從信號與系統到控制

單元：離散F級數-4

離散時間 週期方波 的 傅立葉級數

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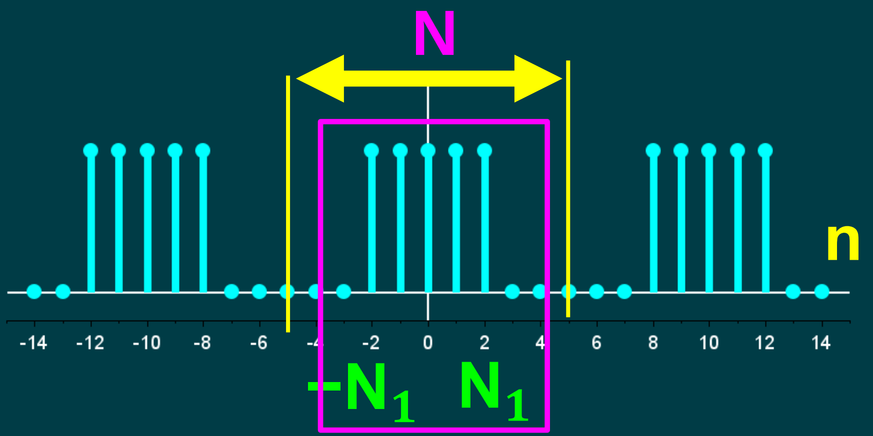
單元學習目標與大綱

- 針對 離散時間 週期性方波函數，
  找出對應的傅立葉級數

離散時間週期方波函數

$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{others in } \langle N \rangle \end{cases}$$



$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jkw_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \; e^{-jkw_0 n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jkw_0}\right)^n$$

離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

$$k = 0, \pm N, \pm 2N, \ldots$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jkw_0}\right)^n = \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-jN\frac{2\pi}{N}}\right)^n$$

$$= \frac{1}{N}\sum_{n=-N_1}^{N_1}\left(e^{-j2\pi}\right)^n = \frac{1}{N}\sum_{n=-N_1}^{N_1}(1)^n$$

$$= \frac{1}{N}(2N_1+1)$$

$$e^{js} = \cos(s) + j\sin(s)$$

離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

$k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} (e^{-jkw_0})^{n}$$

$$= \frac{1}{N} [(e^{-jkw_0})^{-N_1} + (e^{-jkw_0})^{-N_1+1} + (e^{-jkw_0})^{-N_1+2}$$

$$+ \ldots + (e^{-jkw_0})^{0} + (e^{-jkw_0})^{1} + \ldots + (e^{-jkw_0})^{N_1}]$$

離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

$k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N}\left[(e^{-jkw_0})^{-N_1} + (e^{-jkw_0})^{-N_1+1} + (e^{-jkw_0})^{-N_1+2} + \ldots + (e^{-jkw_0})^{0} + (e^{-jkw_0})^{1} + \ldots + (e^{-jkw_0})^{N_1} \right]$$

$$= \frac{1}{N}\left[\frac{(e^{-jkw_0})^{-N_1}\left[1 - (e^{-jkw_0})^{2N_1+1}\right]}{\left[1 - (e^{-jkw_0})\right]} \right]$$

離散時間週期方波函數

$$a_k = \frac{1}{N}\left[\frac{\left(e^{-jkw_0}\right)^{-N_1}\left[1-\left(e^{-jkw_0}\right)^{2N_1+1}\right]}{\left[1-\left(e^{-jkw_0}\right)\right]}\right]$$

$$= \frac{1}{N}\left[\frac{\left(e^{jkw_0N_1}\right)\left[1-\left(e^{-jkw_0(2N_1+1)}\right)\right]}{\left[1-\left(e^{-jkw_0}\right)\right]}\right]$$

$$\left(e^{s}\right)\left(e^{-s}\right) = 1$$

$$1 = \left(e^{jkw_0(2N_1+1)/2}\right)\left(e^{-jkw_0(2N_1+1)/2}\right)$$

離散時間週期方波函數

$$a_k = \frac{1}{N}\left[\frac{\left(e^{jkw_0N_1}\right)\left[1-\left(e^{-jkw_0(2N_1+1)}\right)\right]}{\left[1-\left(e^{-jkw_0}\right)\right]}\right]$$

$$\left(e^{jkw_0(2N_1+1)/2}\right)\left(e^{-jkw_0(2N_1+1)/2}\right) - \left(e^{-jkw_0(2N_1+1)/2}\right)\left(e^{-jkw_0(2N_1+1)/2}\right)$$

$$\left(e^{-jkw_0(2N_1+1)/2}\right)\left[\left(e^{jkw_0(2N_1+1)/2}\right)-\left(e^{-jkw_0(2N_1+1)/2}\right)\right]$$

$$\left(e^{-jkw_0(N_1+1/2)}\right)$$

$$\left(e^{-jkw_0N_1}\right)\left(e^{-jkw_0(1/2)}\right)$$

離散時間週期方波函數

$$a_k = \frac{1}{N}\left[\frac{(e^{-jkw_0(1/2)})\,[(e^{jkw_0(N_1+1/2)}) - (e^{-jkw_0(N_1+1/2)})]}{[1 - (e^{-jkw_0})]}\right]$$

$$(e^{s})(e^{-s}) = 1$$

$$1 = (e^{jkw_0/2})(e^{-jkw_0/2})$$

$$(e^{jkw_0/2})(e^{-jkw_0/2}) - (e^{-jkw_0/2})(e^{-jkw_0/2})$$

$$(e^{-jkw_0/2})\;[\;(e^{jkw_0/2}) - (e^{-jkw_0/2})\;]$$

離散時間週期方波函數

$$a_k = \frac{1}{N}\left[\frac{(e^{-jkw_0/2})}{(e^{-jkw_0/2})}\frac{[(e^{jkw_0(N_1+1/2)}) - (e^{-jkw_0(N_1+1/2)})]}{[(e^{jkw_0/2}) - (e^{-jkw_0/2})]}\right]$$

$$= \frac{1}{N}\left[\frac{[(e^{jkw_0(N_1+1/2)}) - (e^{-jkw_0(N_1+1/2)})]\ /\ 2j}{[(e^{jkw_0/2}) - (e^{-jkw_0/2})]\ /\ 2j}\right]$$

$$= \frac{1}{N}\left[\frac{\sin(kw_0(N_1+1/2))}{\sin(kw_0/2)}\right]$$

$$\sin(s) = \frac{1}{2j}(e^{js} - e^{-js})$$

離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N}(2N_1 + 1) \qquad k = 0, \pm N, \pm 2N, \ldots$$

$$= \frac{1}{N}\frac{\sin(kw_0(N_1 + 1/2))}{\sin(kw_0/2)} \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

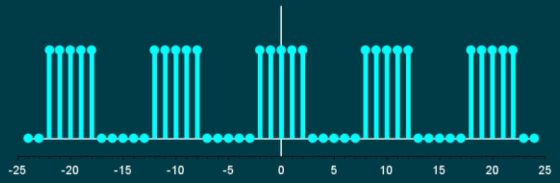
$2N_1 + 1 = 5$

$N = 10$

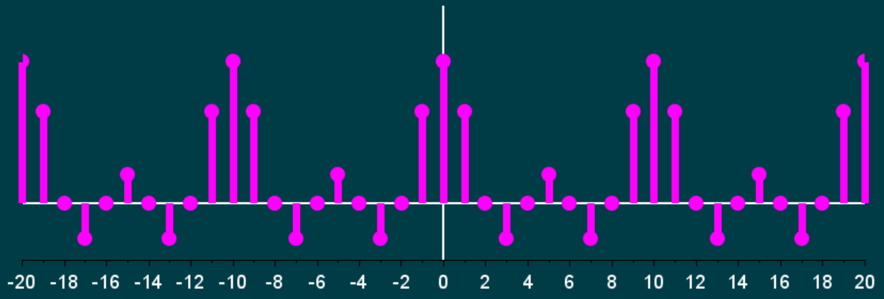
$$a_k = \frac{1}{N}(2N_1 + 1) \qquad k = 0, \pm N, \pm 2N, \ldots$$

$$= \frac{1}{N} \frac{\sin(k w_0 (N_1 + 1/2))}{\sin(k w_0 / 2)} \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

x[n]



a_k



離散時間週期方波函數

$$w_0 = \frac{2\pi}{N}$$

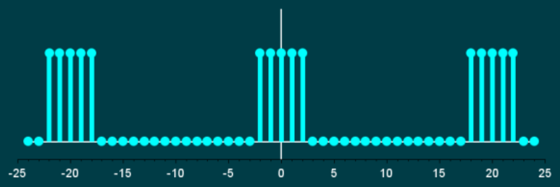
$$2N_1 + 1 = 5$$

$$N = 20$$

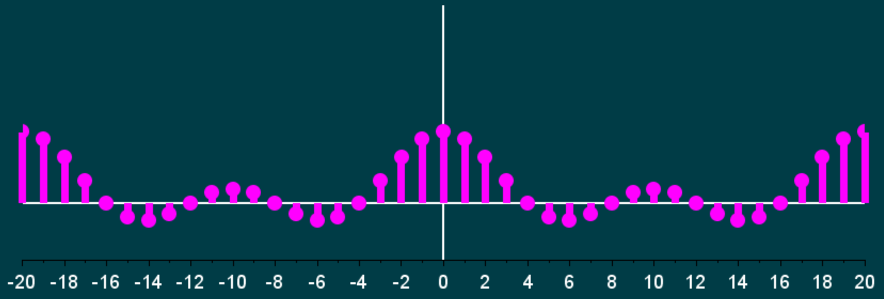
$$a_k = \frac{1}{N}(2N_1 + 1) \qquad k = 0, \pm N, \pm 2N, \ldots$$

$$= \frac{1}{N}\frac{\sin(kw_0(N_1+1/2))}{\sin(kw_0/2)} \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

x[n]

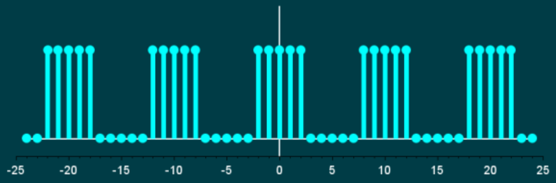


a_k

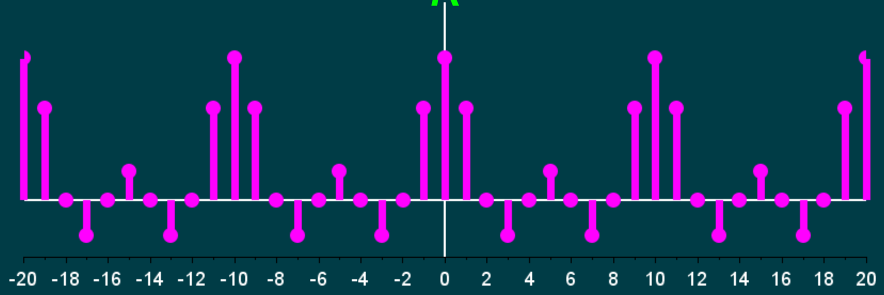


離散時間週期方波函數

$x[n]$ $2N_1+1 = 5$



a_k

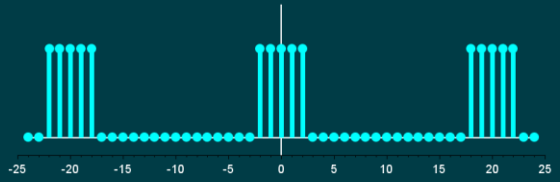


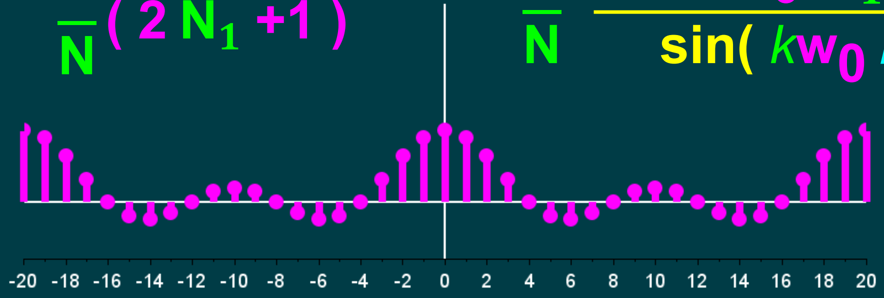
$N = 10$

$N = 20$

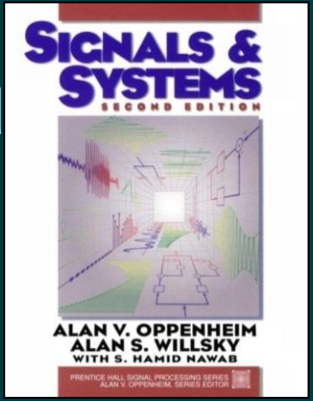
$\frac{1}{N}(2N_1+1)$

$\frac{1}{N} \frac{\sin(k\omega_0(N_1+1/2))}{\sin(k\omega_0/2)}$





參考文獻



- **Alan V. Oppenheim, Alan S. Willsky, S. Hamid**
  Signals & Systems,
  Prentice Hall, 2nd Edition, 1997

- **SciLab:**
  Open source software for numerical computation
  http://www.scilab.org/