boxed{\frac{1}{3}e^{-j\frac{\pi}{4}}}\right)^n\right]u[n]$$

$$X(z) = \frac{1}{2j}\left[\frac{z}{z-\boxed{\frac{1}{3}e^{j\frac{\pi}{4}}}} - \frac{z}{z-\boxed{\frac{1}{3}e^{-j\frac{\pi}{4}}}}\right]$$

$$|z| > \frac{1}{3} \qquad |z| > \frac{1}{3}$$

三角函數 的 Z轉換

$$|z| > \frac{1}{3}$$

$$X(z) = \frac{1}{2j}\left[\frac{\boxed{z}}{\boxed{z - \frac{1}{3}e^{j\frac{\pi}{4}}}} - \frac{\boxed{z}}{\boxed{z - \frac{1}{3}e^{-j\frac{\pi}{4}}}}\right]$$

$$= \frac{1}{2j}\,\frac{\boxed{z}\left(z - \boxed{\frac{1}{3}}\boxed{e^{-j\frac{\pi}{4}}}\right) - \boxed{z}\left(z - \boxed{\frac{1}{3}}\boxed{e^{j\frac{\pi}{4}}}\right)}{\left(z - \frac{1}{3}e^{j\frac{\pi}{4}}\right)\left(z - \frac{1}{3}e^{-j\frac{\pi}{4}}\right)}$$

$$= \frac{z\,\frac{1}{3}\boxed{\frac{1}{2j}\left[e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}}\right]}}{\left(z - \frac{1}{3}e^{j\frac{\pi}{4}}\right)\left(z - \frac{1}{3}e^{-j\frac{\pi}{4}}\right)}$$

$$\boxed{\sin(s) = \frac{1}{2j}\left(e^{js} - e^{-js}\right)}$$

三角函數 的 Z轉換

$$X(z) = \frac{z\,\frac{1}{3}\,\sin\left(\frac{\pi}{4}\right)}{\left(z - \frac{1}{3}e^{j\frac{\pi}{4}}\right)\left(z - \frac{1}{3}e^{-j\frac{\pi}{4}}\right)} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$= \frac{\frac{\sqrt{2}}{6}\boxed{z}}{\left(\boxed{z - \frac{1}{3}e^{j\frac{\pi}{4}}}\right)\left(\boxed{z - \frac{1}{3}e^{-j\frac{\pi}{4}}}\right)} \qquad \begin{array}{l} = 0 \\ = 0 \end{array}$$

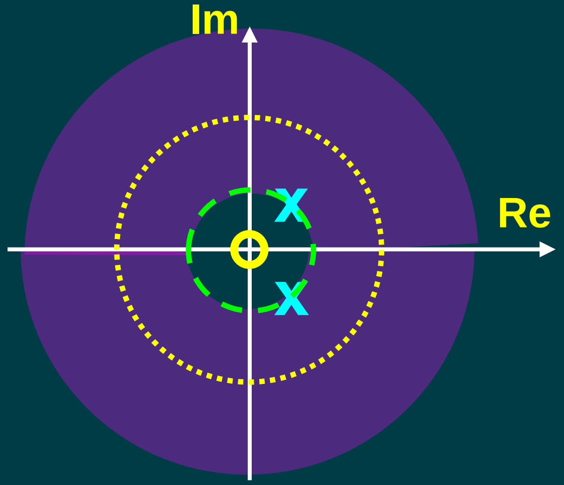
- 零點 (Zero)： $z = 0$
- 極點 (Pole)： $z = \frac{1}{3}e^{j\frac{\pi}{4}}$ $\quad z = \frac{1}{3}e^{-j\frac{\pi}{4}}$

三角函數 的 Z轉換

$$\left(\frac{1}{3}\right)^{n} \sin\left(\frac{\pi}{4} n\right) u[n] \overset{ZT}{\longleftrightarrow} \frac{\frac{\sqrt{2}}{6} z}{\left(z - \frac{1}{3} e^{j\frac{\pi}{4}}\right)\left(z - \frac{1}{3} e^{-j\frac{\pi}{4}}\right)}$$

$$\boxed{|z| > \frac{1}{3}}$$

- 零點 (Zero) : $\boxed{z = 0}$
- 極點 (Pole) : $z = \frac{1}{3} e^{j\frac{\pi}{4}}$, $z = \frac{1}{3} e^{-j\frac{\pi}{4}}$
- 收斂區間 (ROC)



參考文獻

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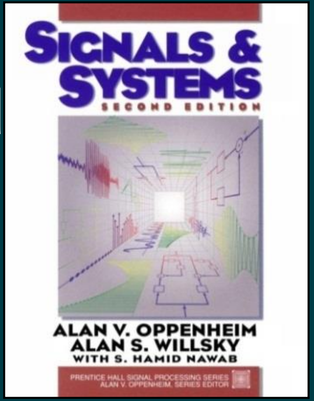

- SciLab:
  Open source software for numerical computation
  http://www.scilab.org/