Spring 2015

信號與系統
Signals and Systems

Chapter SS-10
The z-Transform

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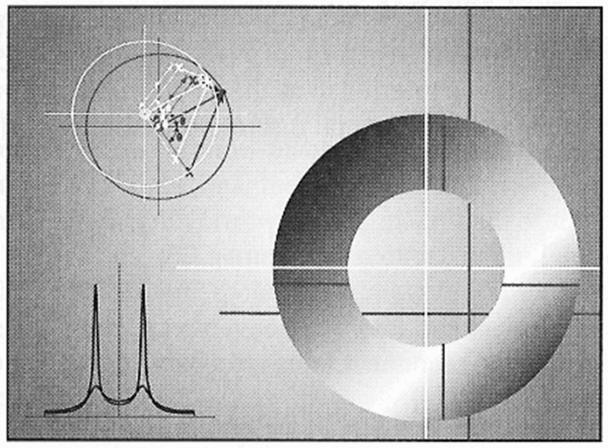
NTU-EE

Feb15 – Jun15



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS — CT, DT (Chap 3)

Aperiodic

FT — CT (Chap 4), DT (Chap 5)

Unbounded/Non-convergent

LT — CT (Chap 9)

zT — DT (Chap 10)

Time-Frequency (Chap 6)

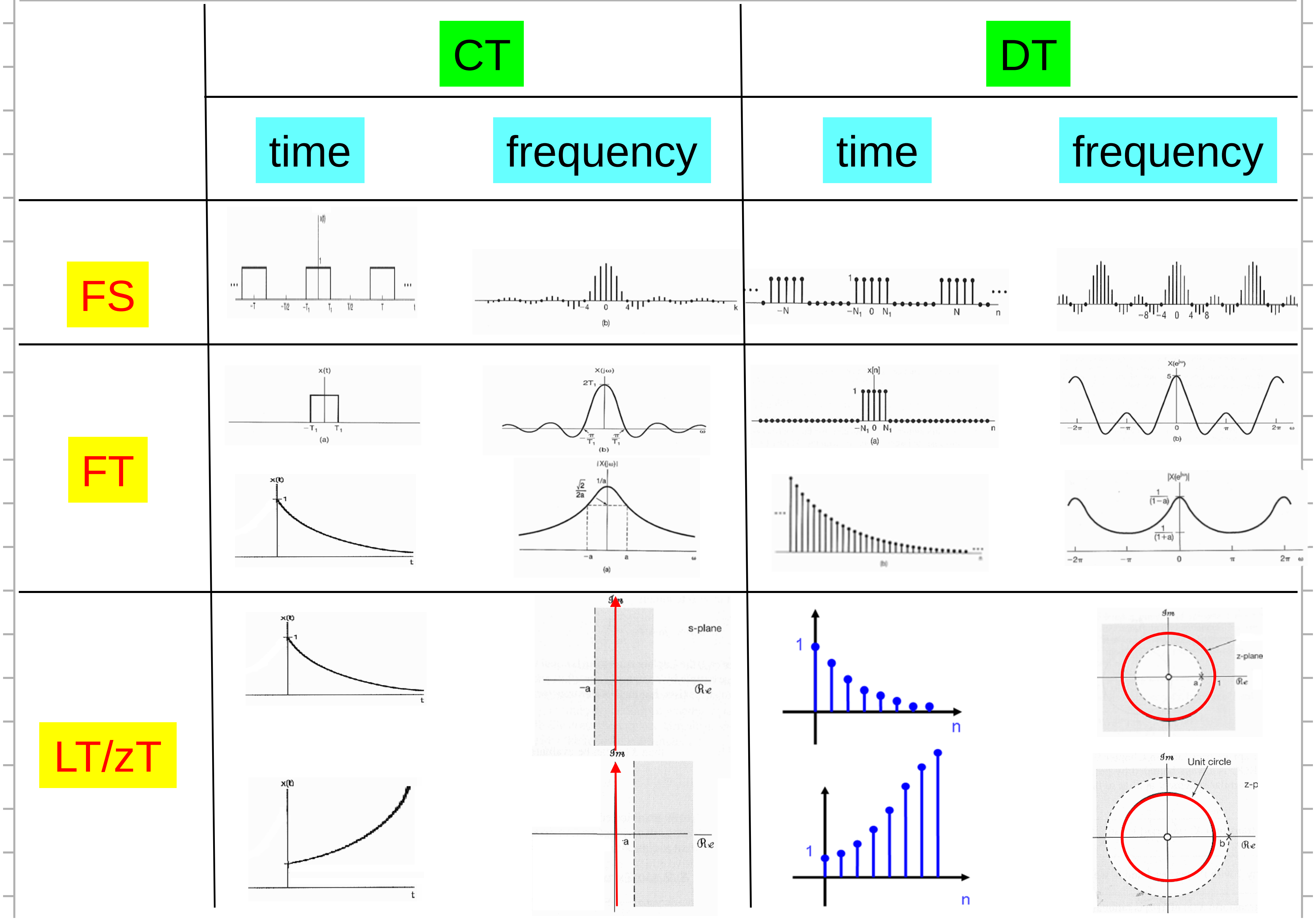
CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)

Fourier Series, Fourier Transform, Laplace Transform, z-Transform



Outline

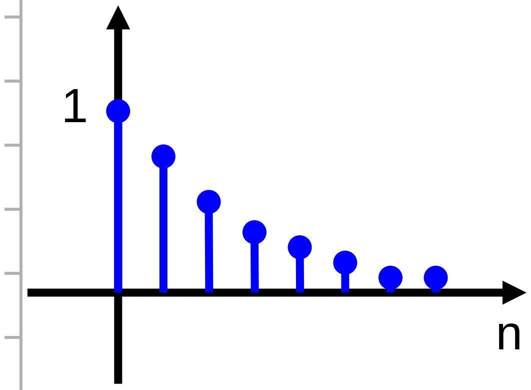
- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

Brief History of the z-Transform

- The z-transform was known to Laplace, and
re-introduced in 1947 by W. Hurewicz
as a tractable way to solve linear, constant-coefficient difference eqns.
- It was later dubbed "the z-transform" by Ragazzini and Zadeh
in the sampled-data control group at Columbia University in 1952
- The name of "the z-transform"
  - The letter "z" being a sampled/digitized version of
the letter "s" in Laplace transforms.
  - Another possible source is the presence of the letter "z"
in the names of both Ragazzini and Zadeh
who published the seminal paper.
- The modified or advanced z-transform was later developed and
popularized by E. I. Jury in 1958, 1973.
- The idea contained within the z-transform is also known as
the method of generating functions around 1730
when it was introduced by DeMoivre with probability theory.
- From a mathematical view
the z-transform can also be viewed as a Laurent series
where one views the sequence of numbers under consideration
as the (Laurent) expansion of an analytic function (the z-transform).

- From the Fourier Transform of DT signals x[n]:

$x[n] = a^n u[n]$

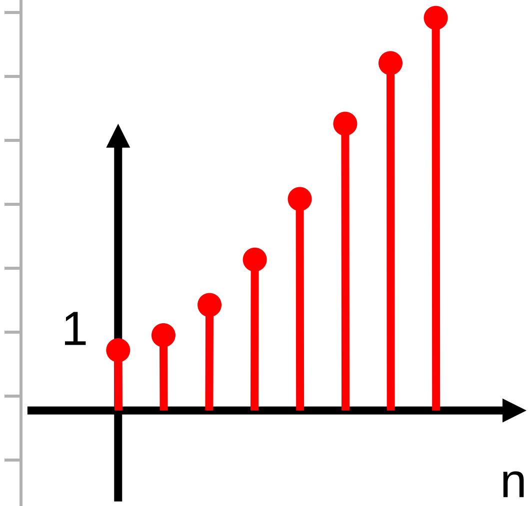


FT

$$X(e^{jw}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{+\infty} x[n] \left(e^{jw}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (ae^{-jw})^n \qquad = \frac{1}{1 - ae^{-jw}}$$



$$\longrightarrow \quad \sum_{n=-\infty}^{+\infty} \left(x[n] \qquad \right) \quad \left(e^{jw}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(\qquad e^{jw} \right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(\qquad \right)^{-n}$$

zT

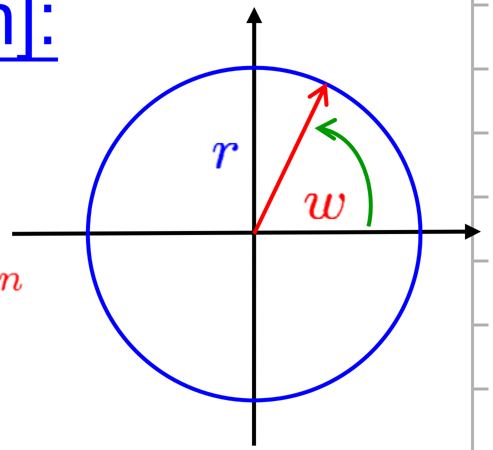
The z-Transform of a General Signal x[n]:

FT $z = e^{jw}$

$$X(e^{jw}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

zT $z = re^{jw}$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$



$$e^{jw} =$$

$$re^{jw} =$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) \qquad x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$X(e^{jw}) = \mathcal{F}\left\{x[n]\right\} \qquad X(z) = \mathcal{Z}\left\{x[n]\right\}$$

$$x[n] = \mathcal{F}^{-1}\left\{X(e^{jw})\right\} \qquad x[n] = \mathcal{Z}^{-1}\left\{X(z)\right\}$$

$$X(z)\Big|_{z=e^{jw}} = \mathcal{Z}\left\{x[n]\right\}\Big|_{z=e^{jw}} = \mathcal{F}\left\{x[n]\right\} = X(e^{jw})$$

▪ z-Transform & Fourier Transform:

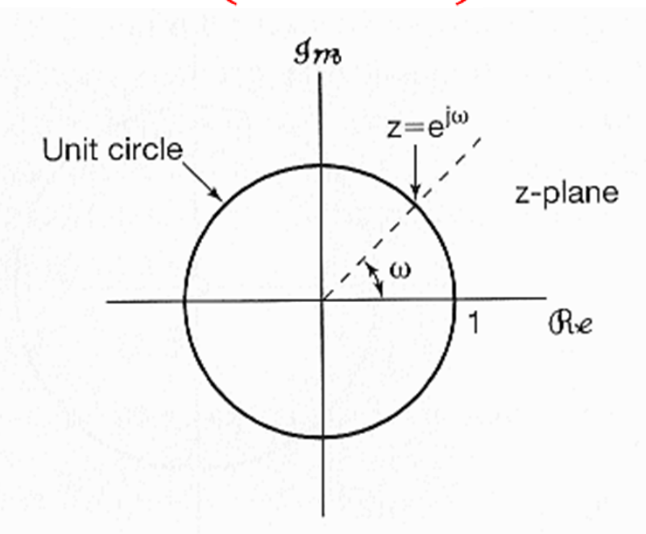
$$\mathcal{Z}\left\{x[n]\right\}\Big|_{z=re^{jw}} = X(re^{jw})$$

$$= \sum_{n=-\infty}^{+\infty} x[n](re^{jw})^{-n}$$

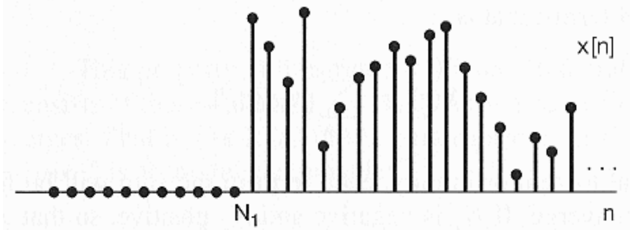
$$= \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-jwn}$$

$$= \mathcal{F}\left\{x[n]r^{-n}\right\}$$

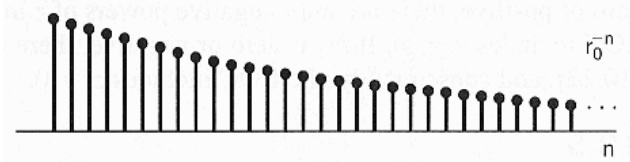
$$z = e^{jw}$$



$x[n]$

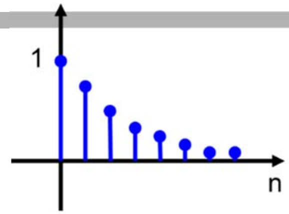


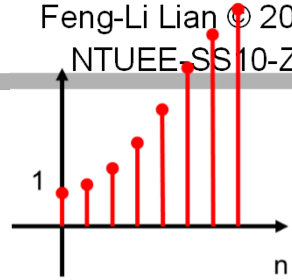
r^{-n}



Example 10.1:

$$x[n] = a^n u[n]$$

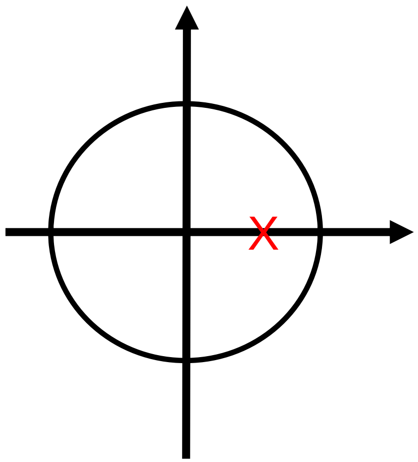


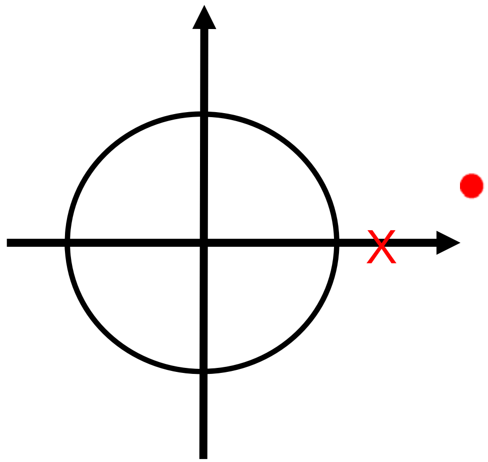


$$\Rightarrow \; X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1 - ae^{-jw}}$$

$$\Rightarrow \; X(z) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \qquad |az^{-1}| < 1$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

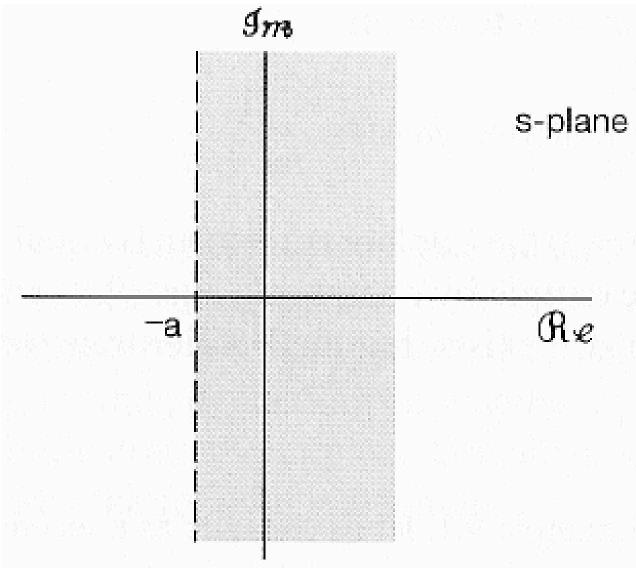


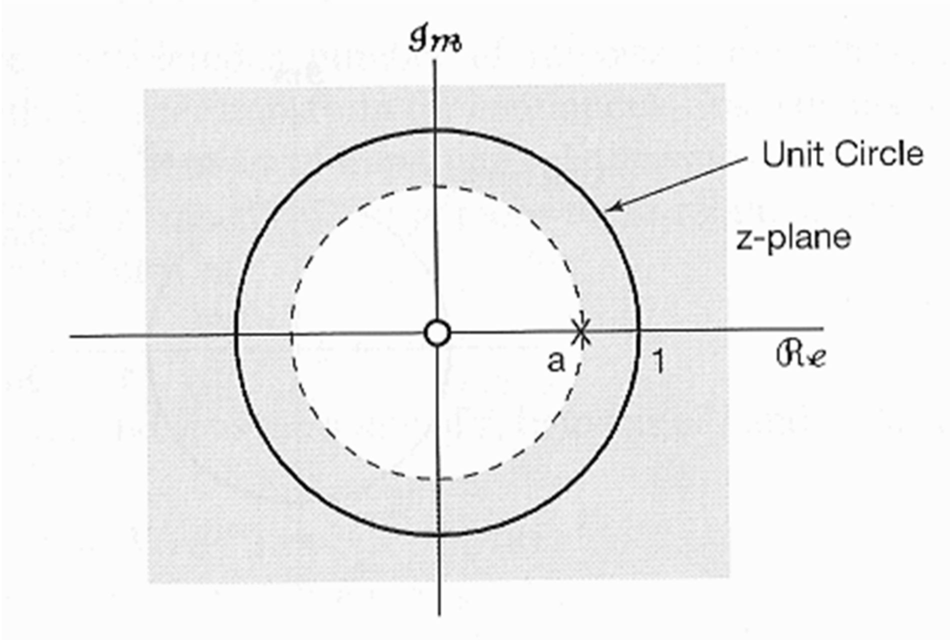


- For $|a| > 1$,

  ROC does not include the unit circle,

  $\mathcal{F}\left\{ a^n u[n] \right\}$ does not converge





$$s = \sigma + jw$$

$$z = re^{jw}$$

$$e^{-at}u(t) \qquad e^{-\sigma t} \quad e^{-jwt}$$

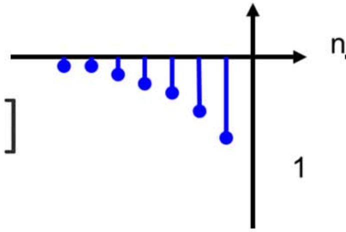
$$e^{-st}$$

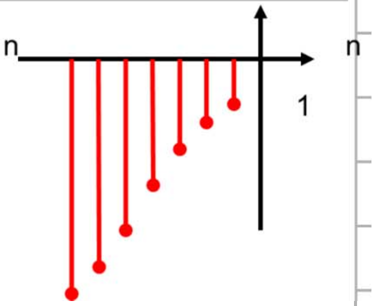
$$a^n u[n] \qquad r^{-n} \quad (e^{jw})^{-n}$$

$$(z)^{-n}$$

- Example 10.2:

$$x[n] = -a^n u[-n-1]$$





$$\Rightarrow X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

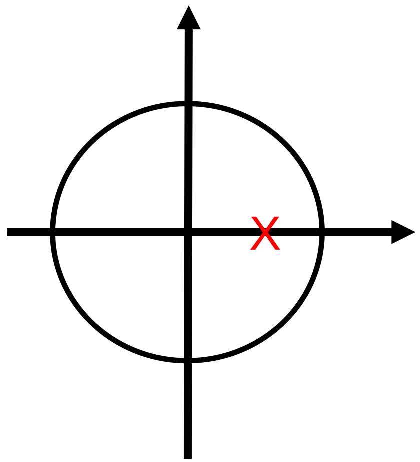
$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

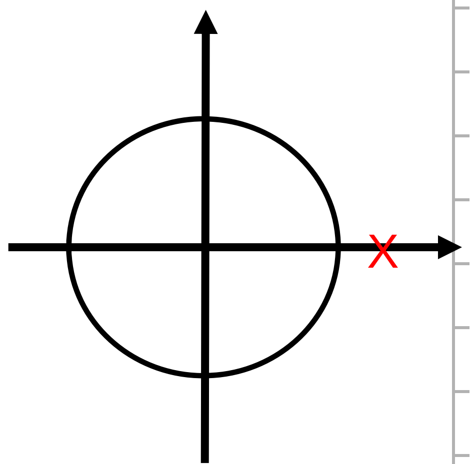
$|a^{-1}z| < 1$

$$= 1 - \frac{1}{1 - a^{-1}z}$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

$$|z| < |a|$$



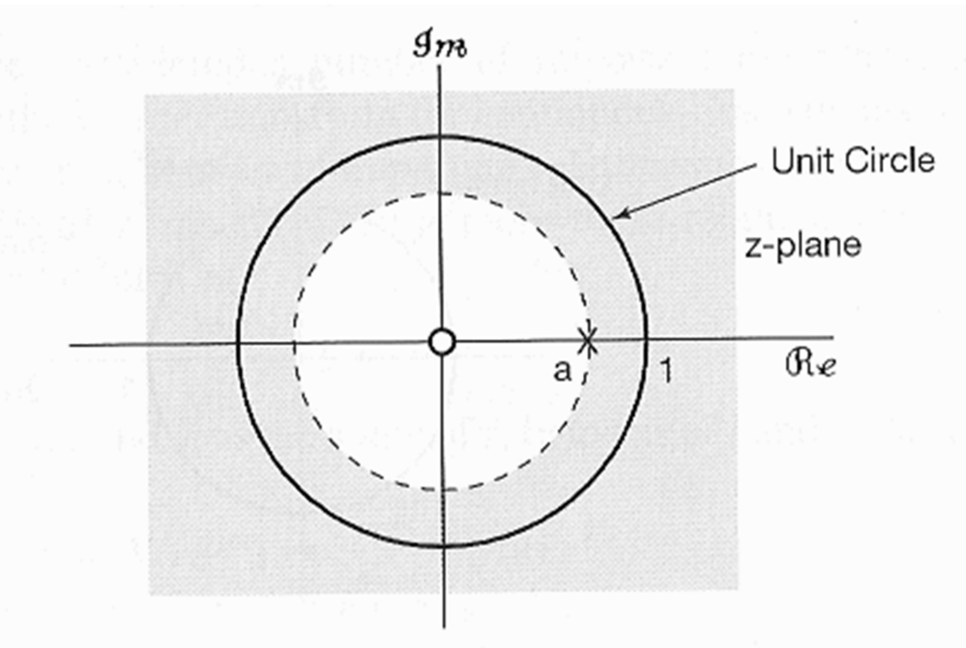


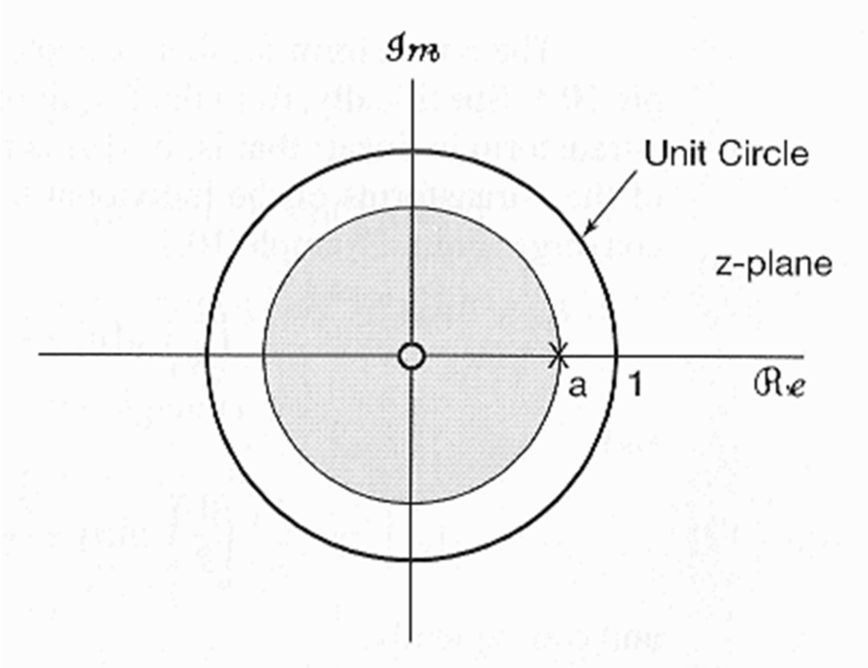
Region of Convergence (ROC):

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| > |a|$$

$$-a^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| < |a|$$

where Fourier transform of $x[n]r^{-n}$ converges





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- Example 10.3:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{ \qquad\qquad\qquad\qquad \right\} z^{-n}$$

$$= \qquad \sum_{n=-\infty}^{+\infty} \left\{ \qquad\qquad \right\} z^{-n} \qquad \sum_{n=-\infty}^{+\infty} \left\{ \qquad\qquad \right\} z^{-n}$$

$$= \frac{\qquad}{\qquad} \qquad \frac{\qquad}{\qquad}$$

$$7 \cdot \left(\frac{1}{3}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} 7 \cdot \frac{\qquad}{\qquad} = 7 \cdot \frac{\qquad}{\qquad}, \quad |z|$$

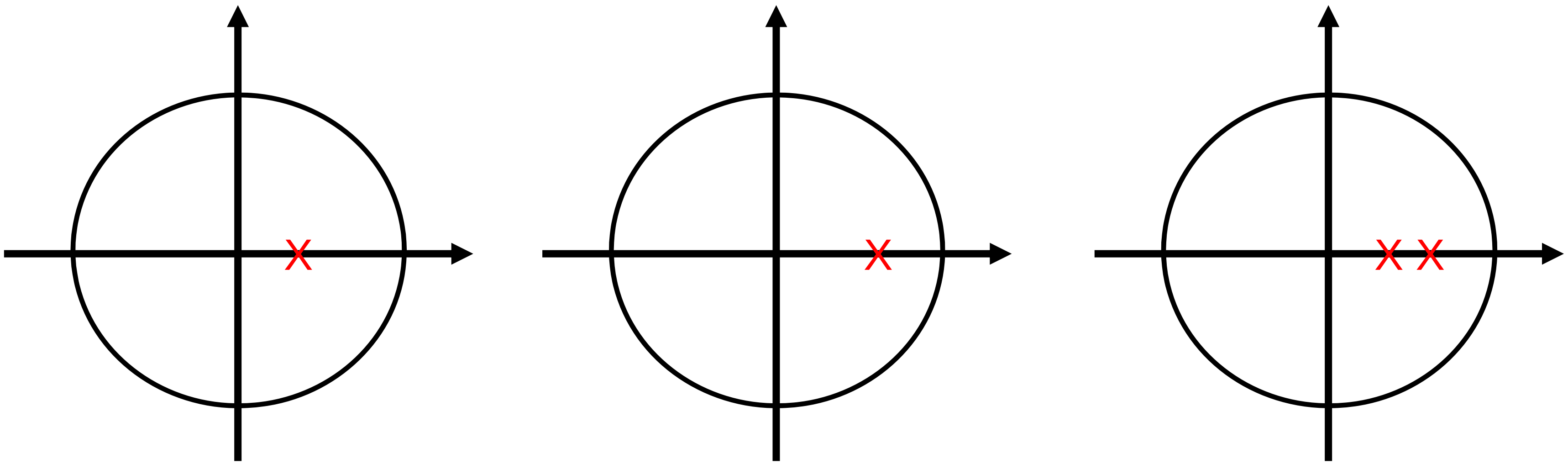
$$6 \cdot \left(\frac{1}{2}\right)^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} 6 \cdot \frac{\qquad}{\qquad} = 6 \cdot \frac{\qquad}{\qquad}, \quad |z|$$

- Example 10.3:

$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{7}{\qquad} \qquad \frac{6}{\qquad}, \quad |z|$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \underline{\qquad\qquad\qquad\qquad}, \quad |z|$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \underline{\qquad\qquad\qquad\qquad}, \quad |z|$$



- Example 10.4:

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{\quad}{\quad}\left(e^{j\quad} \qquad e^{-j\quad}\right)$$

$$= \frac{\quad}{\quad}\left(\left(\qquad\right)^n \quad \left(\qquad\right)^n\right)$$

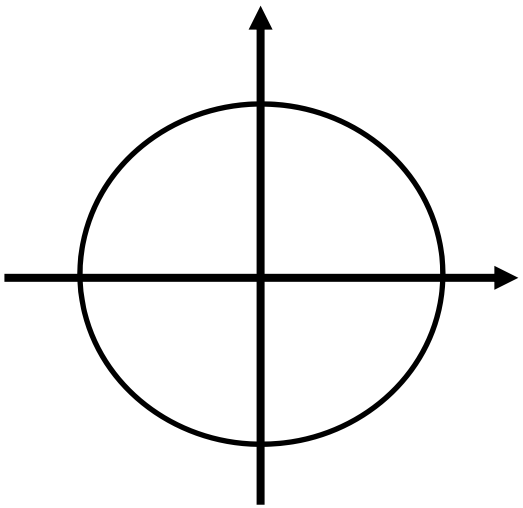
$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] = \frac{\quad}{\quad}\left(\left(\qquad\right)^n \quad \left(\qquad\right)^n\right)$$

- Example 10.4: $a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| > |a|$

$$\left(\frac{1}{3} e^{j\pi/4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \underline{\qquad\qquad\qquad}, \quad |z|$$

$$\left(\frac{1}{3} e^{-j\pi/4}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \underline{\qquad\qquad\qquad}, \quad |z|$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4} n\right) u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \underline{\qquad}\left(\underline{\qquad\qquad} \quad \underline{\qquad\qquad}\right), \quad |z|$$



Outline

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
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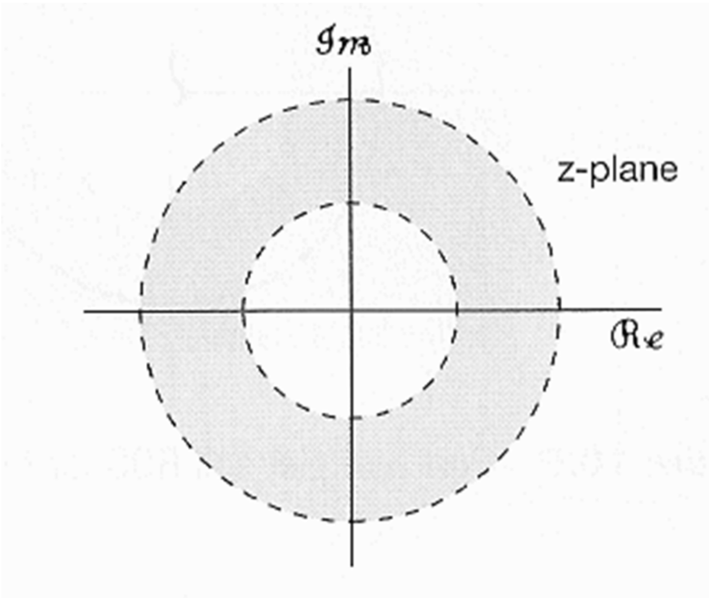
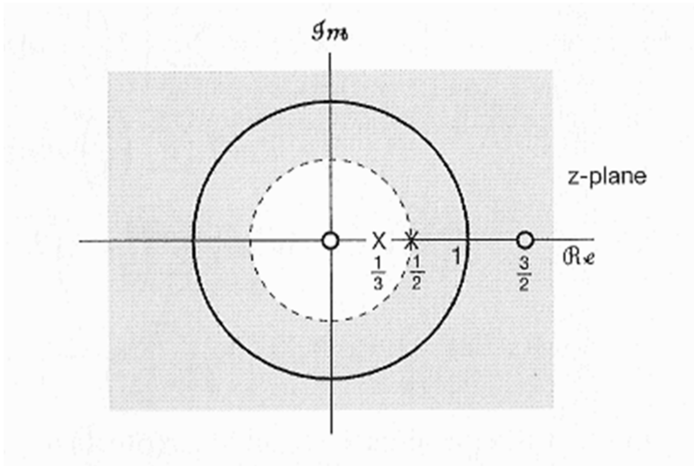
Properties of ROC:

1. The ROC of X(z) consists of a ring in the z-plane centered about the origin

2. The ROC does not contain any poles

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \qquad |z| > \frac{1}{2}$$





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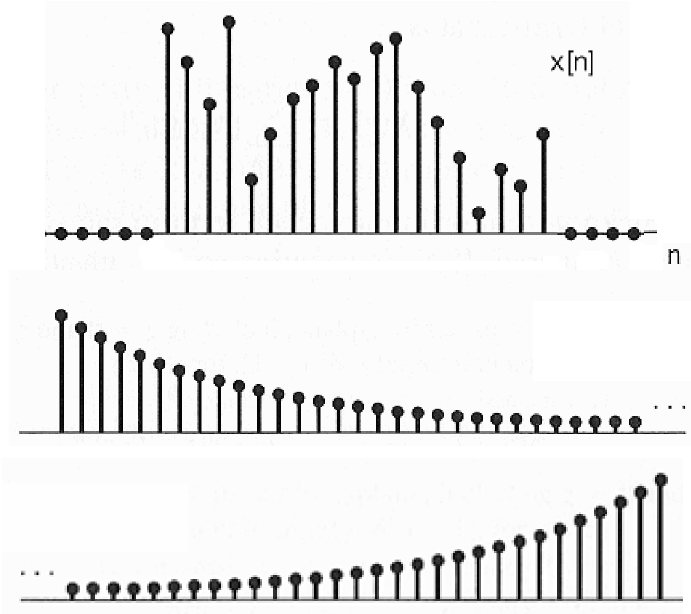
Properties of ROC:



3. If x[n] is of finite duration,
then the ROC is the entire z-plane,
except possibly $z = 0$ and/or $z = \infty$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=N_1}^{N_2} x[n] z^{-n} \quad \text{is bounded}$$

$$= \cdots + x[-3] z^{+3} + \cdots + x[5] z^{-5}$$

$$= \cdots + x[-3] z^{+3} + \cdots + x[5] \left(\frac{1}{z}\right)^{5}$$

- However, $|z| \to 0 \quad \Rightarrow |z|^N \to \infty$ if N is negative
$|z| \to \infty \quad \Rightarrow |z|^N \to \infty$ if N is positive

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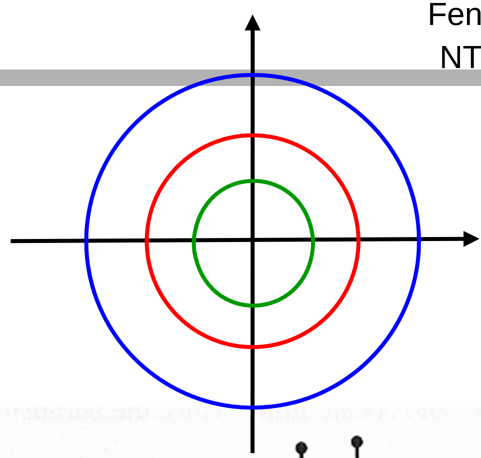
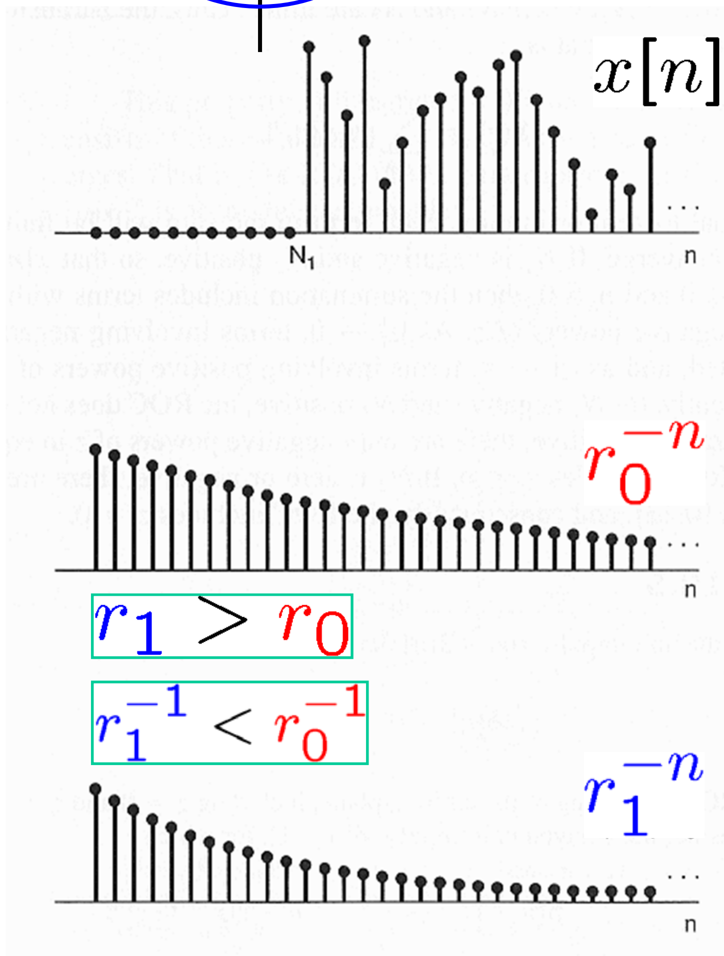
- Properties of ROC:

  4. If x[n] is right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC



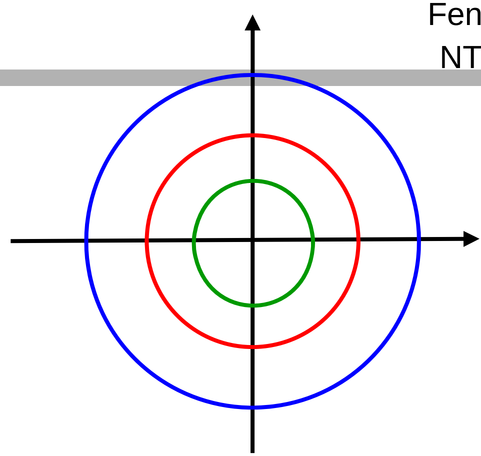


$$X(r_0 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

$$X(r_1 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_1^{-n} \right\} e^{-jwn}$$

$$\boxed{r_1 > r_0} \qquad \boxed{r_1^{-1} < r_0^{-1}}$$

$$< \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

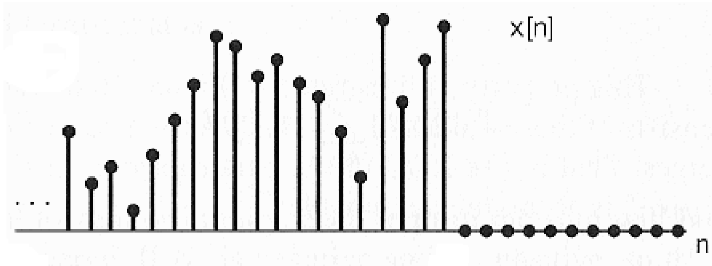


▪ Properties of ROC:

5. If x[n] is left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which

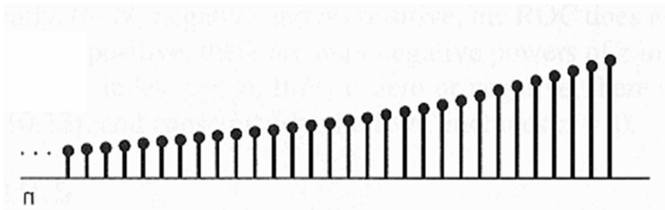
   $0 < |z| < r_0$

   will also be in the ROC



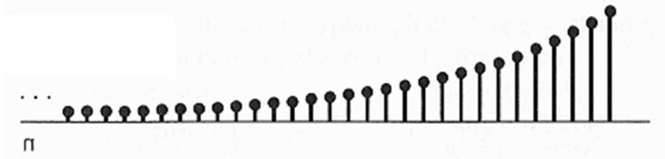
$$X(r_0 e^{jw}) = \sum_{n=-\infty}^{N_2} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

$$X(r_1 e^{jw}) = \sum_{n=-\infty}^{N_2} \left\{ x[n] r_1^{-n} \right\} e^{-jwn}$$

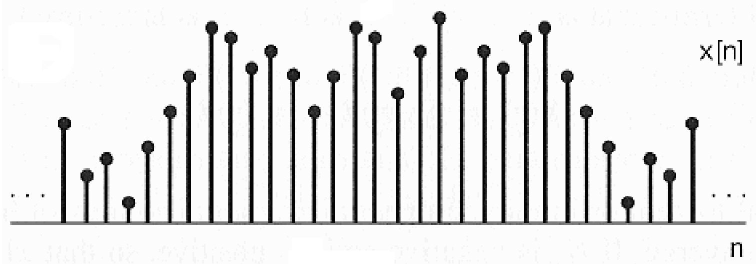


$\boxed{0 < r_1 < r_0}$ $\boxed{r_1^m < r_0^m, \quad m > 0}$

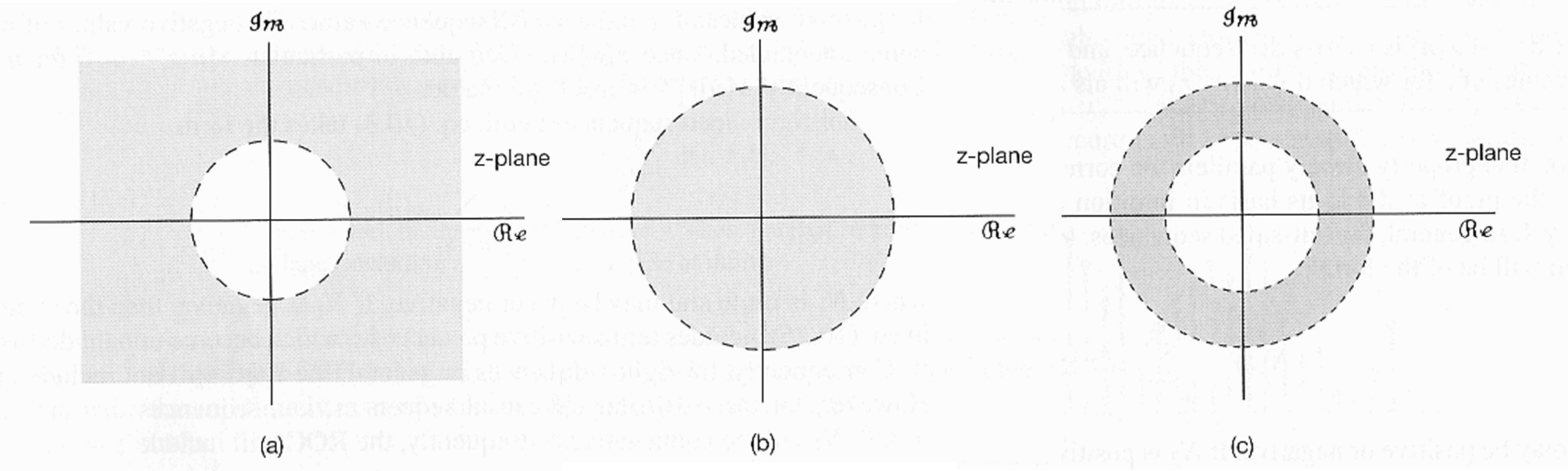
$$< \sum_{n=-\infty}^{N_2} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$



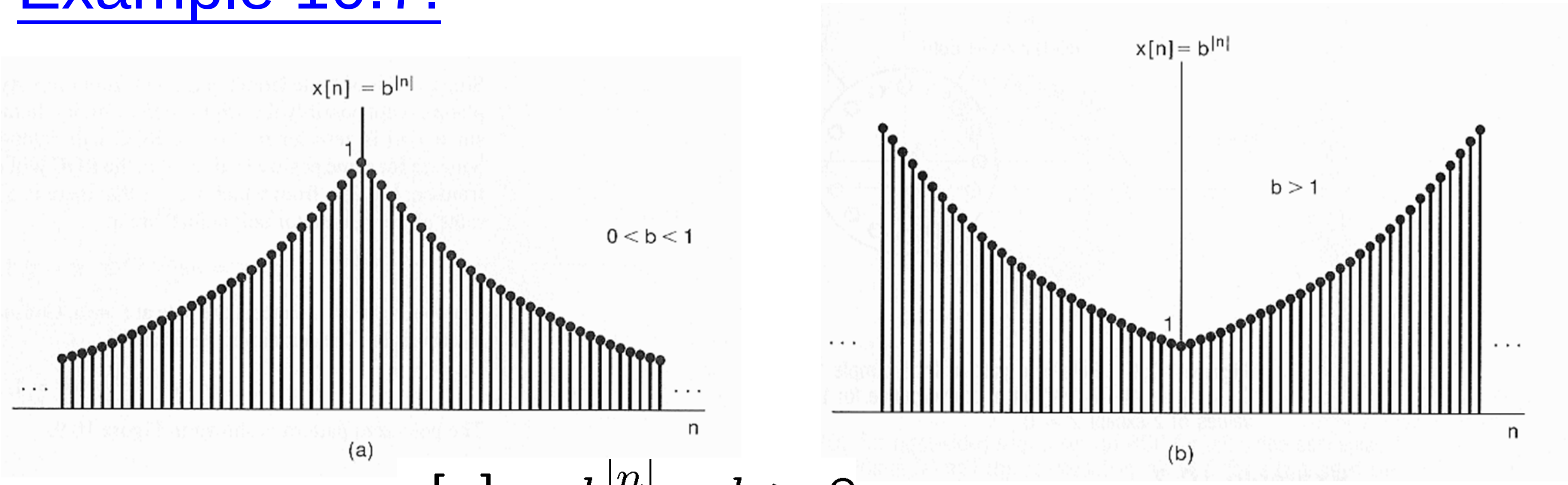
- Properties of ROC:



6. If $x[n]$ is two-sided, and
   if the circle $|z| = r_0$ is in the ROC,
   then the ROC will consist of a ring
   in the z-plane that includes the circle $|z| = r_0$



- Example 10.7:



$$x[n] = b^{|n|}, \quad b > 0$$

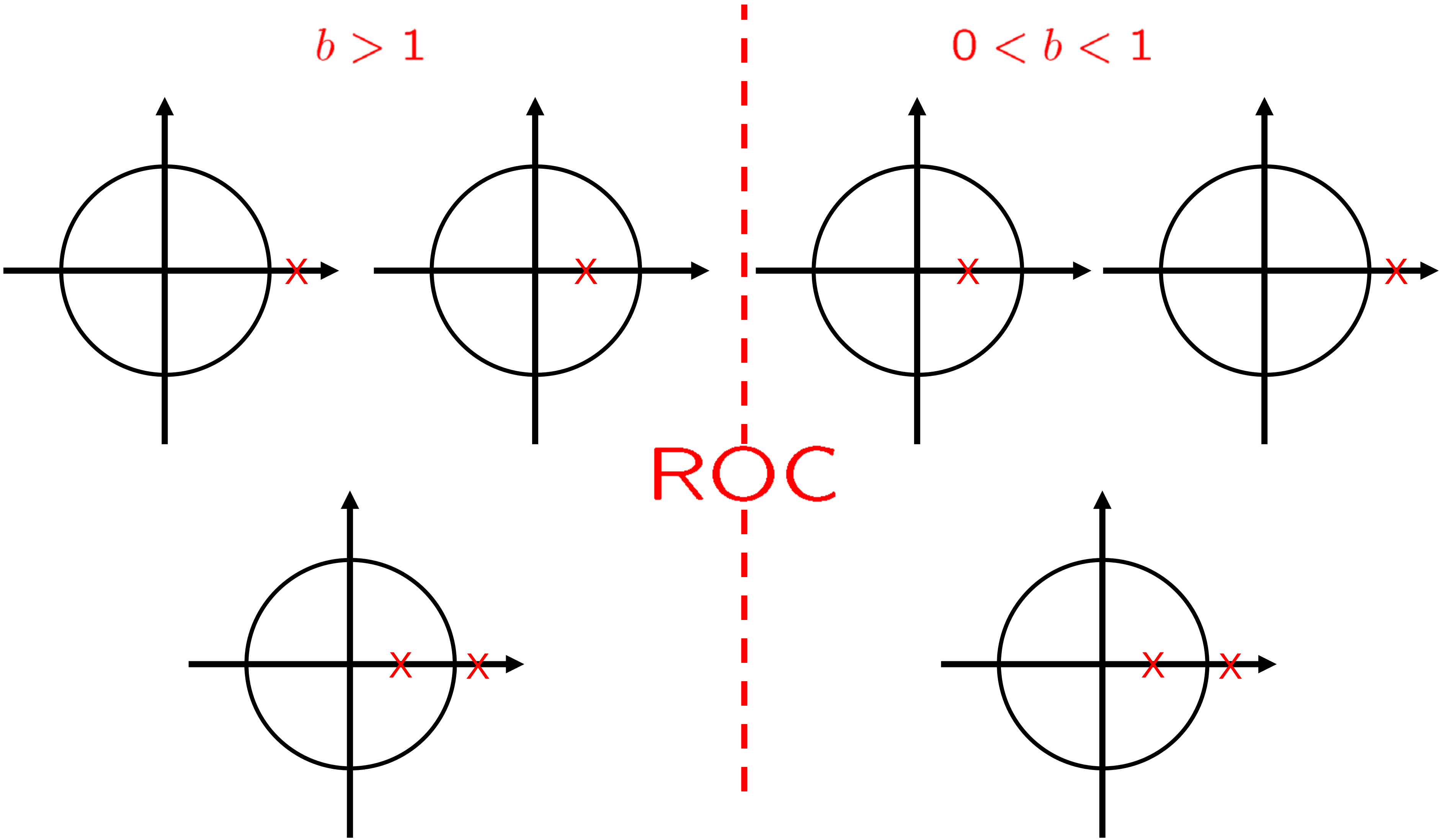
$$= b^{n}u[n] + b^{-n}u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$= \underline{\qquad\qquad} \qquad \underline{\qquad\qquad}, \qquad |z|$$

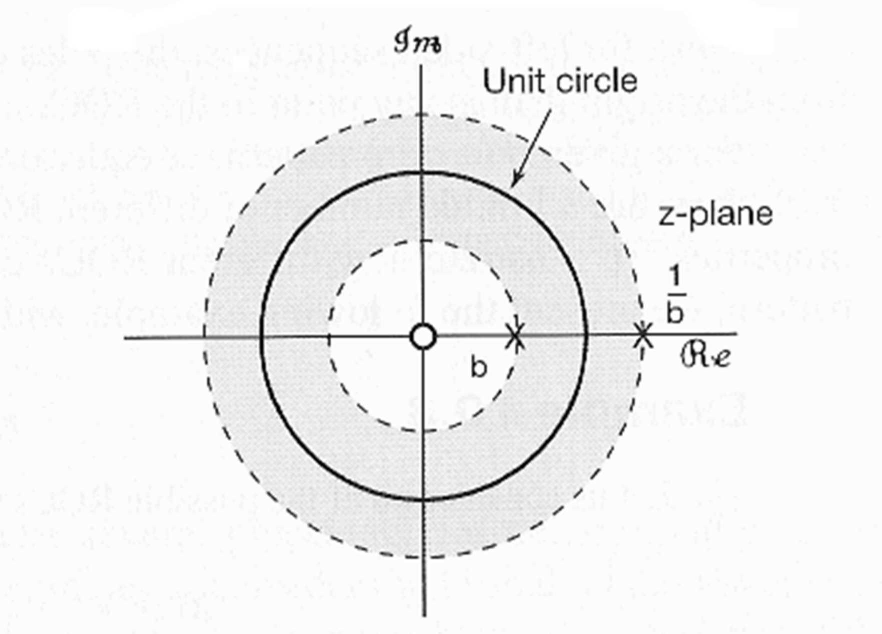
$$= \qquad\qquad \underline{\qquad\qquad\qquad\qquad}, \qquad |z|$$

- Example 10.7:



- Properties of ROC:

7. If the z-transform X(z) of x[n] is rational,
   then its ROC is bounded by poles or extends to ∞



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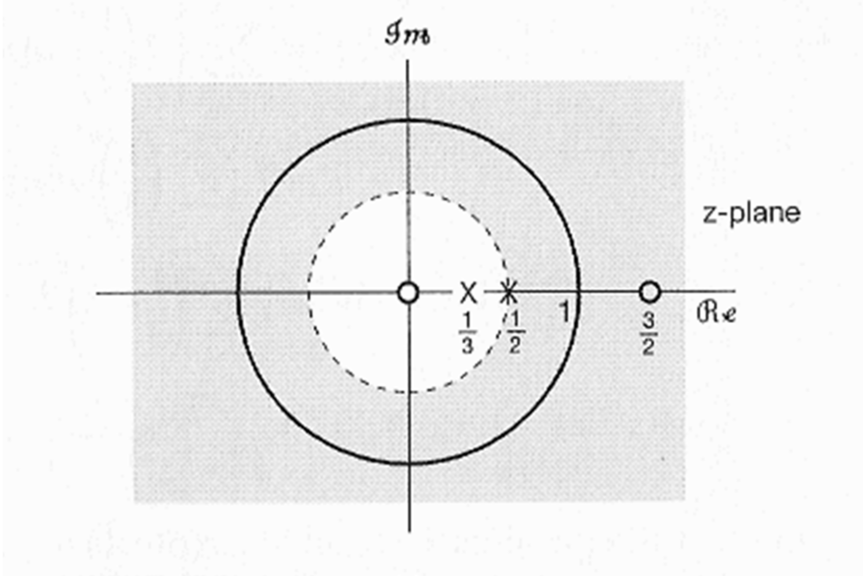

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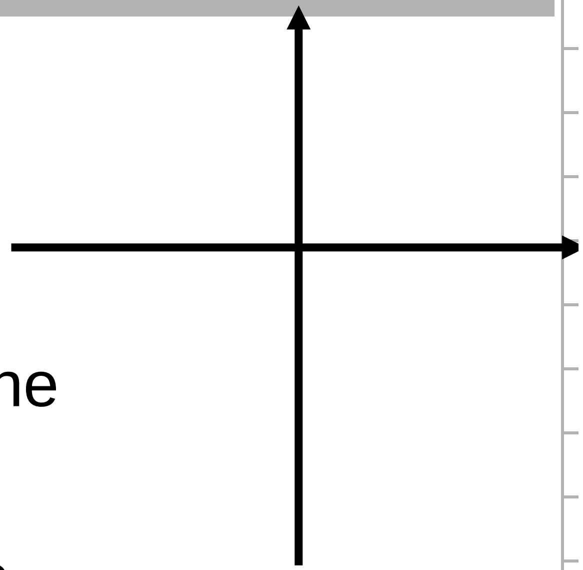
$$X(z) = \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$

- Properties of ROC:

8. If the z-transform X(z) of x[n] is rational



– If x[n] is right sided,
  then the ROC is the region in the z-plane
  outside the outermost pole ---
  i.e., outside the circle of radius equal to
  the largest magnitude of the poles of X(z)

– Furthermore, if x[n] is causal,
  then the ROC also includes $z = \infty$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=0}^{+\infty} x[n] \left(\frac{1}{z}\right)^{n}$$

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- Properties of ROC:

9. If the z-transform X(z) of x[n] is rational

   and if x[n] is left sided,

   then the ROC is the region in the z-plane

   inside the innermost pole ---

   i.e., inside the circle of radius equal to

   the smallest magnitude of the poles of X(z)

   other than any at z = 0

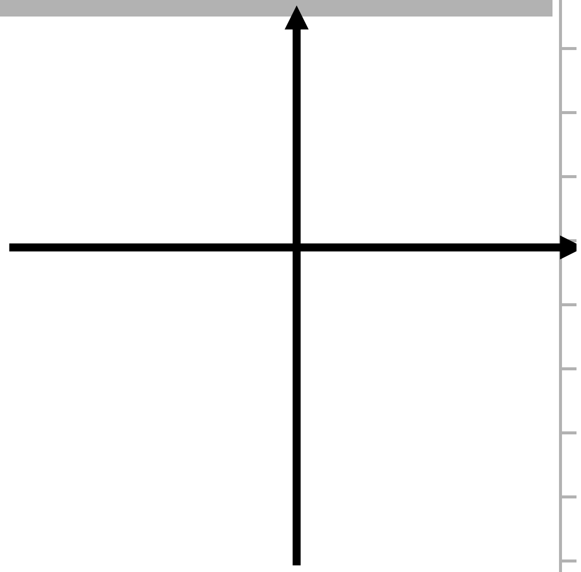
   and extending inward and

   possibly including z = 0

   - In particular, if x[n] is anti-causal,

     (i.e., if it is left sided and = 0 for n > 0),

     then the ROC also includes z = 0

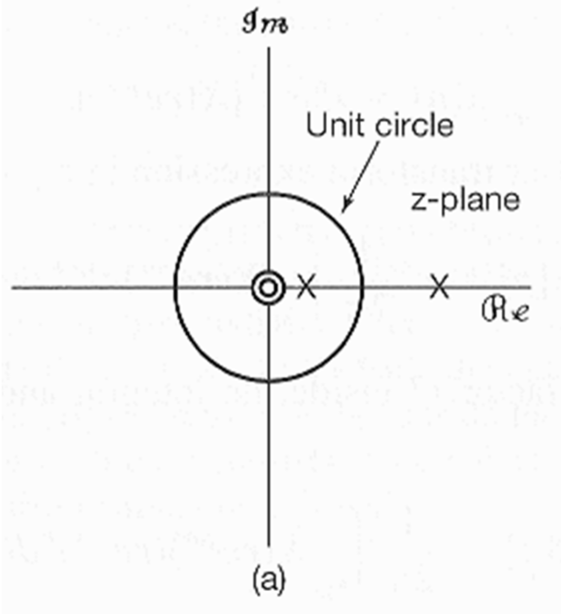


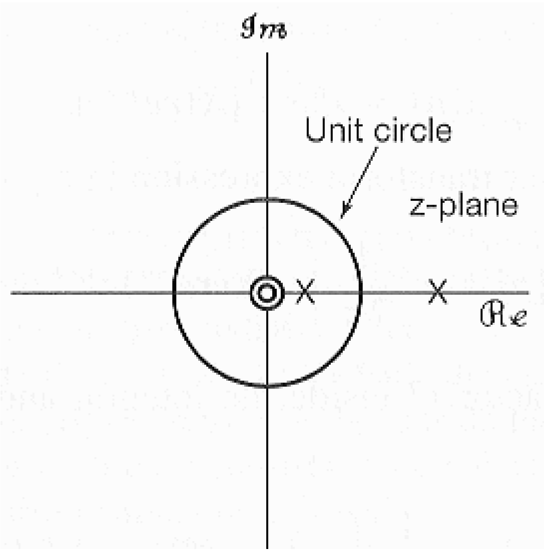
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=-\infty}^{0} x[n]\, z^{-n} = \sum_{m=0}^{\infty} x[-m]\, z^{m}$$

- Example 10.8:

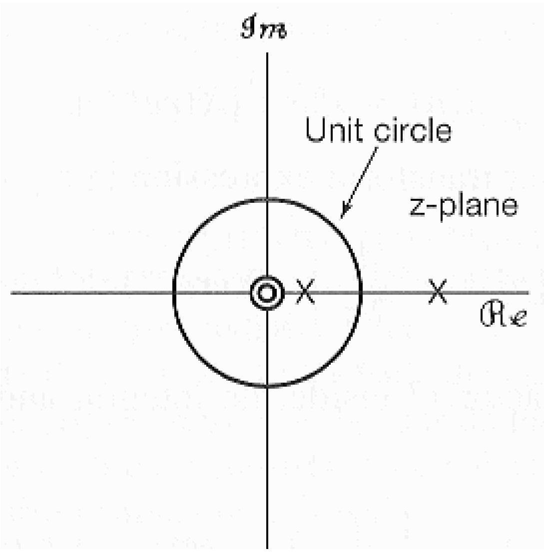
$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

$$= \frac{\qquad}{(\,z \qquad\qquad)(\,z \qquad\qquad)}$$

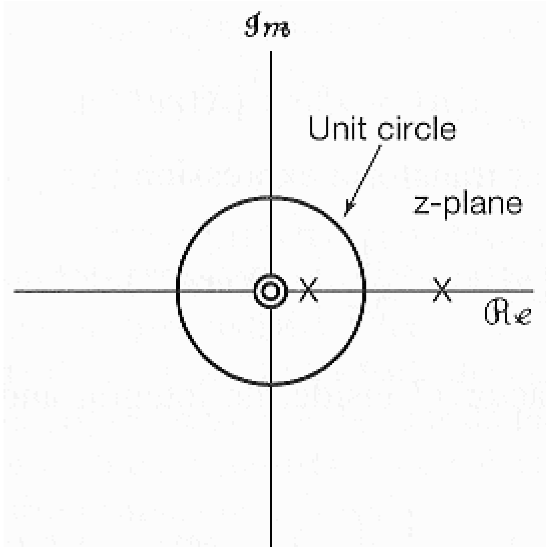




right-sided



left-sided



two sided

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

The Inverse z-Transform:

- By the use of contour integration

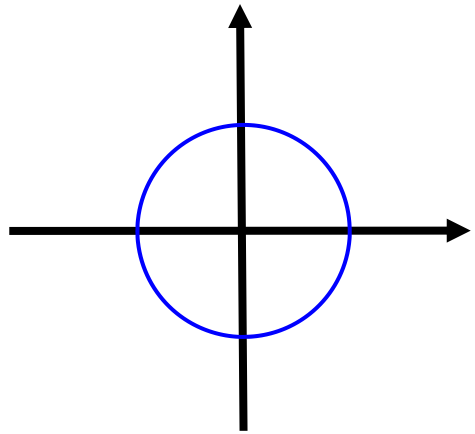
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(re^{jw}) = \mathcal{F}\left\{ x[n] r^{-n} \right\}$$

$\forall z = re^{jw}$ in the ROC

$$x[n] r^{-n} = \mathcal{F}^{-1}\left\{ X(re^{jw}) \right\}$$

$$x[n] = r^n \mathcal{F}^{-1}\left\{ X(re^{jw}) \right\}$$



$$= r^n \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) e^{jwn} dw$$

$$= \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) \left(re^{jw}\right)^n dw$$

$$z = re^{jw}$$

$$dz = jre^{jw} dw = jz dw$$

$$\Rightarrow \quad x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$dw = \frac{1}{jz} dz$$

▪ The Inverse z-Transform:

$$a^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-a^n u[-n-1] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

- By the technique of partial fraction expansion

$$X(z) = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} + \cdots + \frac{M}{1-mz^{-1}}$$

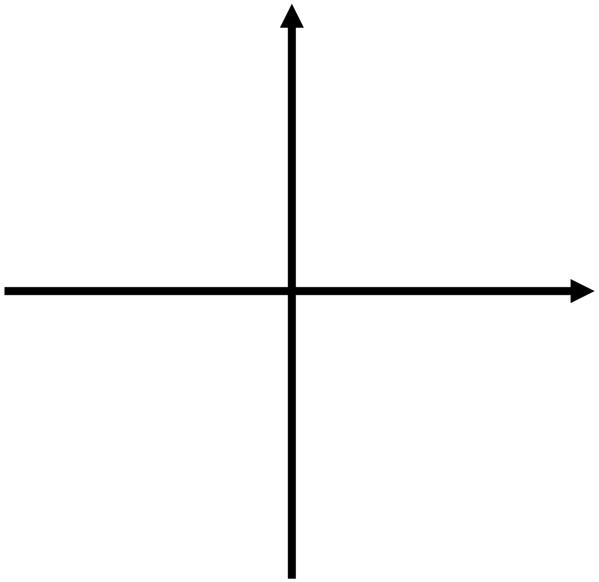
$$x[n] = A\,a^n\,u[n] - B\,b^n\,u[-n-1] + \cdots + x_m[n]$$

(if ROC outside $z = a$) (if ROC inside $z = b$)

- Example 10.9:

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

$$= \frac{(z \qquad\qquad)}{(z \qquad\qquad)(z \qquad\qquad)}$$



$$X(z) = \frac{\qquad}{(1 - \qquad z^{-1})} + \frac{\qquad}{(1 - \qquad z^{-1})}$$

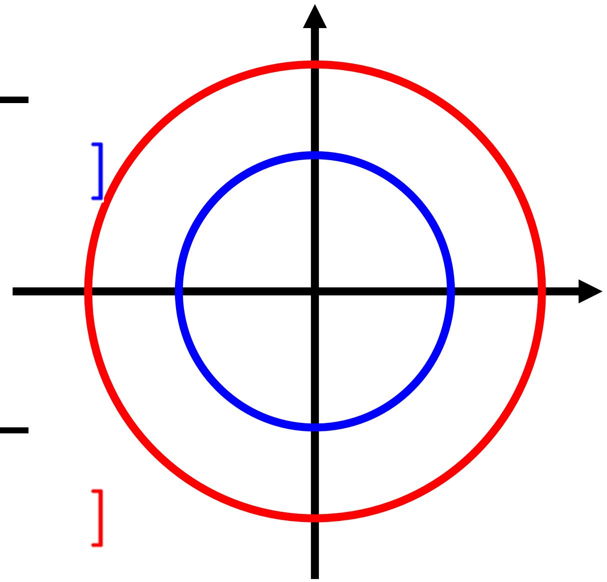
$$u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{(1 \qquad z^{-1})}, \quad |z|$$

$$u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{(1 \qquad z^{-1})}, \quad |z|$$

$$\Rightarrow \quad x[n] =$$

- Examples 10.9, 10.10, 10.11:

	$|z| < \frac{1}{4}$	$\frac{1}{4} < |z|$
$\frac{1}{(1-\frac{1}{4}z^{-1})}$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$



	$|z| < \frac{1}{3}$	$\frac{1}{3} < |z|$
$\frac{1}{(1-\frac{1}{3}z^{-1})}$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$

	$|z| < \frac{1}{4}$	$\frac{1}{4} < |z| < \frac{1}{3}$	$\frac{1}{3} < |z|$
$\frac{1}{(1-\frac{1}{4}z^{-1})}$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$
$\frac{1}{(1-\frac{1}{3}z^{-1})}$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$	$(\quad\quad)^n u[\quad\quad]$

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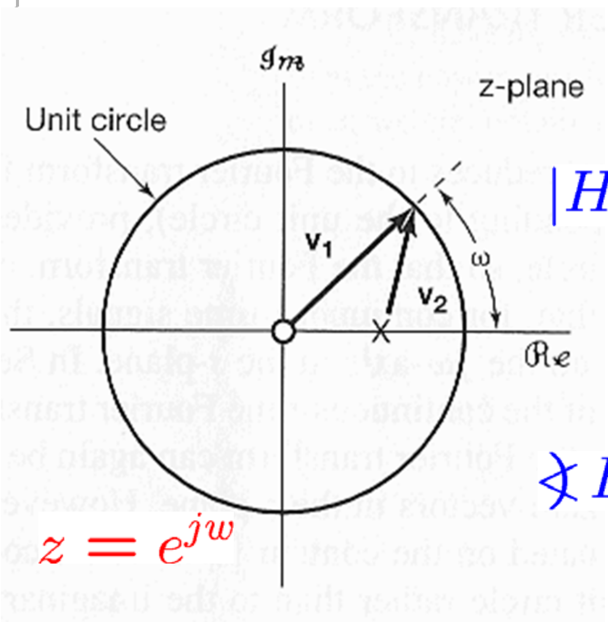
First-Order Systems:

$$h[n] = a^n u[n]$$

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- For $|a| < 1$, ROC includes $|z| = 1$

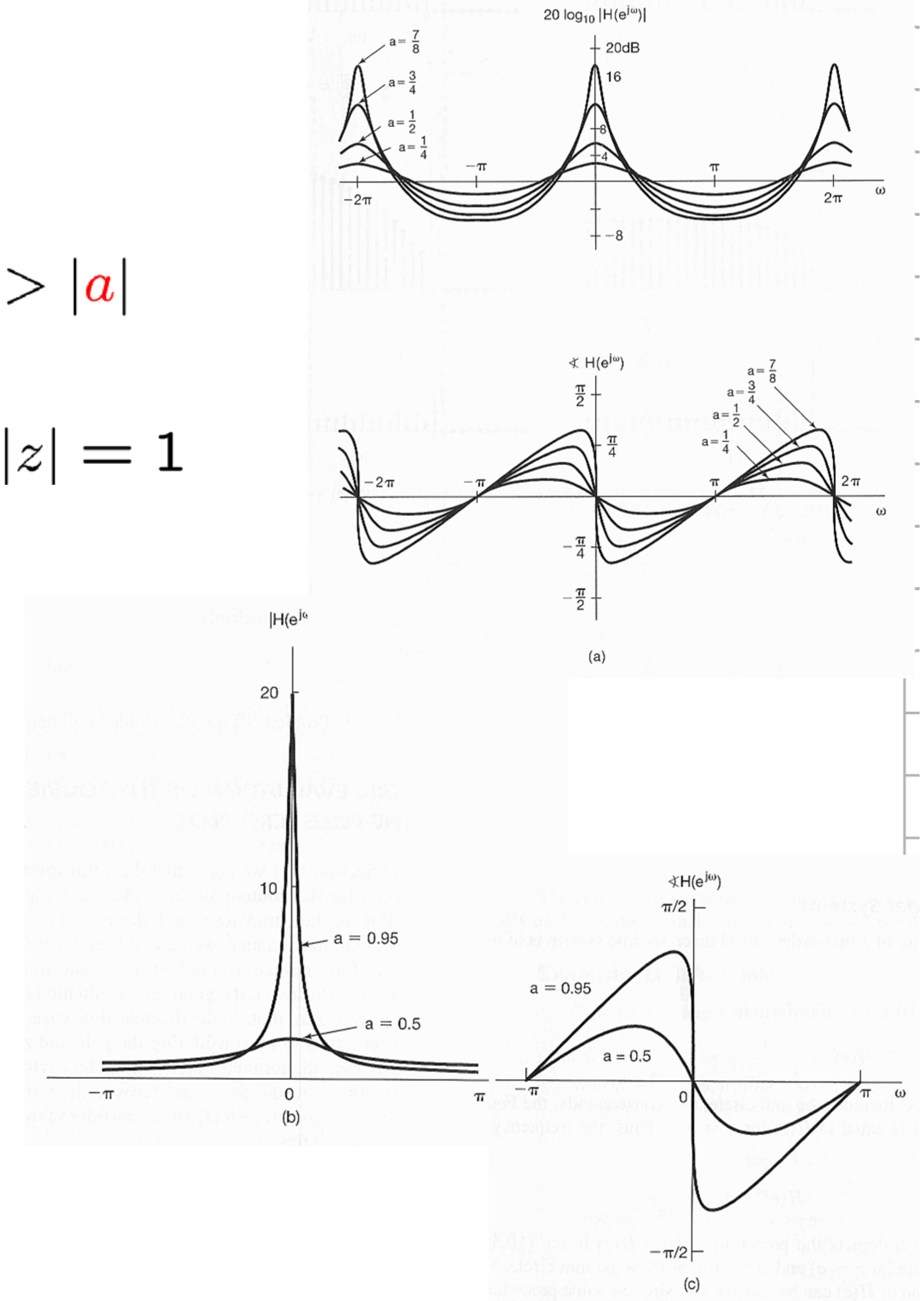
$$\Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$



$$z = e^{jw}$$

$$|H| = \rule{3cm}{0.4pt}$$

$$\sphericalangle H =$$

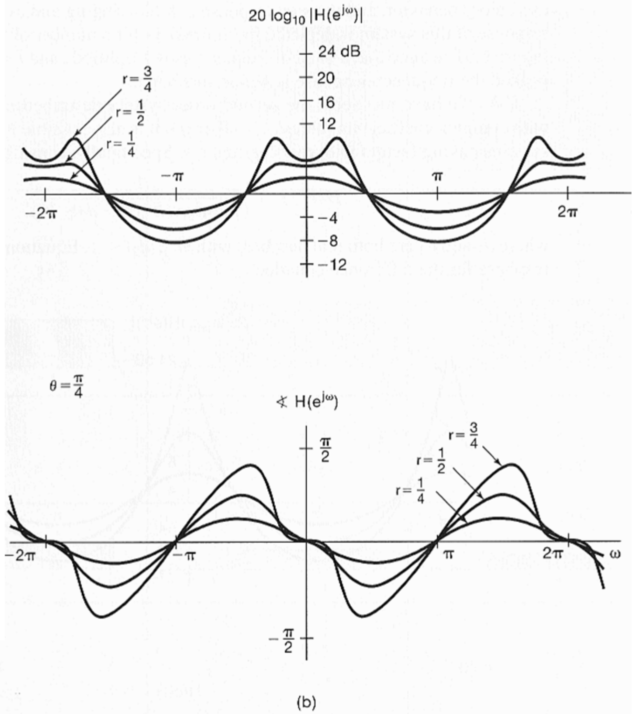


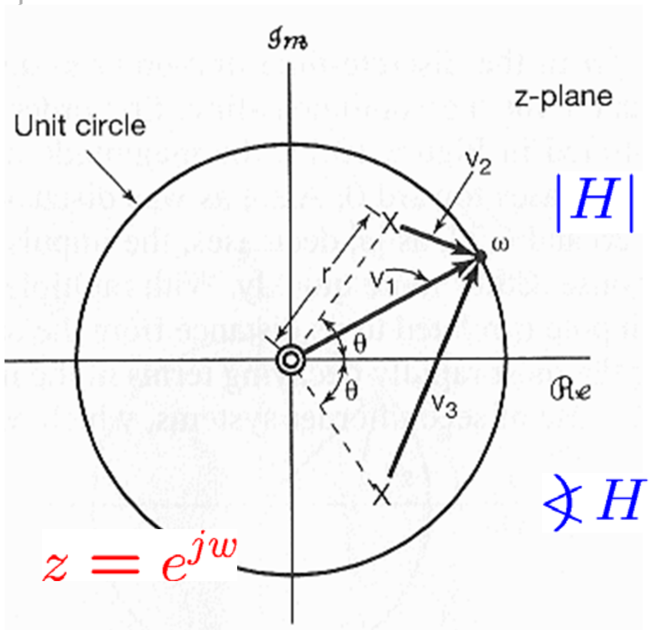
Second-Order Systems:

$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+(r^2)z^{-2}}$$

$$= \frac{z^2}{(z-p_1)(z-p_2)}, \quad |z| > |r|$$

poles : $\begin{cases} p_1 = re^{j\theta} \\ p_2 = re^{-j\theta} \end{cases}$ zeros : $z_1 = z_2 = 0$

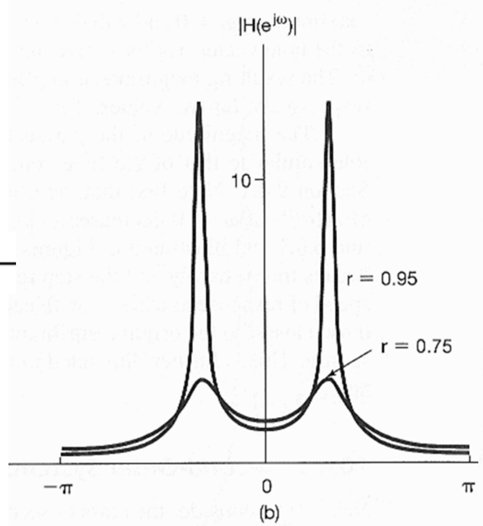


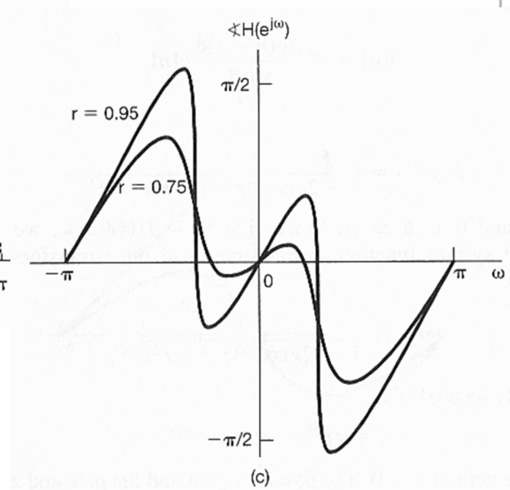


$z = e^{jw}$

$|H| =$ ———

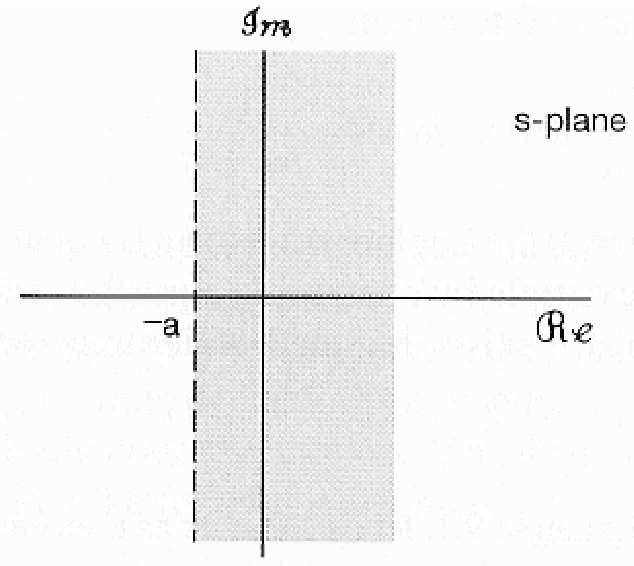
$\sphericalangle H =$



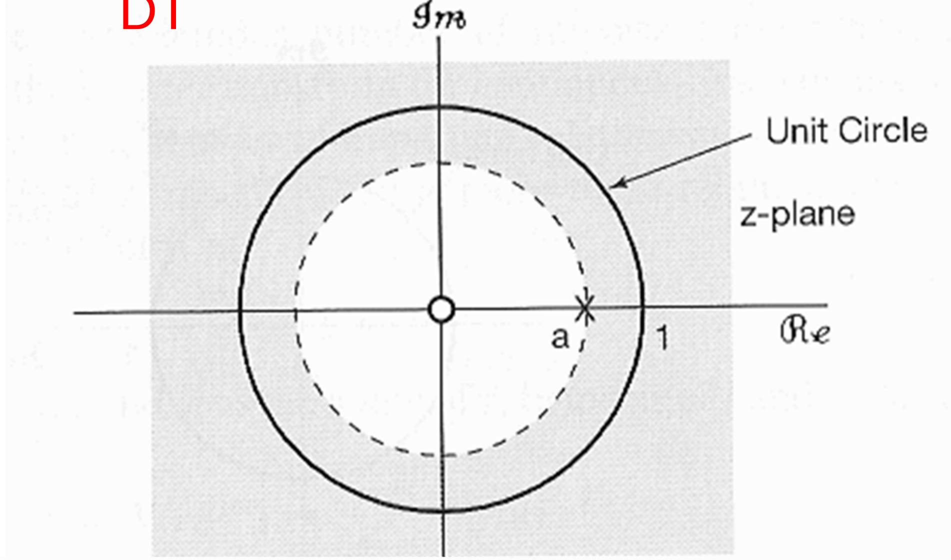


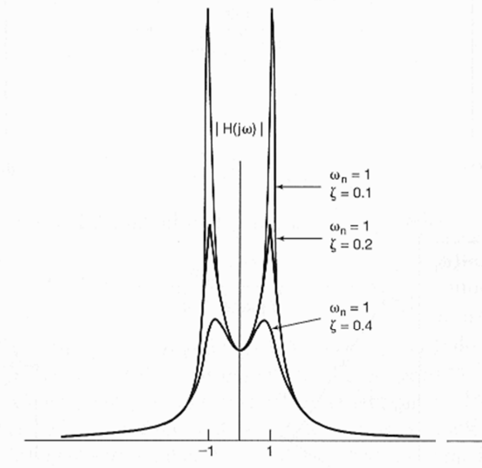
Laplace Transform and The z-Transform

CT $s = \sigma + jw$

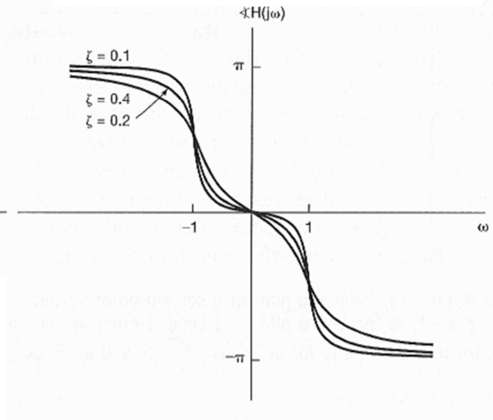


DT $z = re^{jw}$

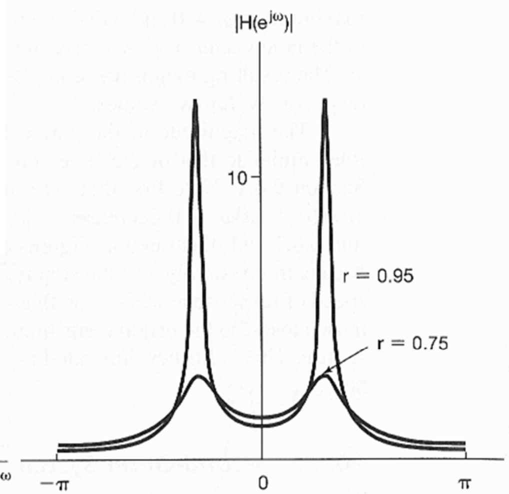




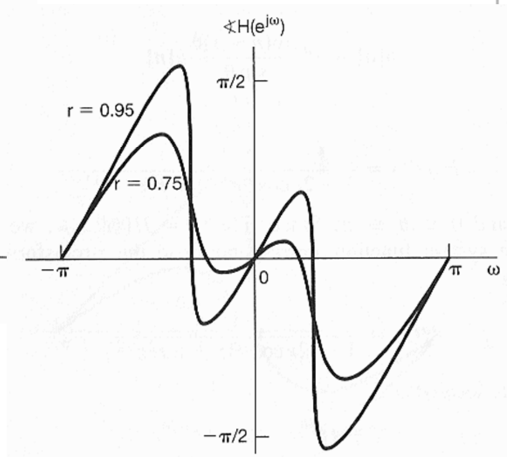
(a)



(b)



(b)



(c)

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Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

- Linearity of the z-Transform:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad ROC = R_1$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad ROC = R_2$$

$$a x_1[n] + b x_2[n] \overset{z}{\longleftrightarrow} a X_1(z) + b X_2(z),$$

with ROC containing $R_1 \cap R_2$

$$\sum_{n=-\infty}^{+\infty} \quad z^{-n} \qquad \sum_{n=-\infty}^{+\infty} \quad z^{-n}$$

$$\sum_{n=-\infty}^{+\infty} \quad z^{-n} \qquad \sum_{n=-\infty}^{+\infty} \quad z^{-n}$$

$$X_1(\quad) \qquad X_2(\quad)$$

- Time Shifting:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x[n-n_0] \stackrel{z}{\longleftrightarrow} z^{-n_0} X(z), \quad ROC = R$$

except for the possible addition or deletion of the origin or infinity

$$\sum_{=-\infty}^{+\infty} x[\quad] \, z^{-(\quad)}$$

$$\sum_{=-\infty}^{+\infty} x[\quad] \, z^{-(\quad)}$$

$$X\left(\quad\right)$$

Scaling in the z-Domain:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$z_0^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \quad ROC = |z_0| R$$

$$\sum_{n=-\infty}^{+\infty} z_0^n x[n] z^{-n}$$

$$\frac{1}{2\pi j} \oint X\left(\frac{z}{z_0}\right) z^{n-1} dz$$

$$\sum_{n=-\infty}^{+\infty} x[n] \left(\frac{\quad}{\quad}\right)^{-n}$$

$$\frac{1}{2\pi j} \oint X\left(\quad\right)\left(\quad\right)^{n-1} d$$

$$\sum_{n=-\infty}^{+\infty} x[n] \left(\quad\right)^{-n}$$

$$\frac{1}{2\pi j} \oint X\left(\quad\right)\left(\quad\right)^{n-1}\left(\quad\right)^{n-1} d$$

$$\frac{1}{2\pi j} \oint X\left(\quad\right)\left(\quad\right)^{n-1} d$$

$$X\left(\quad\right) \qquad X\left(\quad\right)$$

$$x[n]$$

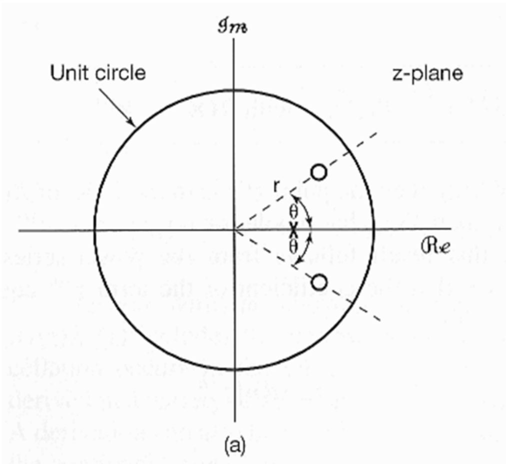
Scaling in the z-Domain:

$$a^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

$$(\quad a)^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(\frac{z}{b}) = \frac{\frac{z}{\ \ }}{\frac{z}{\ \ } - a}, \quad |\frac{z}{\ \ }| > |a|$$

$$e^{jw_0 n} x[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X\left(e^{-jw_0} z\right), \quad ROC = R$$

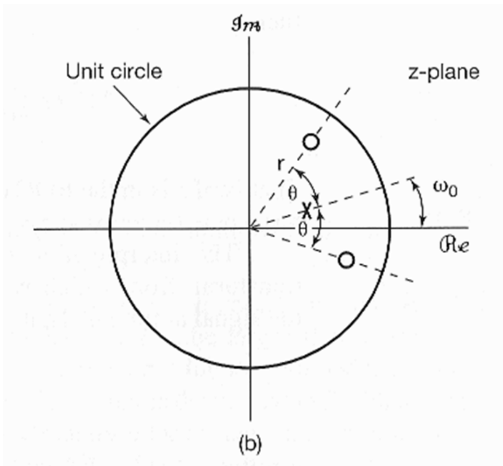
$$\left(e^{jw_0}\right)^n$$



(a)

$ae^{\ jb}$

$ae^{\ jb}$



(b)

$ae^{\ jb} \quad e^{\ jw_0}$

$ae^{\ jb} \quad e^{\ jw_0}$

Time Reversal:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x[-n] \stackrel{z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

$$\sum_{m=+\infty}^{-\infty} x[m] \ (z)^{+m}$$

$$\sum_{m=-\infty}^{+\infty} x[m] \ (\quad)^{-m}$$

$$X(\quad)$$

Time Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

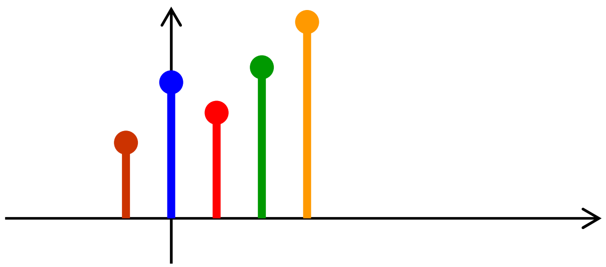
$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = k \cdot m \\ 0, & \text{otherwise} \end{cases}$$

k is a constant

m is a new time variable

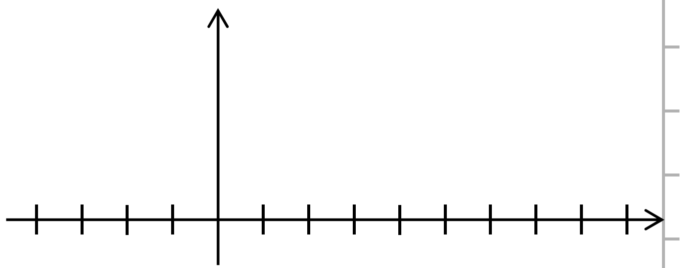
$$x_{(k)}[n] \stackrel{z}{\longleftrightarrow} X(z^k), \quad ROC = R^{1/k}$$

$$\sum_{n=-\infty}^{+\infty} x\left[\quad \right] z^{-n}$$



$$\sum_{m=-\infty}^{+\infty} x\left[\quad \right] z^{-(\quad)}$$

$$\sum_{m=-\infty}^{+\infty} x\left[\quad \right] (z \quad)^{-(\quad)} \qquad X\left(\quad \right)$$



- **Conjugation:**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x^*[n] \overset{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \; z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x \;\; [n] \; z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x \;\; [n] \; (\;\; z \;\;)^{-n}$$

$$X(\;\; z \;\;) = \sum_{n=-\infty}^{+\infty} x \;\; [n] \; (\;\; z \;\;)^{-n}$$

- **Convolution Property:**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_1[n] \stackrel{z}{\longleftrightarrow} X_1(z), \quad ROC = R_1$$

$$x_2[n] \stackrel{z}{\longleftrightarrow} X_2(z), \quad ROC = R_2$$

$$x_1[n] * x_2[n] \quad \stackrel{z}{\longleftrightarrow} \quad X_1(z) X_2(z),$$

with ROC containing $R_1 \cap R_2$

$R_1 \cap R_2$ may be larger if pole-zero cancellation occurs in the product

$$= \sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{+\infty} x_1[m] \; x_2[n-m] \right) z^{-n}$$

$$= \sum_{m=-\infty}^{+\infty} x_1[m] \left(\sum_{n=-\infty}^{+\infty} x_2[n-m] z^{-n} \right)$$

$$= \sum_{m=-\infty}^{+\infty} x_1[m] \left(\sum_{=-\infty}^{+\infty} x_2[\quad] z^{-(\quad)} \right)$$

$$= \left(\sum_{m=-\infty}^{+\infty} x_1[m] \, z^{-(\quad)} \right) \left(\sum_{=-\infty}^{+\infty} x_2[\quad] z^{-(\quad)} \right) \qquad X_1(\quad) X_2(\quad)$$

Differentiation in the z-Domain:

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad ROC = R$$

$$nx[n] \xleftrightarrow{\mathcal{Z}} -z\frac{d}{dz}X(z), \quad ROC = R$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \quad z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \quad z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \quad z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \quad z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \quad z^{-n}$$

If $x[n] = 0$ for $n < 0$

- The Initial-Value Theorem:

$$\Rightarrow \ x[0] = \lim_{z \to \infty} X(z)$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] \, z^{-n} \qquad = x[0] + x[1] \, z^{-1} + x[2] \, z^{-2}$$

- The Final-Value Theorem:

$$\Rightarrow \ x[\infty] = \lim_{z \to 1} (1 - z^{-1}) X(z) \qquad X(z) - (z^{-1}) X(z)$$

$$X(z) = x[0] + x[1] \, z^{-1} + x[2] \, z^{-2}$$

$$-(z^{-1}) X(z) = \qquad -x[0] \, z^{-1} - x[1] \, z^{-2} - x[2] \, z^{-3}$$

Properties of the z-Transform

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9		Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

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- The z-Transform
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- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

Some z-Transform Pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

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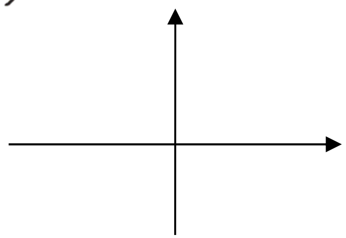
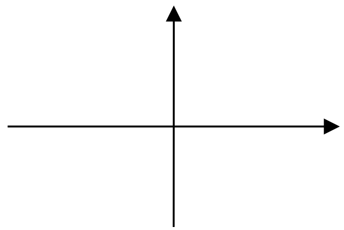
Analysis & Characterization of LTI Systems:

$h[n]$

$x[n]$ → LTI System → $y[n] = h[n] * x[n]$

$X(z)$ → LTI System → $Y(z) = H(z)\, X(z)$

$H(z) = \mathcal{Z}\{h[n]\}$ $\quad H(z)$: system function or transfer function

	$h[n]$	$H(z)$
▪ Causality $x[\ \ ]$ $\quad y[n]$	$h[n]$	
▪ Stability $x[n]$ $\quad y[n]$	$h[n]$	

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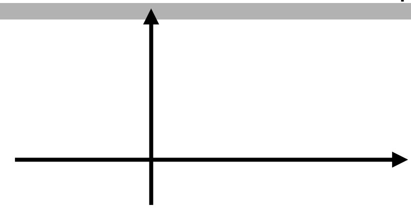
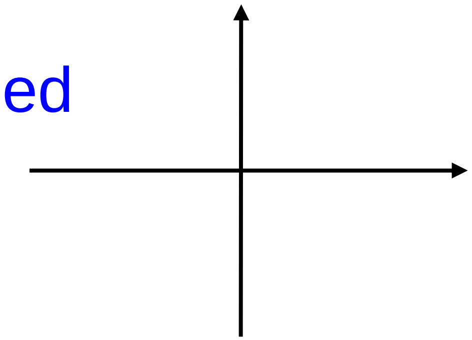
- Causality: $\sum_{n=0}^{+\infty} h[n] z^{-n}$



  - For a causal LTI system,
    h[n] = 0 for n < 0,   and   thus is right-sided

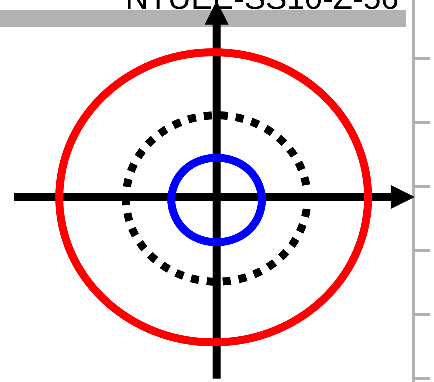


  - A DT LTI system is causal
    if and only if
    the ROC of the system function H(z)
    is   the exterior of a circle in the z-plane, including infinity

  - A DT LTI system with a rational H(z) is causal
    if and only if
    (a) ROC is exterior of a circle outside outermost pole;
    and infinity must be in the ROC; and
    (b) order of numerator <= order of denominator

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N} = \frac{Y(z)}{X(z)} \qquad y[n+N] + \cdots = x[n+M] + \cdots$$

- ## Example 10.21:



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$$H(z) = \frac{\qquad}{(1- \quad z^{-1})(1- \quad z^{-1})} = \frac{\qquad}{(z- \quad)(z- \quad)}$$

$\Rightarrow$ *ROC*: the .... of a circle of outside the outermost pole

$\Rightarrow$ the impulse response is ....-sided

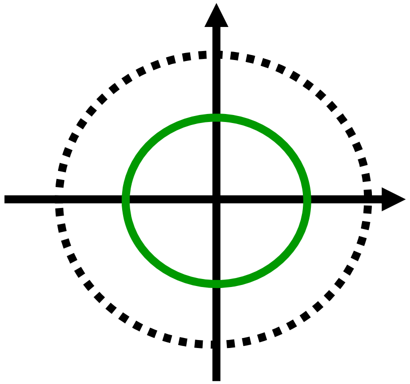
$\Rightarrow$ deg of num of $H(z)$ ..... deg of den of $H(z)$

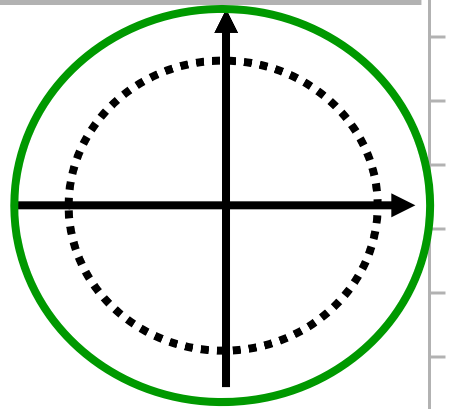
$\Rightarrow$ the system is ....

$\Rightarrow$ $h[n] =$

$\Rightarrow$ $h[n] = 0, n$

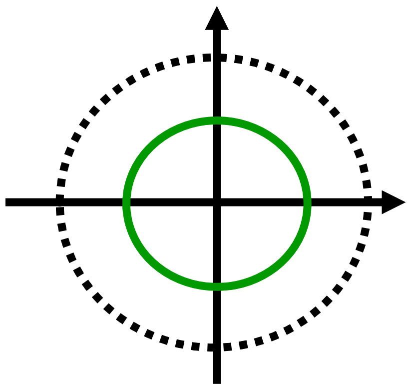
Stability:



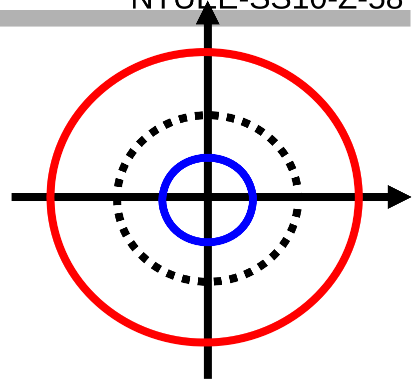


- An DT LTI system is stable
  if and only if
  the ROC of $H(z)$ includes the unit circle [i.e., $|z| = 1$]

- A causal LTI system with rational $H(z)$ is stable
  if and only if
  all of the poles of $H(z)$ lie in the inside the unit circle,
  i.e., all of the poles have magnitude < 1



- Example 10.22:



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$\Rightarrow$ ROC .... include the unit circle $\Rightarrow$ .... stable

$\Rightarrow$ i.e., $h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right] u[n] \rightarrow$, as $n \rightarrow \infty$

- If $ROC = |z| < 1/2$ $\Rightarrow h[n] =$

  $\Rightarrow$ the system is .... causal .... stable

- If $ROC = 1/2 < |z| < 2$ $\Rightarrow h[n] =$

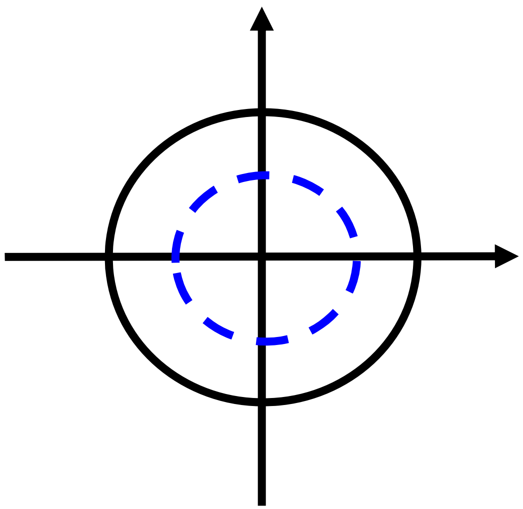
  $\Rightarrow$ the system is .... causal .... stable

Example 10.24:

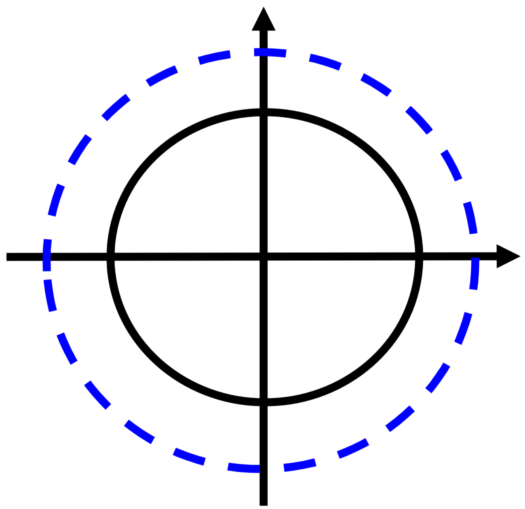
$$H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}} \qquad = \frac{z^2}{z^2 - (2r\cos\theta)z + r^2}$$

$$\Rightarrow \ z_1 = \qquad\qquad z_2 =$$

If it is causal, $|z| \quad |r|$



$r < 1$, ... stable



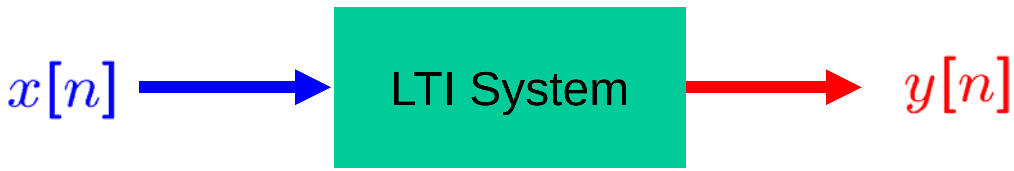
$r > 1$, ... stable

- LTI Systems by Linear Constant-Coef Difference Equations:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



$$Y(z) = X(z) H(z) \qquad H(z) = \frac{Y(z)}{X(z)}$$

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$$

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\left\{y[n-k]\right\} = \sum_{k=0}^{M} b_k \mathcal{Z}\left\{x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

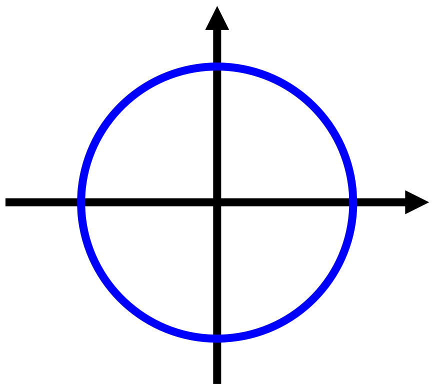
zeros

poles

- **Example 10.25:**

$$y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{3} x[n-1]$$

$$\Rightarrow Y(z) \qquad z^{-1} Y(z) = X(z) \qquad z^{-1} X(z)$$



$$\Rightarrow H(z) = \frac{}{} = \frac{}{1 - \quad z^{-1}} + \frac{}{1 - \quad z^{-1}}$$

$$= \frac{z}{z} = \frac{}{z} + \frac{}{z}$$

- If $ROC = \left\{ |z| > 1/2 \right\}$, $\Rightarrow h[n]$ is ....-sided

$$\Rightarrow h[n] =$$

- If $ROC = \left\{ |z| < 1/2 \right\}$, $\Rightarrow h[n]$ is ....-sided

$$\Rightarrow h[n] =$$

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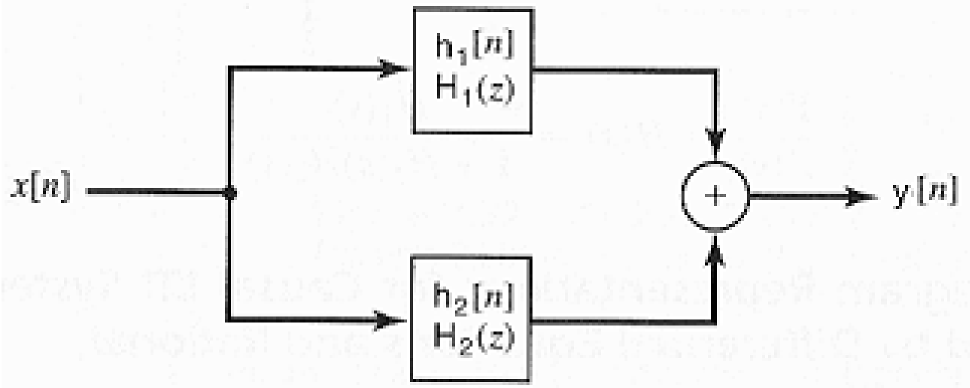
▪ System Function Blocks:

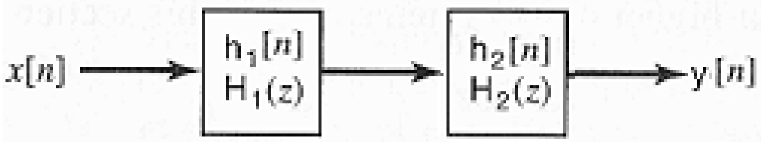
- parallel interconnection



$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

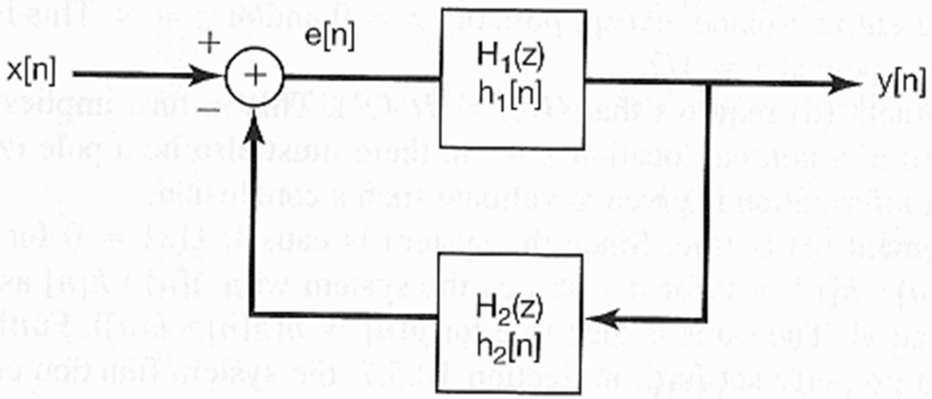
- series interconnection



$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z)\ H_2(z)$$

- feedback interconnection



$$Y = H_1\ E$$

$$Z = H_2\ Y$$

$$E = X - Z$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z) H_2(z)}$$

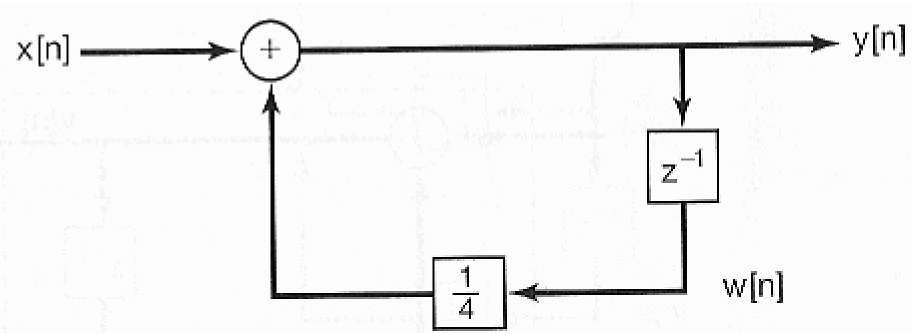
Example 10.28:

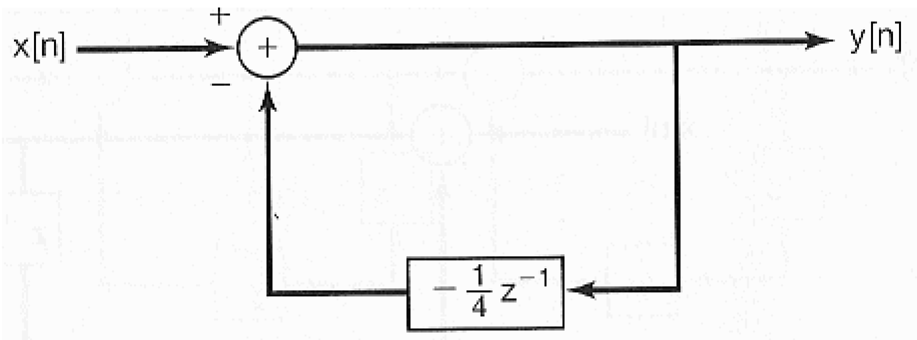
- Consider a causal LTI system with system function

$$H(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$\Rightarrow \; y[n] = \frac{1}{1 - \frac{1}{4} z^{-1}} x[n] \quad \Rightarrow \; y[n] - \frac{1}{4} y[n-1] = x[n]$$

$$\Rightarrow \; y[n] = \frac{1}{4} y[n-1] + x[n]$$



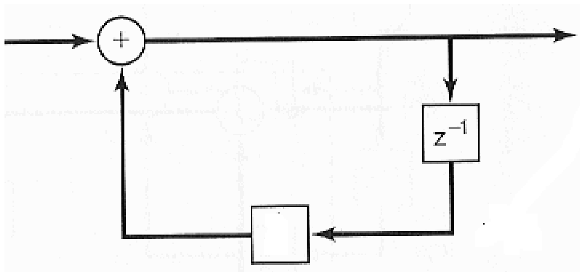


$$w[n] = y[n-1]$$

- Example 10.29:

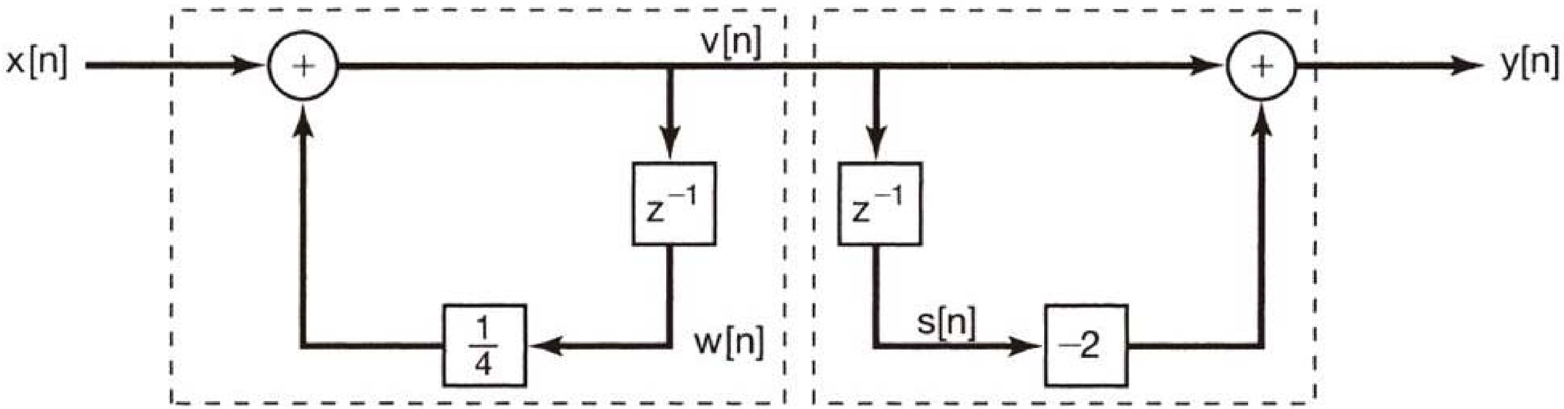
$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{\quad}{\quad}\right)(\qquad)$$

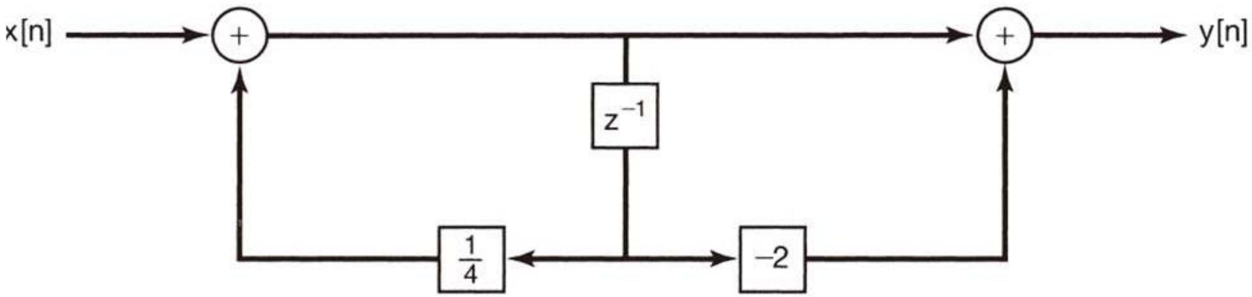
$$Y(z) = (\qquad)V(z) \qquad V(z) = \left(\frac{\quad}{\quad}\right)X(z)$$



- Example 10.29:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1})$$





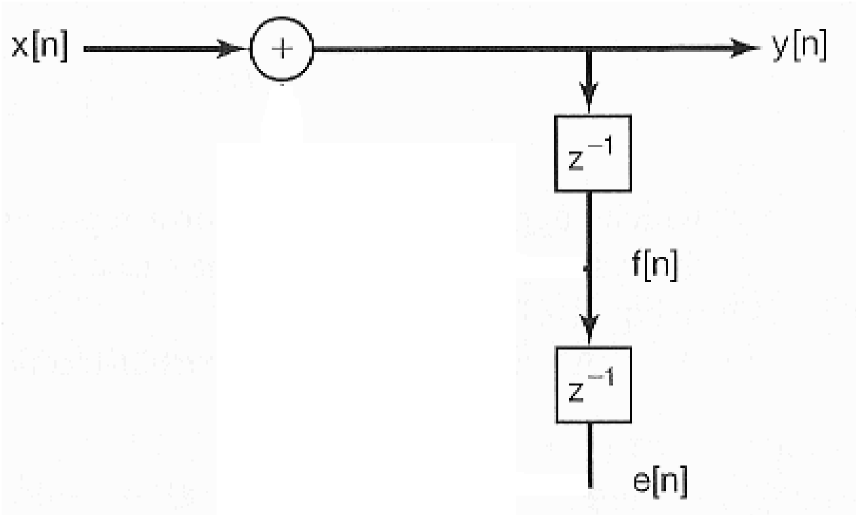
▪ Example 10.30:

$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1 \qquad z^{-1} \qquad z^{-2}}$$

$$\Rightarrow \ y[n] \qquad y[n-1] \qquad y[n-2] = x[n]$$

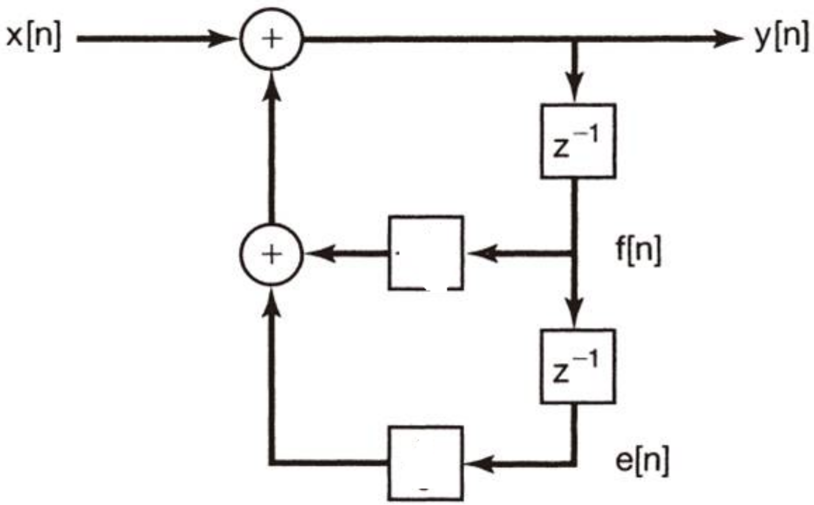
$$\Rightarrow \ y[n] = \qquad y[n-1] \qquad y[n-2] + x[n]$$

$$\Rightarrow \begin{cases} y[n-1] = f[n] \\ y[n-2] = e[n] = f[n-1] \end{cases}$$



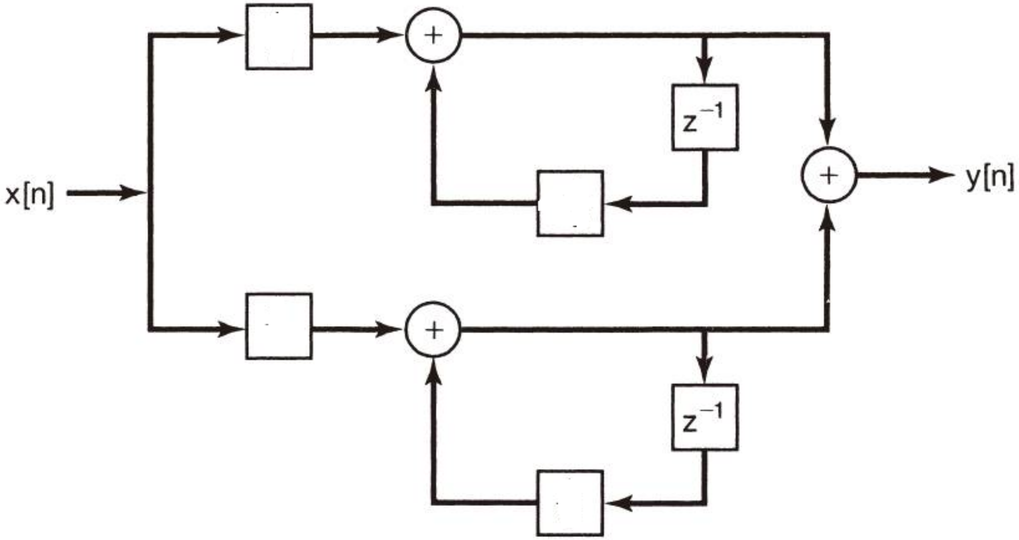
- Example 10.30:

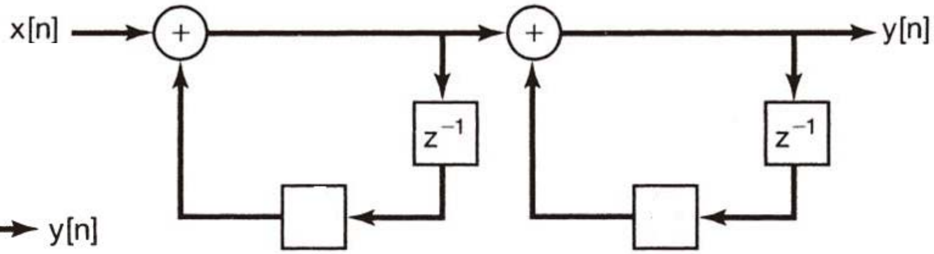
$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$



$$H(z) = \left(\frac{1}{1 + \frac{1}{2}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)$$

$$H(z) = \frac{\frac{2}{3}}{(1 + \frac{1}{2}z^{-1})} + \frac{\frac{1}{3}}{(1 - \frac{1}{4}z^{-1})}$$





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▪ The Unilateral z-Transform of x(t):

bilateral zT

unilateral zT

for causal system & with nonzero init. cond.

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$\mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} x[n] z^{-n} + \sum_{n=0}^{+\infty} x[n] z^{-n}$$

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$x[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z)$$

$$X(z) = \mathcal{Z}\{x[n]\}$$

$$\mathcal{X}(z) = \mathcal{UZ}\{x[n]\}$$

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

$$x[n] = \mathcal{UZ}^{-1}\{\mathcal{X}(z)\}$$

ROC : exterior of a circle

Time-Shifting Property

$$x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z)$$

$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n] z^{-n} \quad = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$x[n-1] \xleftrightarrow{\mathcal{UZ}} z^{-1}\,\mathcal{X}(z) + x[-1]$$

$$\sum_{n=0}^{+\infty} x[n-1] z^{-n} \quad = x[-1] + x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \cdots$$

$$z^{-1}\mathcal{X}(z) \quad = \quad x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \cdots$$

$$x[n-2] \xleftrightarrow{\mathcal{UZ}} z^{-2}\,\mathcal{X}(z) + x[-1]z^{-1} + x[-2]$$

$$x[n+1] \xleftrightarrow{\mathcal{UZ}} z\,\mathcal{X}(z) - zx[0]$$

- Example 10.33: $x[n] = a^{n+1}u[n+1]$

  - since $x[-1] = 1 \neq 0$, $X(z) \neq \mathcal{X}(z)$

$\Rightarrow$ bilateral transform : $X(z) = \dfrac{z}{1 - az^{-1}}, \quad |z| > |a|$

$\Rightarrow$ unilateral transform :
$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n]z^{-n} = \sum_{n=0}^{+\infty} a^{n+1} z^{-n} = \frac{a}{1 - az^{-1}}, \quad |z| > |a|$$

- Example 10.33:

$$x_1[n] = a^n u[n] \qquad \mathcal{X}_1(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

$$x_2[n] = x_1[n+1] \qquad \mathcal{X}_2(z) = z\frac{z}{z-a} - z\,x_1[0]$$

$$= a^{n+1}u[n+1] \qquad = \frac{z^2}{z-a} - z \cdot 1$$

$$= \frac{z^2 - z^2 + az}{z-a}$$

$$= \frac{az}{z-a}$$

$$= \frac{a}{1-z^{-1}a}$$

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$\boxed{z^{-1}\mathcal{X}(z) + x[-1]}$
Time advance	$x[n+1]$	$\boxed{z\mathcal{X}(z) - zx[0]}$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$ $z_0^n x[n]$ $a^n x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$ $\mathcal{X}(z/z_0)$ $\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \text{ for any } m \end{cases}$	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$\boxed{(1 - z^{-1})\mathcal{X}(z) - x[-1]}$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

$z^{-1}\mathcal{X}(z) + x[-1]$

$z\mathcal{X}(z) - zx[0]$

$x_1[n] = x_2[n] \equiv 0, \quad n < 0$

$(1 - z^{-1})\mathcal{X}(z) - x[-1]$

Summary of Fourier Transform and z Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

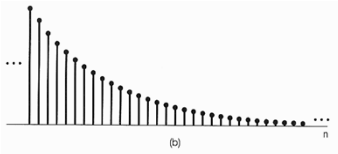
$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

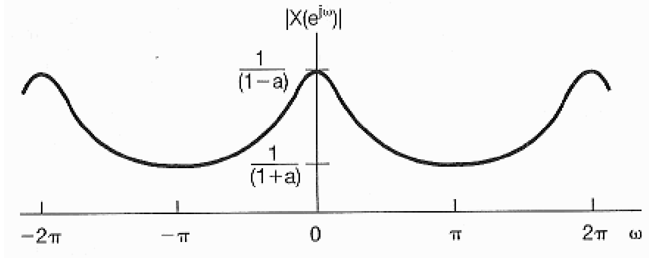
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(z) = X(re^{jw}) = \mathcal{Z}\left\{ x[n] \right\} = \mathcal{F}\left\{ x[n] r^{-n} \right\}$$

$$X(e^{jw}) = \mathcal{F}\left\{ x[n] \right\} = \mathcal{Z}\left\{ x[n] \right\}\Big|_{z=e^{jw}} = X(z)\Big|_{z=e^{jw}}$$

Summary of Fourier Transform and z Transform





$$x[n] = a^n\, u[n], \quad |a| < 1 \qquad \stackrel{\mathcal{F}}{\longleftrightarrow} \qquad X(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$x[n] = a^n u[n] \qquad \stackrel{\mathcal{L}}{\longleftrightarrow} \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

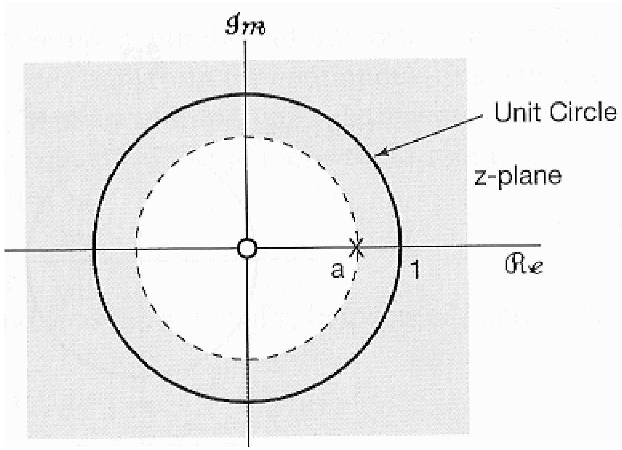
$$|z| > |a|$$

definition
theorem
property

Causality

Stability

ROC



- The z-Transform
- The ROC for z-T
- The Inverse z-T
- Geometric Evaluation of the FT
- Properties of the z-T
  - Linearity
  - Time Shifting
  - Shifting in the z-Domain
  - Time Reversal
  - Time Expansion
  - Conjugation
  - Convolution
  - First Difference
  - Accumulation
  - Differentiation in the z-Domain
  - Initial-Value Theorems
- Some Common z-T Pairs
- Analysis & Charac. of LTI Systems Using the z-T
- System Function Algebra, Block Diagram Repre.
- The Unilateral z-T

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS CT DT (Chap 3)

Aperiodic

FT CT (Chap 4)

DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)