Spring 2013

信號與系統
Signals and Systems

Chapter SS-9
The Laplace Transform

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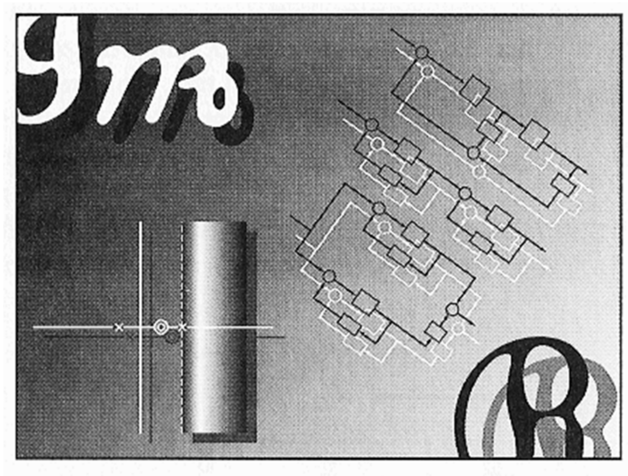
NTU-EE

Feb13 – Jun13



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS CT DT (Chap 3)

Aperiodic

FT CT (Chap 4)

FT DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6)

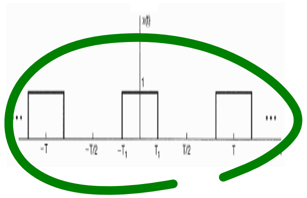
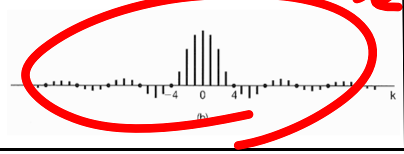
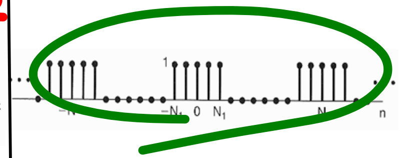
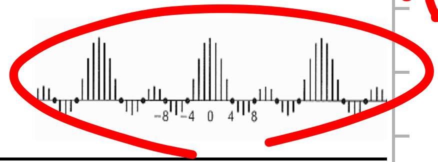
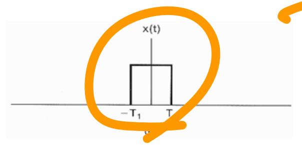
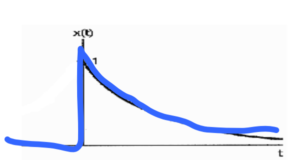
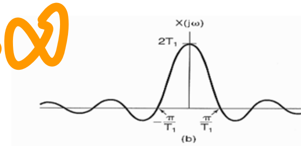
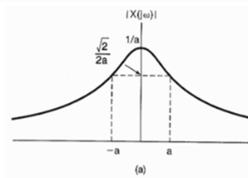
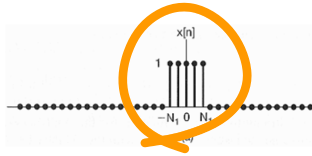
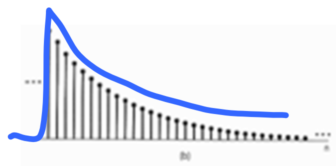
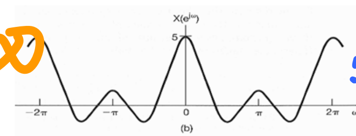
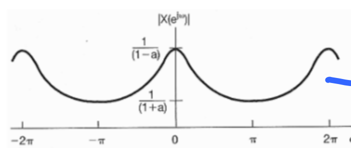
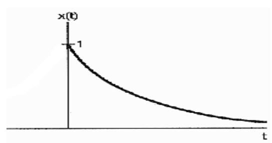
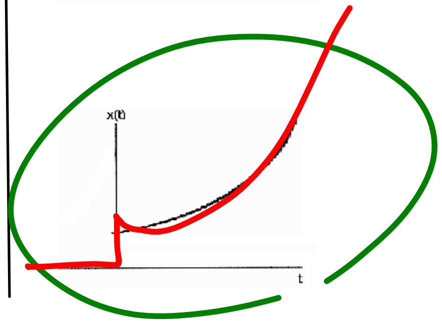
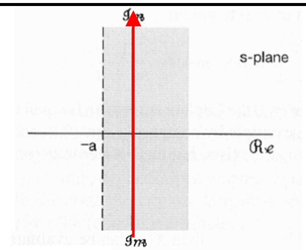
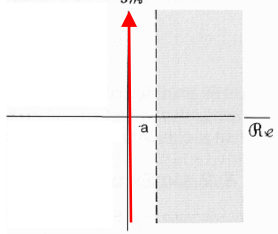
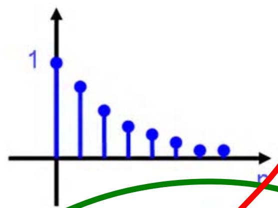
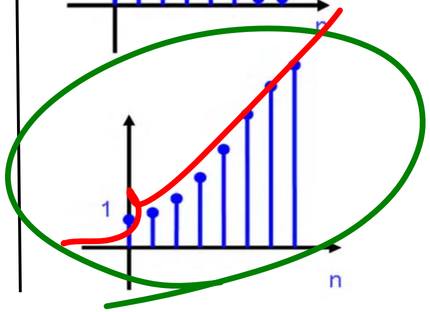
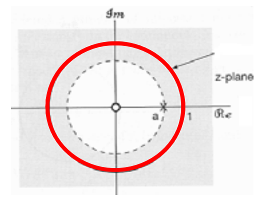
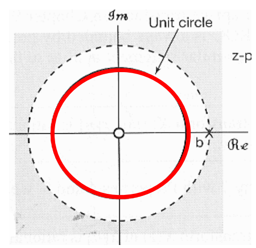
CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)

Fourier Series, Fourier Transform, Laplace Transform, z-Transform

	CT		DT	
	time	frequency	time	frequency
FS		 a_k		 a_k
FT	 $T \to \infty$ 	 ω  ω	 $N \to \infty$ 	 Ω  Ω
LT/zT	 	 	 	 

Signal

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs

System

- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

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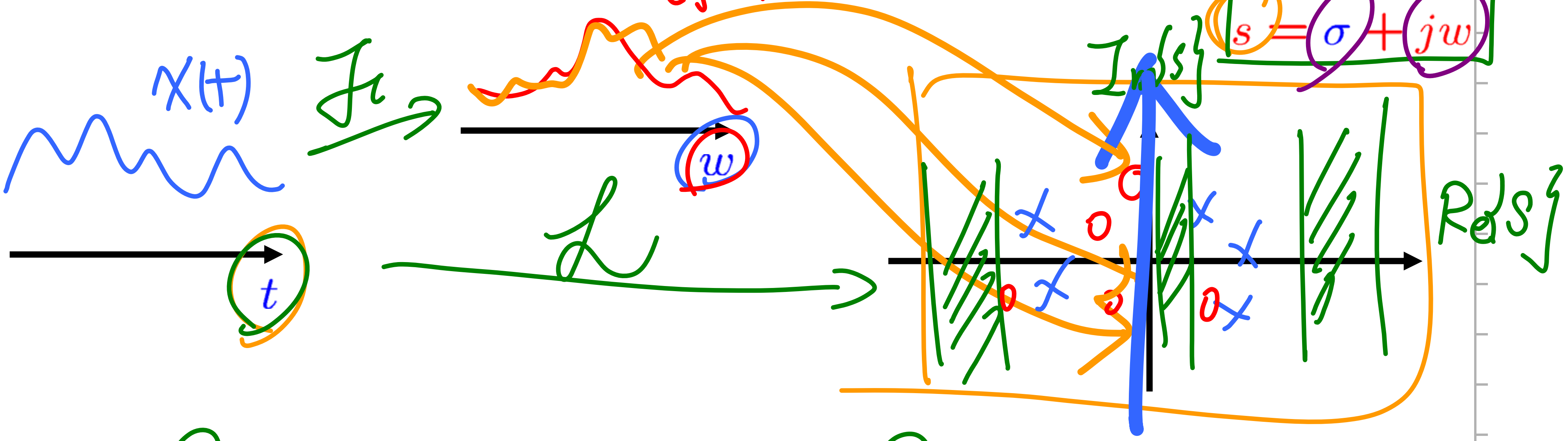
- The Laplace transform of a general signal x(t):

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \qquad X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\boxed{s = \sigma + jw}$$



$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw) \qquad x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(jw) = \mathcal{F}\left\{x(t)\right\} \qquad X(s) = \mathcal{L}\left\{x(t)\right\}$$

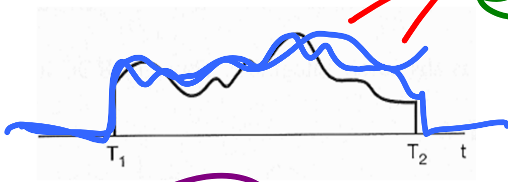
$$x(t) = \mathcal{F}^{-1}\left\{X(jw)\right\} \qquad x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$$

$$X(s)\Big|_{s=jw} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = \mathcal{F}\left\{x(t)\right\} = X(jw)$$

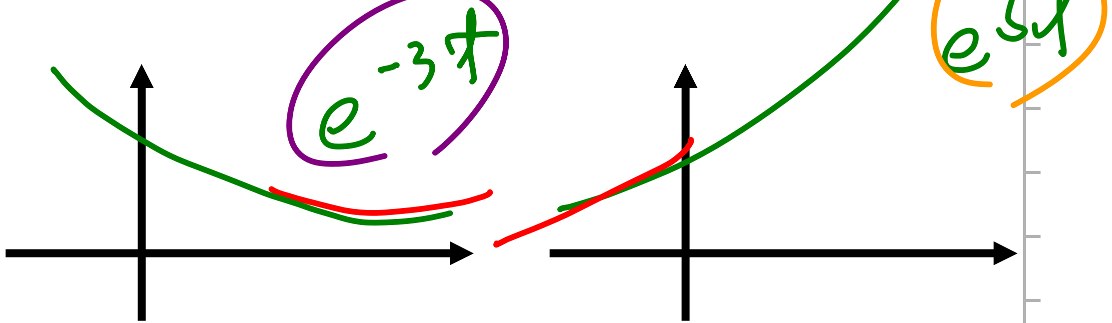
▪ Laplace Transform & Fourier Transform:

$$X(s)\Big|_{s=jw} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = \mathcal{F}\left\{x(t)\right\} = X(jw)$$

$e^{-\sigma t}$



$$\mathcal{L}\left\{x(t)\right\} = X(s) \qquad s = \sigma + jw$$



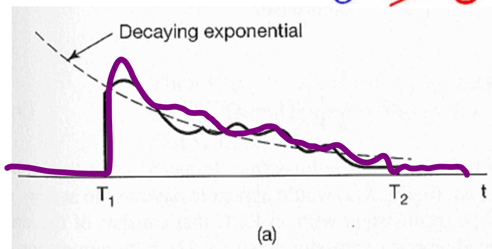
$$= X(\sigma + jw)$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+jw)t}dt$$

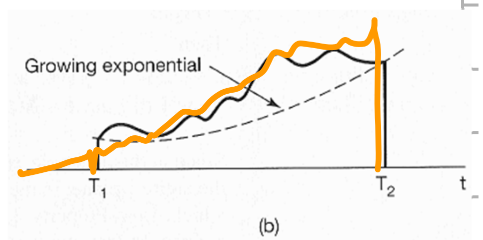
$$= \int_{-\infty}^{\infty} \boxed{\left[x(t)e^{-\sigma t}\right]} e^{-jwt}dt$$

$$= \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$

$\sigma > 0$



$\sigma < 0$

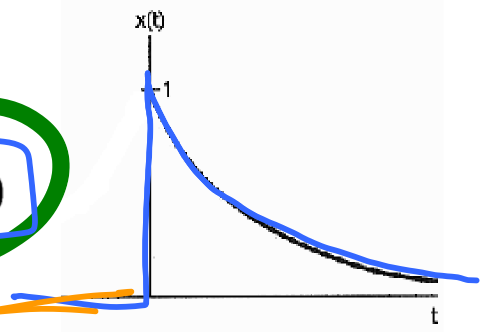


$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

- Example 4.1:

$$x(t) = e^{-at}u(t), \quad \boxed{a > 0}$$

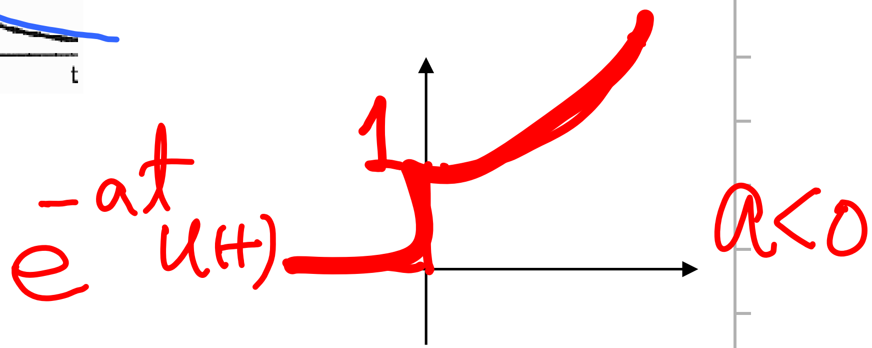


$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$



$$= \boxed{-\frac{1}{a+jw}}\boxed{e^{-(a+jw)t}\Big|_{0}^{\infty}}$$

$+\infty$

$-\infty$

$a>0$

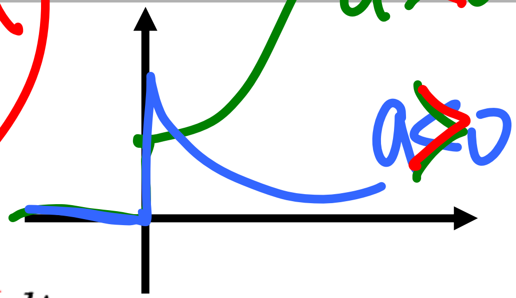
$$= 0 - \left(-\frac{1}{a+jw}e^{-(a+jw)0}\right)$$

$$= \boxed{\frac{1}{a+jw}}, \quad a > 0$$

- **Example 9.1:**

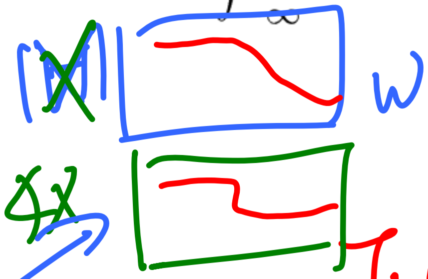
$$x(t) = e^{-at}u(t)$$

e^{2t} $\quad e^{-at}$



$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

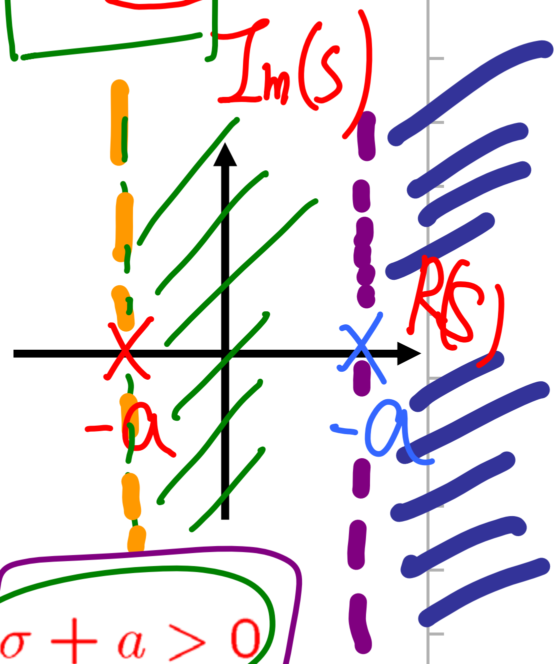
$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$



$$X(jw) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-jwt}dt \qquad = \left.\frac{e^{-(a+jw)t}}{-(a+jw)}\right|_0^{\infty}$$

$$= \frac{1}{jw+a}, \quad a > 0$$



$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \qquad = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(\sigma + jw) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-jwt}dt \qquad = \frac{1}{(\sigma + a) + jw}, \quad \sigma + a > 0$$

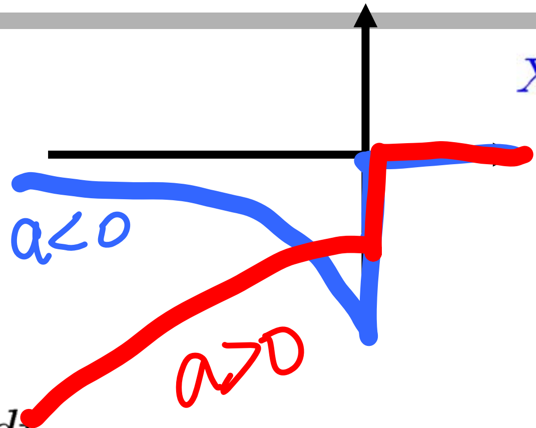
$$= \frac{1}{(\sigma + jw) + a} \qquad = \frac{1}{s + a}, \quad \mathcal{R}e\{s\} > -a$$

$\mathrm{Re}\{s\} > -\mathrm{Re}\{a\}$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- Example 9.2:

$$x(t) = -e^{-at}u(-t)$$



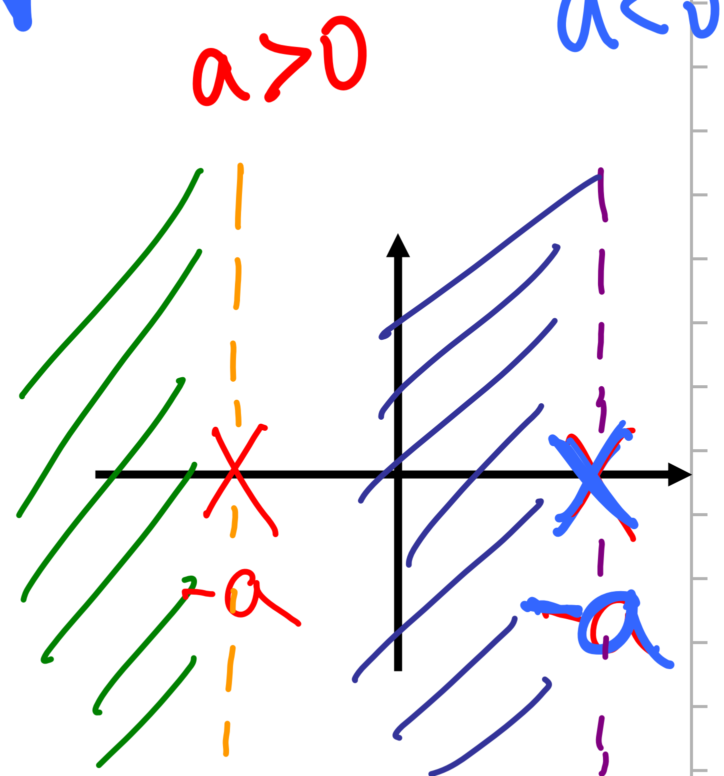
$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}dt$$

e^{-st}

$e^{-\sigma t}$

$$= \boxed{\frac{1}{s+a}}, \quad \mathcal{R}e\{s\} < -a$$



$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

Region of Convergence (ROC)

- **Region of Convergence (ROC):**

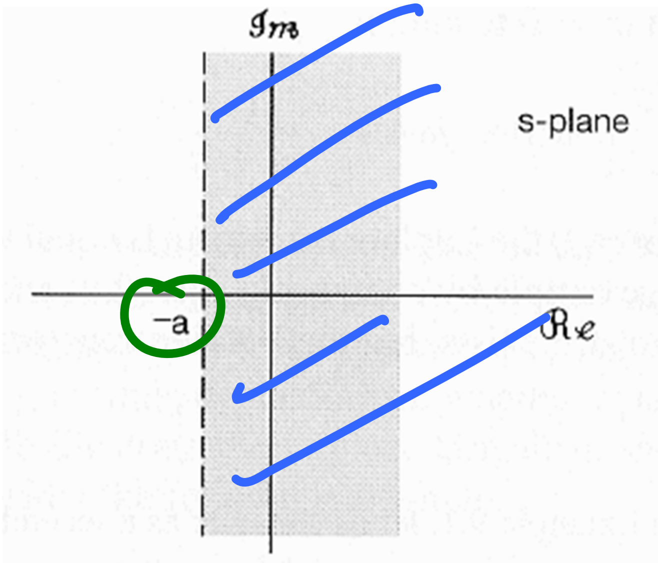
$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

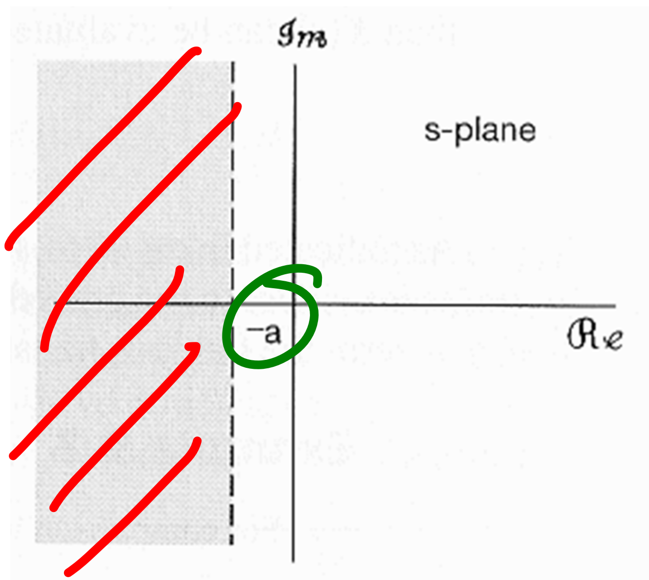
$x(t)e^{-st}$

where Fourier transform of $x(t)e^{-\sigma t}$ converges

$s = \sigma + j\omega$



(a)



(b)

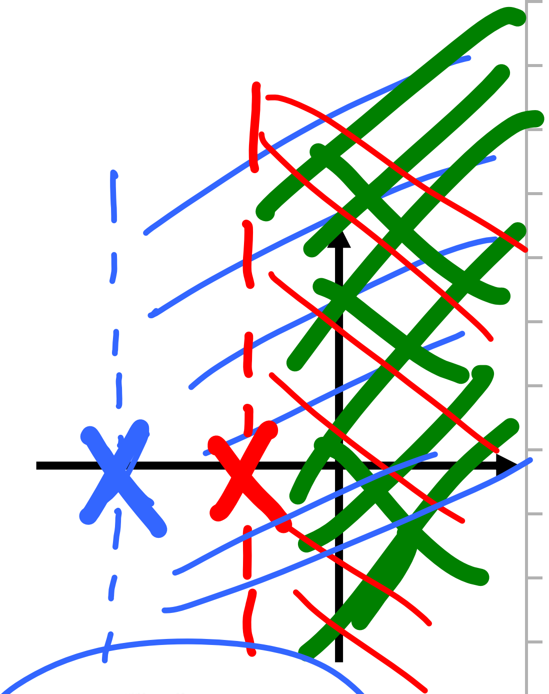
- **Example 9.3:**

$$x(t) = \boxed{3e^{-2t}}u(t) - \boxed{2e^{-t}}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} \boxed{\left[3e^{-2t}u(t) - 2e^{-t}u(t)\right]} e^{-st}dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt$$

$$= 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+1}\right)$$



$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{Re}\{s\} > -2$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \quad \boxed{\mathcal{Re}\{s\} > -1}$$

- Example 9.3:

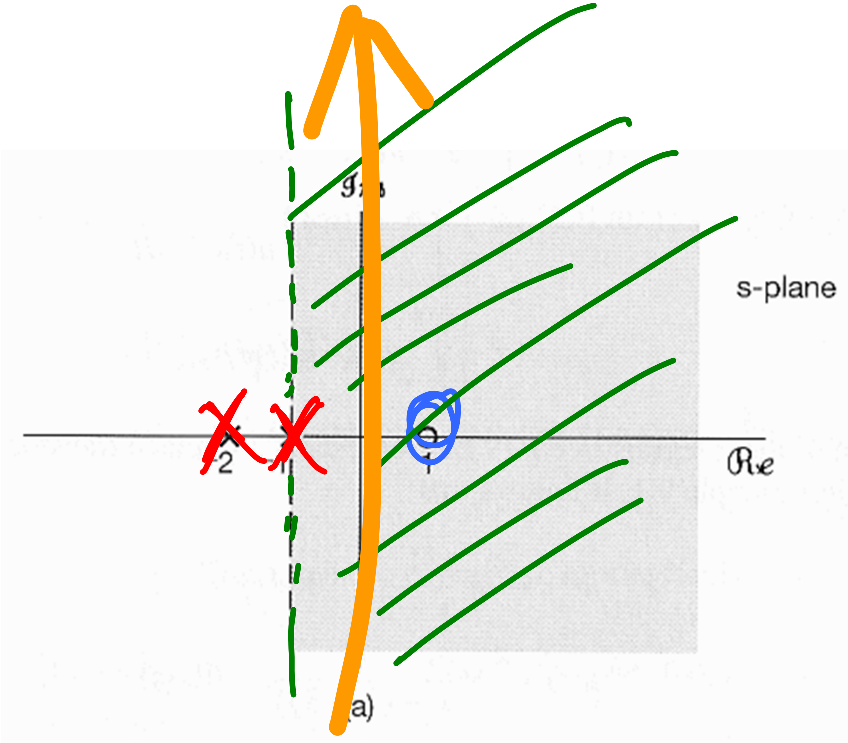
$\mathcal{R}e\{s\} > -2$ $\mathcal{R}e\{s\} > -1$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

$=0$ zeros O

$$\stackrel{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{(s+2)(s+1)}, \quad \boxed{\mathcal{R}e\{s\} > -1}$$

$=0$ poles X



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw $(\sigma = 0)$

- Example 9.4:

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -\mathcal{R}e\{a\}$$

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \qquad \mathcal{R}e\{s\} > -2$$

$$= \left[e^{-2t} + \frac{1}{2}e^{-t}\left(e^{j3t} + e^{-j3t}\right)\right]u(t) \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \qquad \mathcal{R}e\{s\} > -1$$

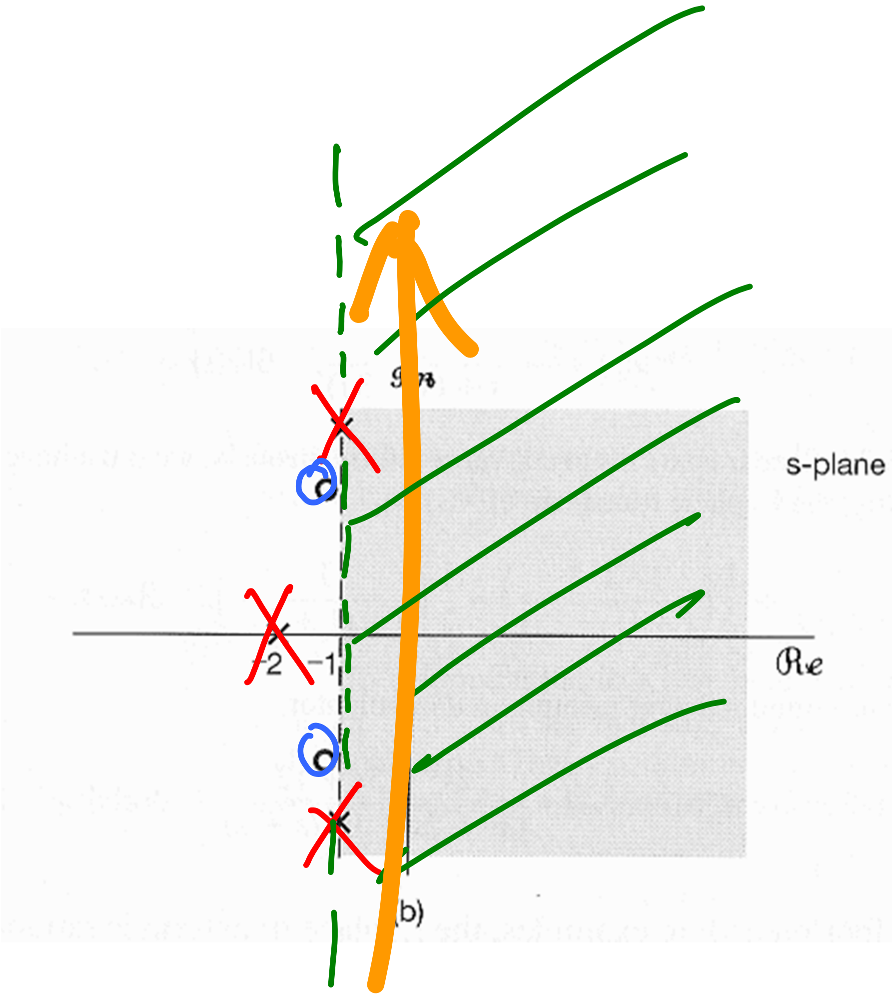
$$X(s) = \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}\right] = \boxed{\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}}$$

- Example 9.4:

$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$

$$e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw

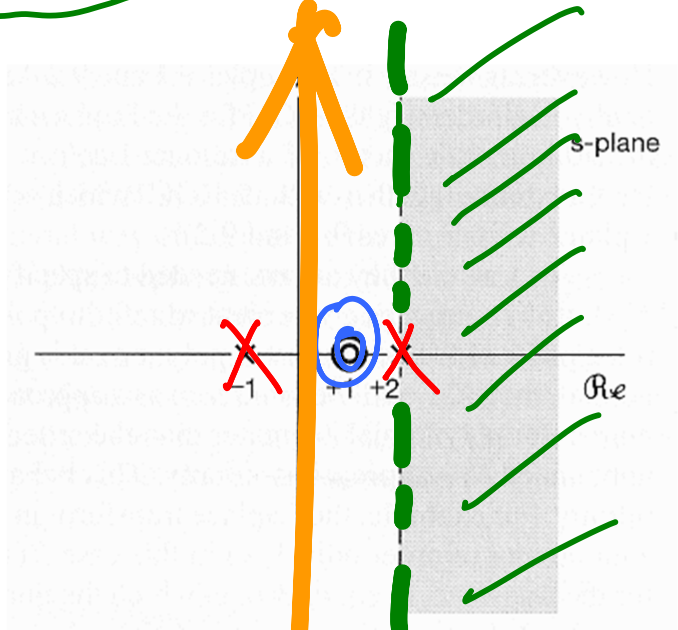
- Example 9.5:

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

$$x(t) = \delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2}, \quad \mathcal{R}e\{s\} > 2$$

$$\delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{R}e\{s\} > 2$$



- The jw-axis is not included in the ROC!
- Fourier transform?
- Why?

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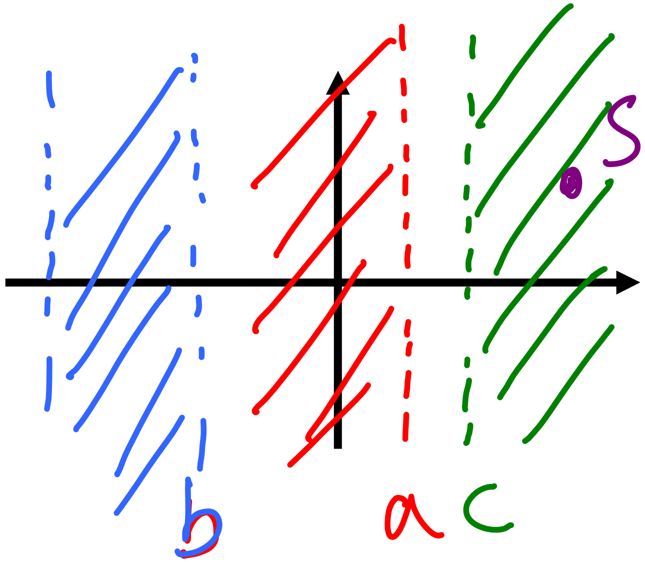
$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw+a}$

$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \boxed{\frac{1}{s+a}}, \boxed{\mathcal{R}e\{s\} > -a}$

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

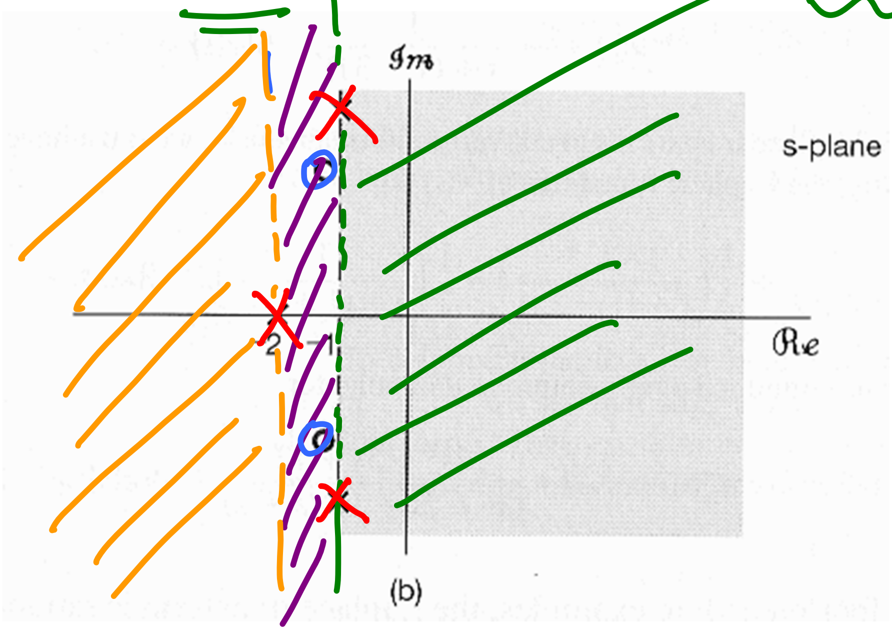
Properties of ROC:

1. The ROC of X(s) consists of strips parallel to the jw-axis in the s-plane



$s = a, b, c$

2. For rational Laplace transforms, the ROC does not contain any poles



(b)

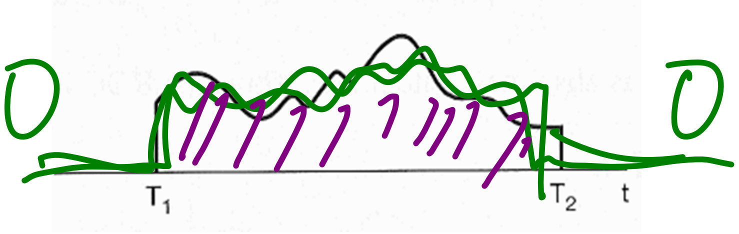
$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$

= 0

= 0

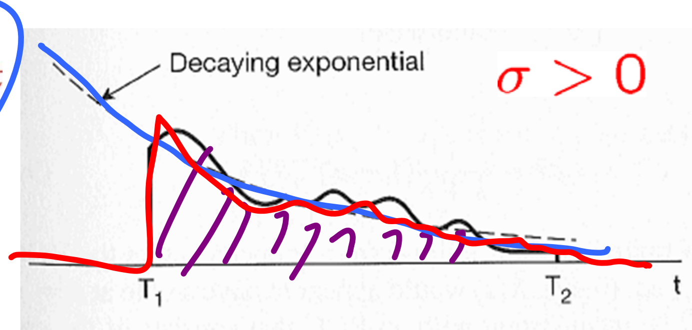
Properties of ROC:

3. If $x(t)$ is of finite duration & is absolutely integrable, then the ROC is the entire s-plane

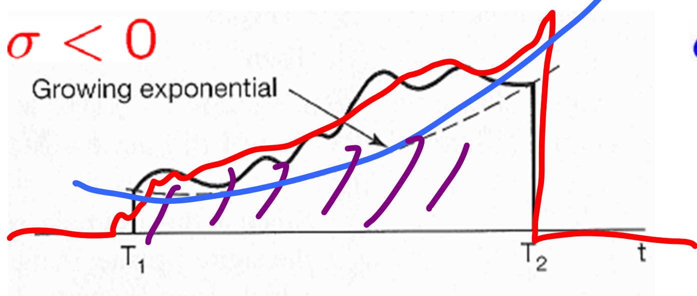


$$\int_{T_1}^{T_2} |x(t)|dt < \infty$$

$e^{-\sigma t}$



(a)



(b)

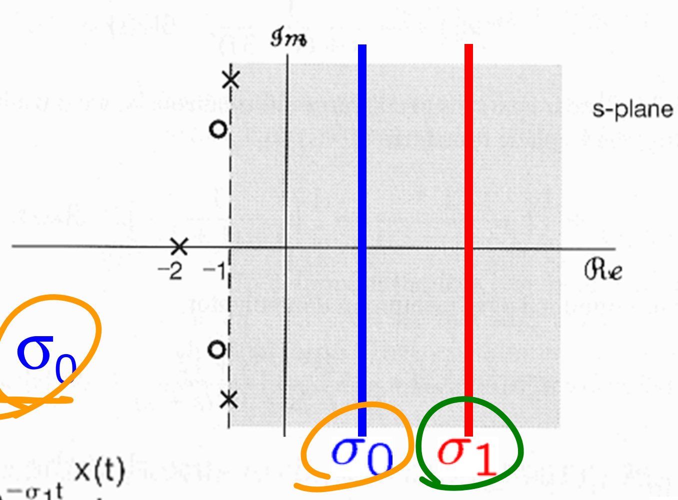
$e^{-\sigma t}$

$$s = \sigma + jw$$

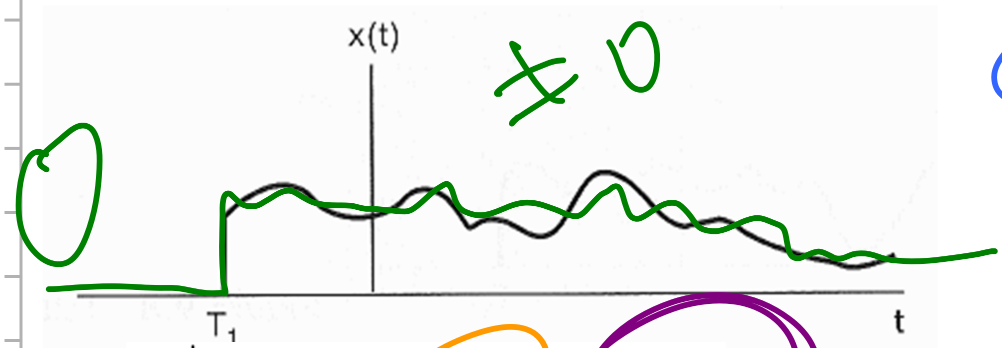
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{T_1}^{T_2} x(t)e^{-st}dt < e^{-\sigma(T_1 \text{ or } T_2)} \int_{T_1}^{T_2} |x(t)|dt$$

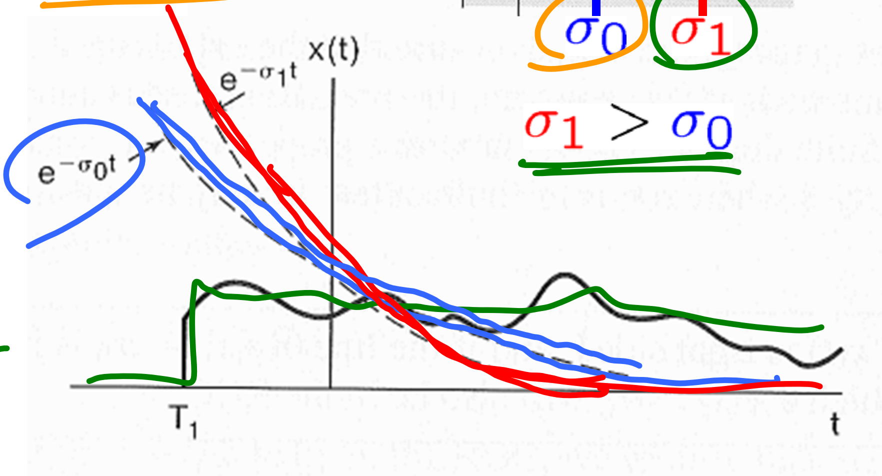
Properties of ROC:

4. If x(t) is right-sided, and
if the line $Re\{s\} = \sigma_0$ is in the ROC,
then all values of s for which $Re\{s\} > \sigma_0$
will also be in the ROC



$\sigma_1 > \sigma_0$





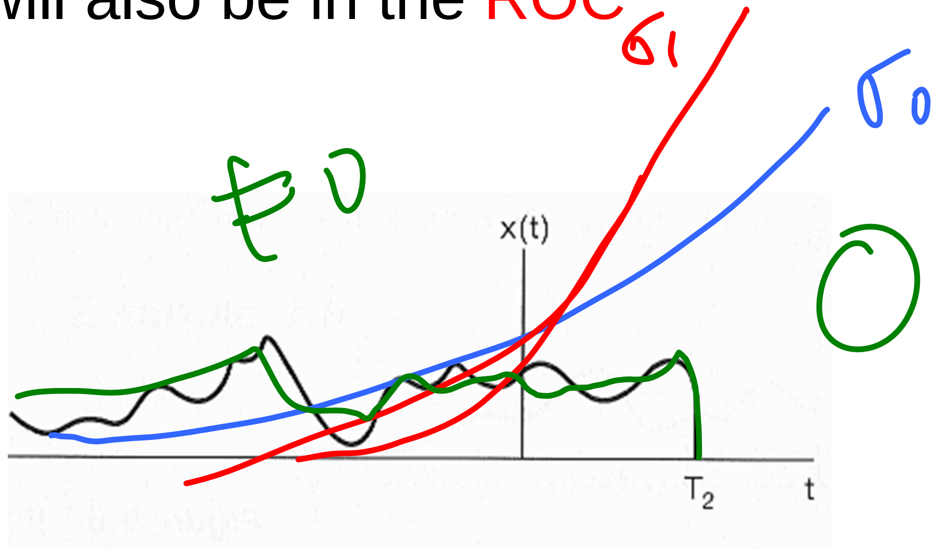
$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

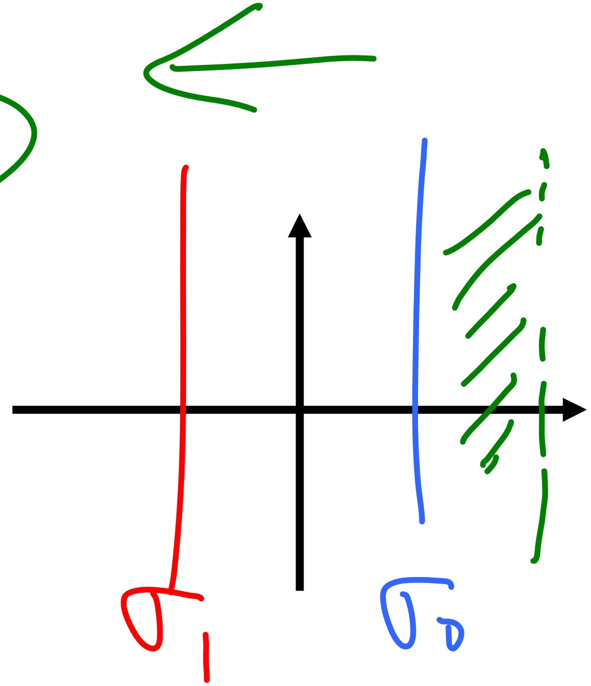
$$\Rightarrow \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{+\infty} |x(t)| e^{-(\sigma_1 + \sigma_0 - \sigma_0)t} dt$$

$$\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

- Properties of ROC:

5. If $x(t)$ is left-sided, and
   if the line $Re\{s\} = \sigma_0$ is in the ROC,
   then all values of s for which $Re\{s\} < \sigma_0$
   will also be in the ROC

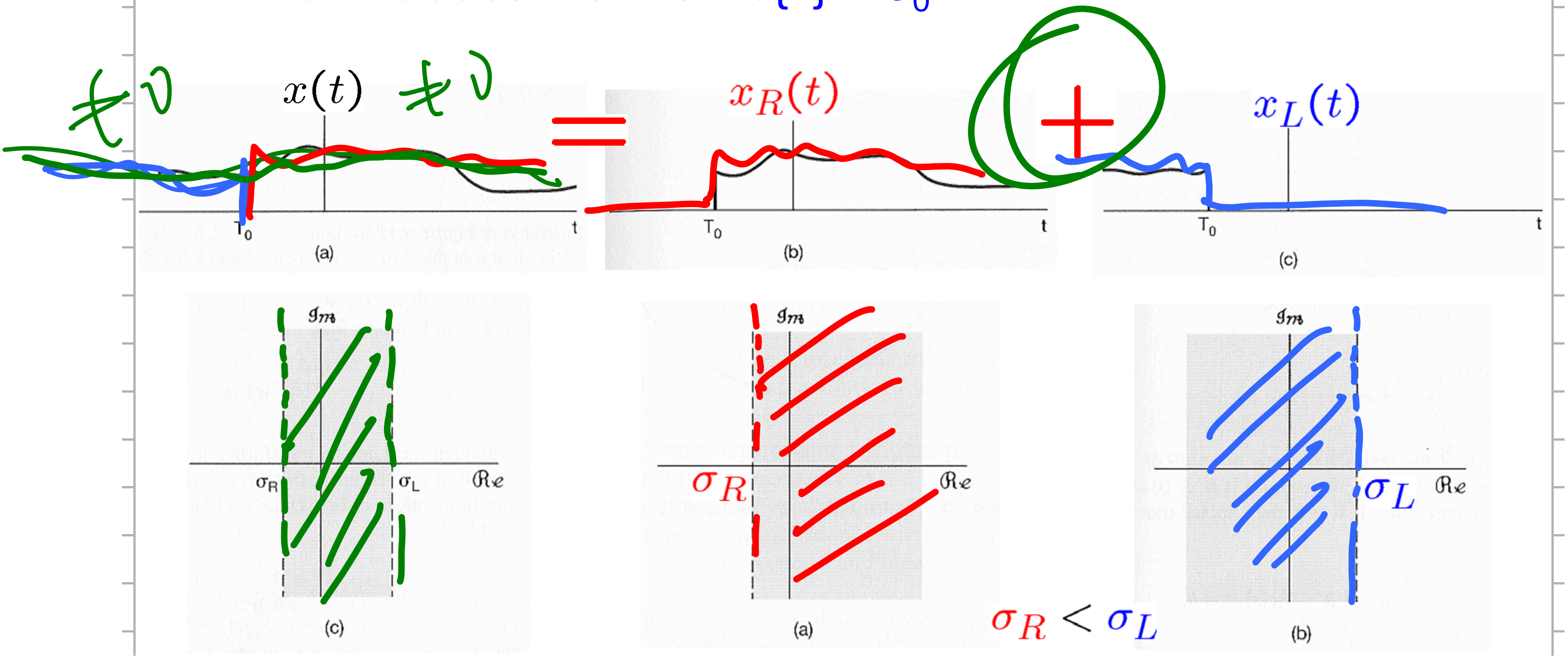




The argument is the similar to that for Property 4.

Properties of ROC:

6. If $x(t)$ is two-sided, and
   if the line $Re\{s\} = \sigma_0$ is in the ROC,
   then the ROC will consist of a strip in the s-plane
   that includes the line $Re\{s\} = \sigma_0$



- **Example 4.2:**

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$x(t) = e^{-a|t|}, \quad \boxed{a > 0}$$

$$\Rightarrow X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt} dt$$

$$= \int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt$$

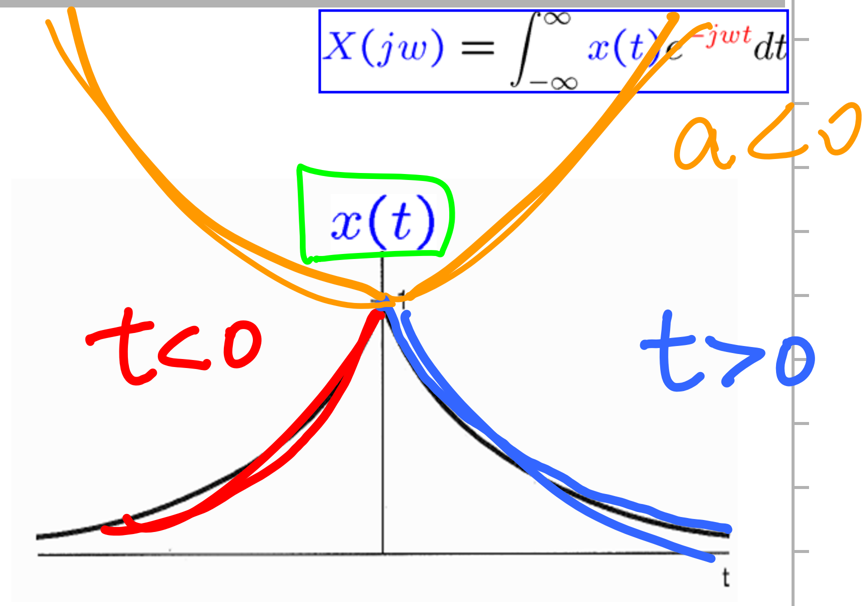
$$= \frac{1}{a - jw} + \frac{1}{a + jw} \quad (a>0)$$

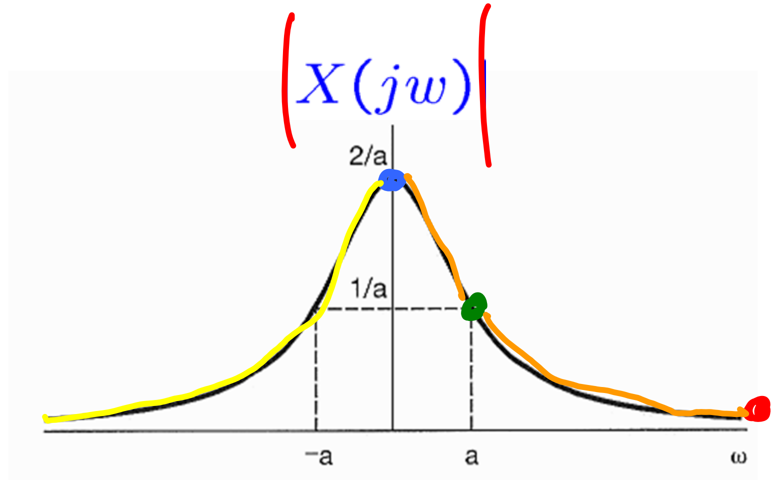
$$= \frac{2a}{a^2 + w^2}$$

$w=0$ $\quad \frac{2a}{a^2}$

$w=a$ $\quad \frac{2a}{a^2+a^2}$

$w \to \infty$ $\quad \frac{2a}{\infty}$





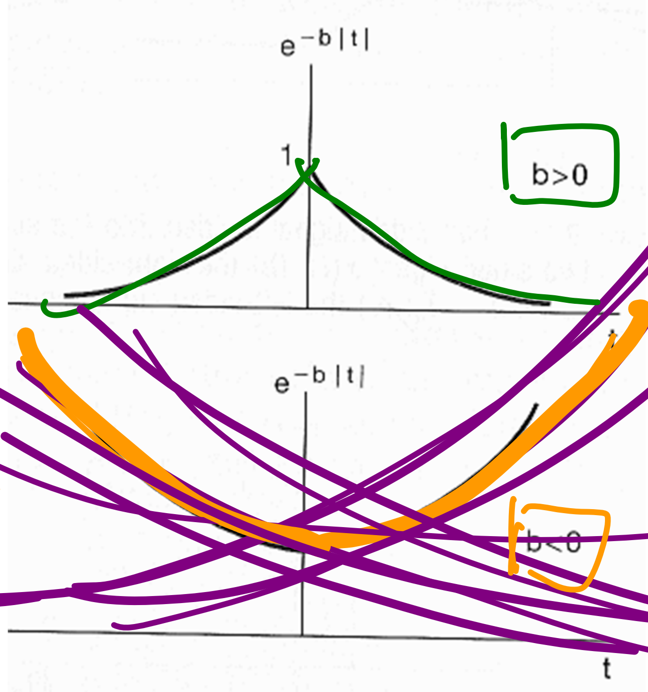
3/21/13
3:11 pm

- Example 9.7:

$$x(t) = e^{-b|t|} \qquad = e^{-bt}u(t) + e^{+bt}u(-t)$$

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \quad \mathcal{R}e\{s\} > -b$$

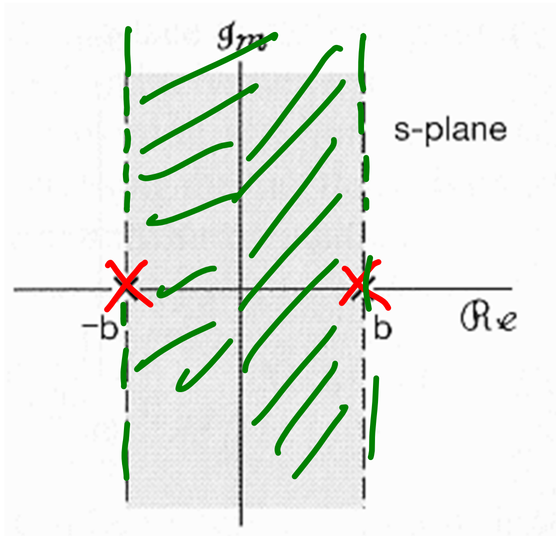
$$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \quad \mathcal{R}e\{s\} < +b$$



- $b > 0$:

$$e^{-b|t|} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} + \frac{-1}{s-b}, \quad -b < \mathcal{R}e\{s\} < +b$$

$$= \frac{-2b}{(s+b)(s-b)}$$



- $b \leq 0$: no common ROC

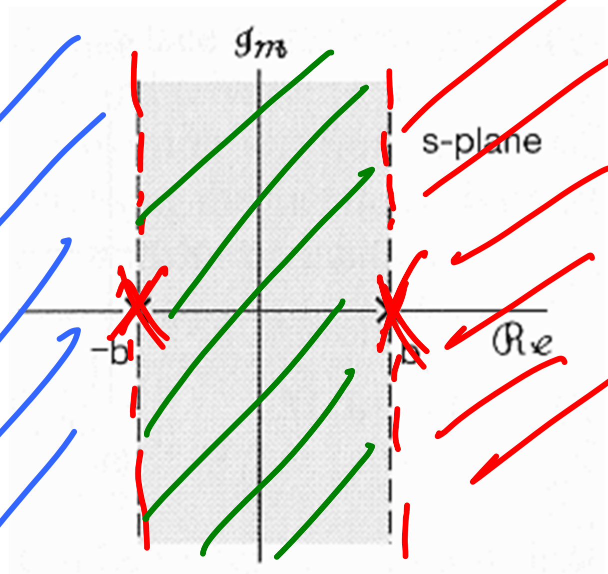
  $x(t)$ has no Laplace transform

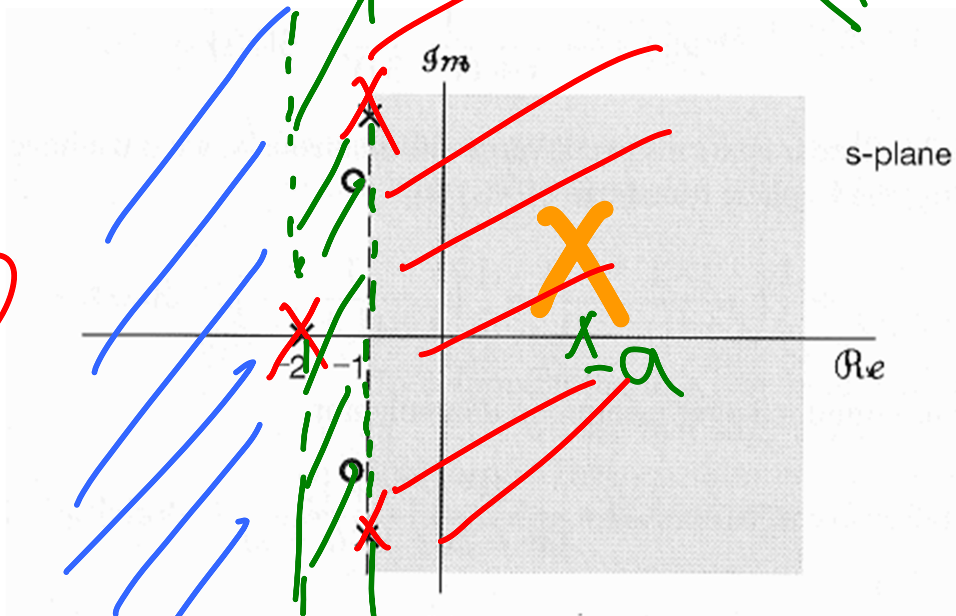
Properties of ROC:

7. If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to ∞. In addition, no poles of X(s) are contained in ROC

$$\frac{-2b}{(s+b)(s-b)}$$

$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$





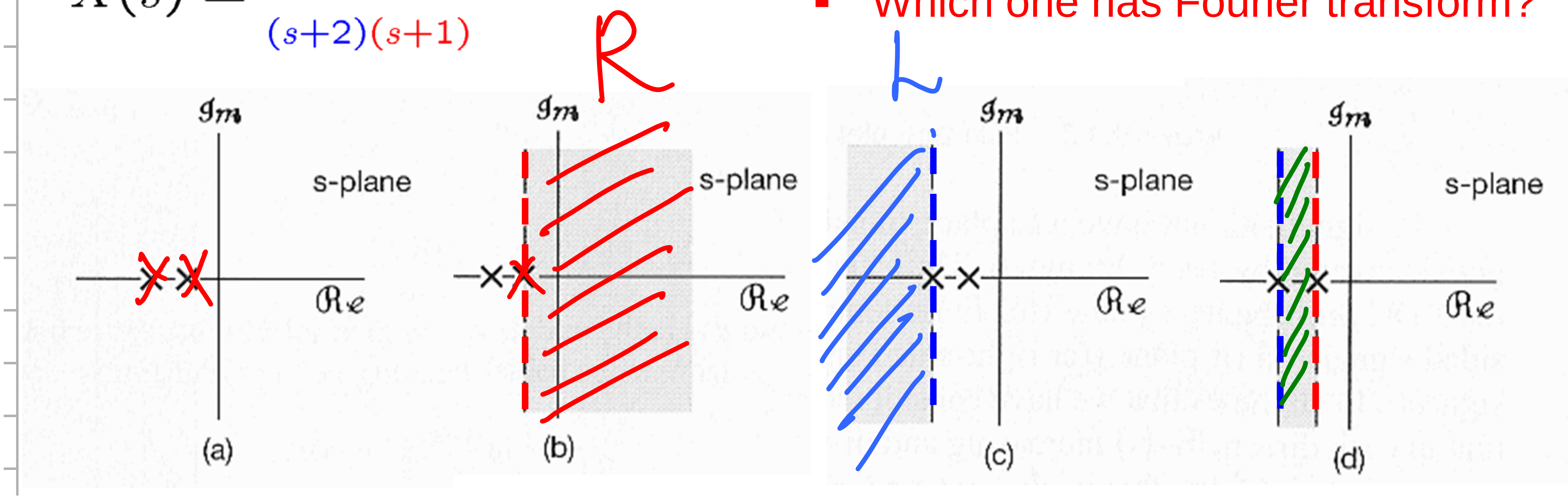
(b)

Properties of ROC:

8. If the Laplace transform X(s) of x(t) is rational
   - If x(t) is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole
   - If x(t) is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole

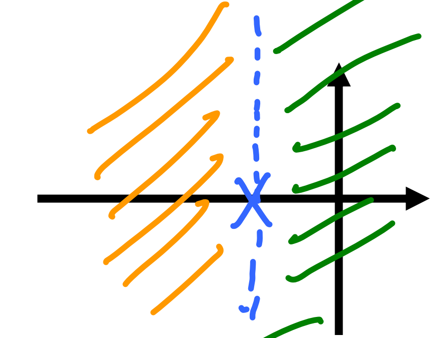
$$X(s) = \frac{1}{(s+2)(s+1)}$$

- Which one has Fourier transform?

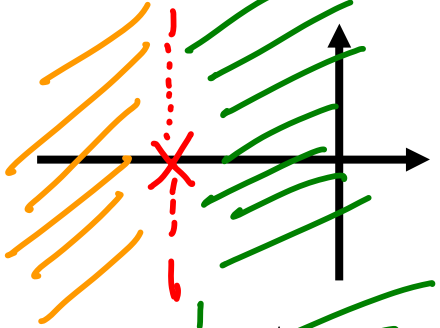


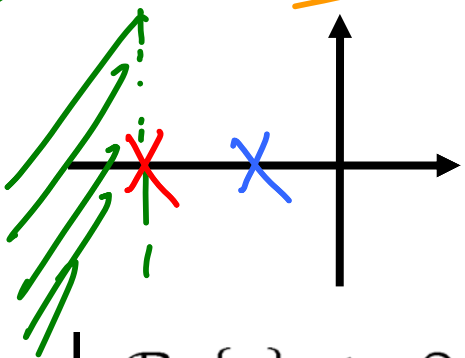
Examples 9.9, 9.10, 9.11:

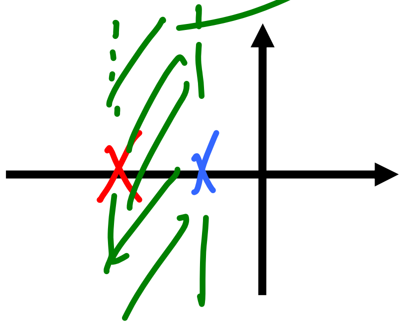
	$\mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)}$	$-\ e^{-t}u(-t)$	$e^{-t}u(t)$

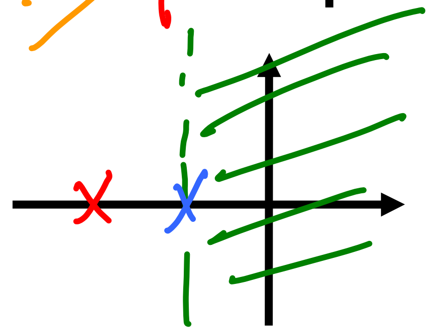


	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\}$
$\frac{1}{(s+2)}$	$-\ e^{-2t}u(-t)$	$e^{-2t}u(t)$









	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)} + \frac{1}{(s+2)}$	$-\ e^{-t}u(-t)$ $+$ $-\ e^{-2t}u(-t)$	$-\ e^{-t}u(-t)$ $+$ $e^{-2t}u(t)$	$e^{-t}u(t)$ $+$ $e^{-2t}u(t)$

$(s+1)(s+2)$

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10:03 am

- The Laplace Transform
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- Properties of the Laplace Transform
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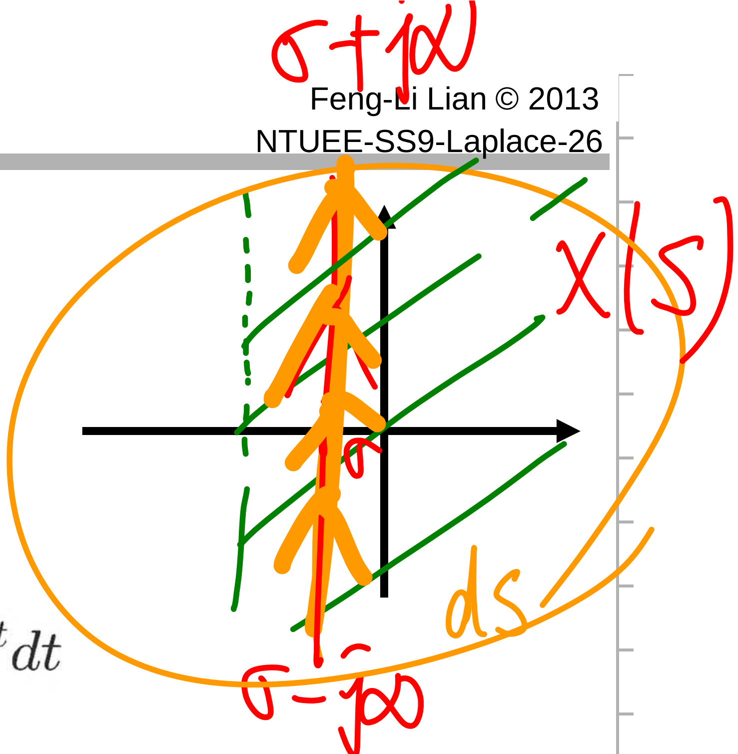
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform



The Inverse Laplace Transform:

- By the use of contour integration

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(\overset{s}{\sigma + jw}) = \mathcal{F}\left\{x(t)e^{-\sigma t}\right\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-jwt}dt$$

$\forall s = \sigma + jw$ in the ROC

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\left\{X(\sigma + jw)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + jw)e^{jwt}dw$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\overset{s}{\sigma + jw})e^{(\overset{s}{\sigma+jw})t}dw$$

$$\boxed{s = \sigma + jw}$$

$$ds = j\,dw$$

I.L.T. $$\Rightarrow \boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds}$$

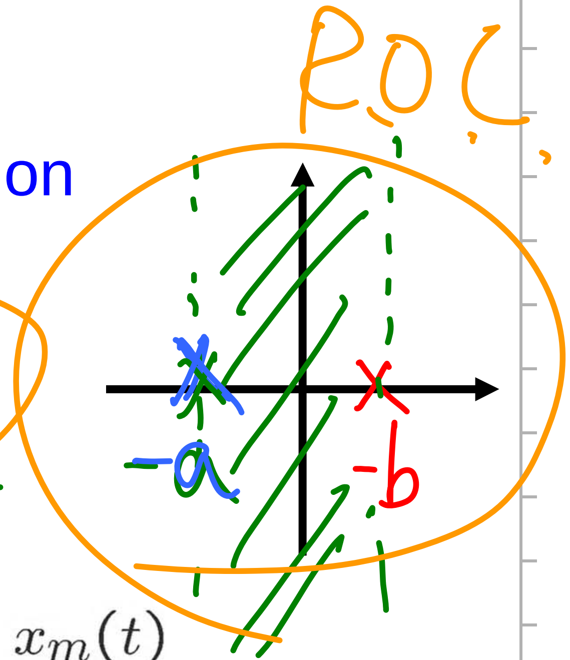
The Inverse Laplace Transform:

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

- By the technique of partial fraction expansion

$$X(s) = \frac{A}{s+a} + \frac{B}{s+b} + \cdots + \frac{M}{s+m}$$



$$x(t) = A e^{-at}u(t) - B e^{-bt}u(-t) + \cdots + x_m(t)$$

(if R.S.) (if L.S.)

- Example 9.9:

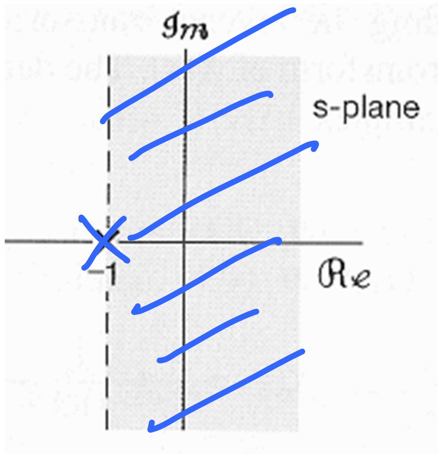
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \boxed{\mathcal{R}e\{s\} > -1}$$

$$= \frac{1}{(s+1)} + \frac{-1}{(s+2)}$$

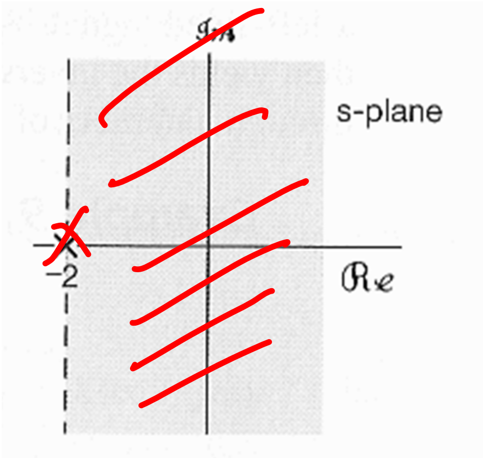
$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{R}e\{s\} > -2$$

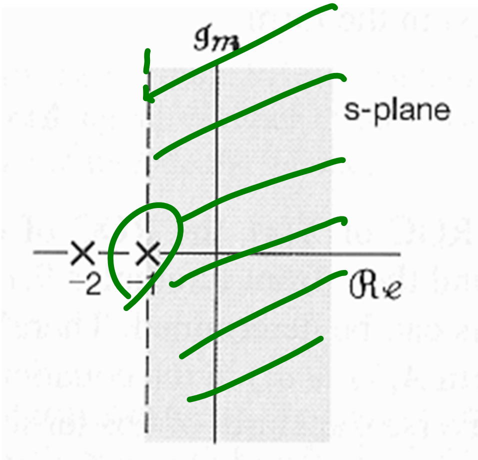
$$\left[e^{-t} + (-1)e^{-2t}\right]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$



(b)



(c)



(a)

- The Laplace Transform
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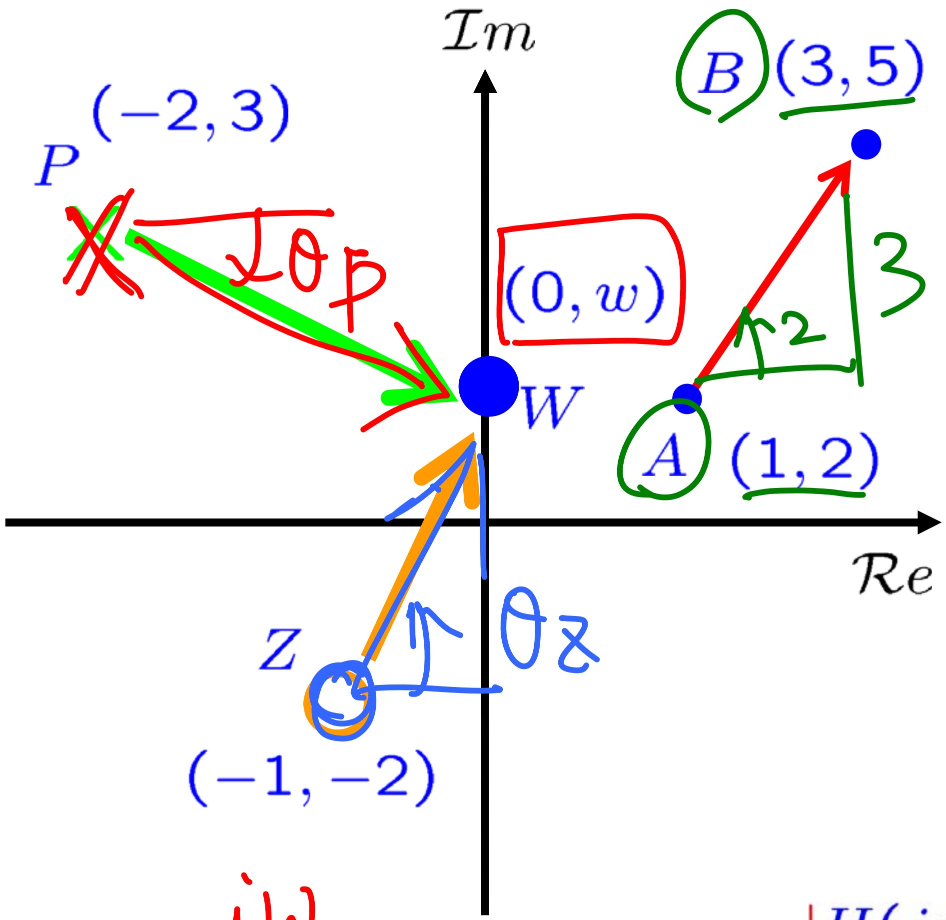
- In s-plane or z-plane:



$$\overrightarrow{AB} = (3+5j) - (1+2j)$$
$$= 2+3j$$

$$\overrightarrow{AB} = (3,5) - (1,2) = (2,3)$$

$$|\overrightarrow{AB}| = \sqrt{2^2+3^2}$$

$$\measuredangle\,\overrightarrow{AB} = \tan^{-1}\frac{(5-2)}{(3-1)}$$

$$H(s) = \frac{s-(-1-2j)}{s-(-2+3j)} \qquad \text{(} s \to jw\text{; } s-(-1-2j)=0\text{, } s-(-2+3j)=0\text{)}$$

$H(jw)$

$$|H(jw)| = \frac{|jw-(-1-2j)|}{|jw-(-2+3j)|} = \frac{|\overrightarrow{ZW}|}{|\overrightarrow{PW}|}$$

$$\measuredangle\, H(jw) = \measuredangle\,\overrightarrow{ZW} - \measuredangle\,\overrightarrow{PW}$$

$$\theta_z - \theta_p$$

$(0, \infty)$

- **First-Order Systems:**

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(jw) = \frac{\frac{1}{\tau}}{jw + \frac{1}{\tau}}$$

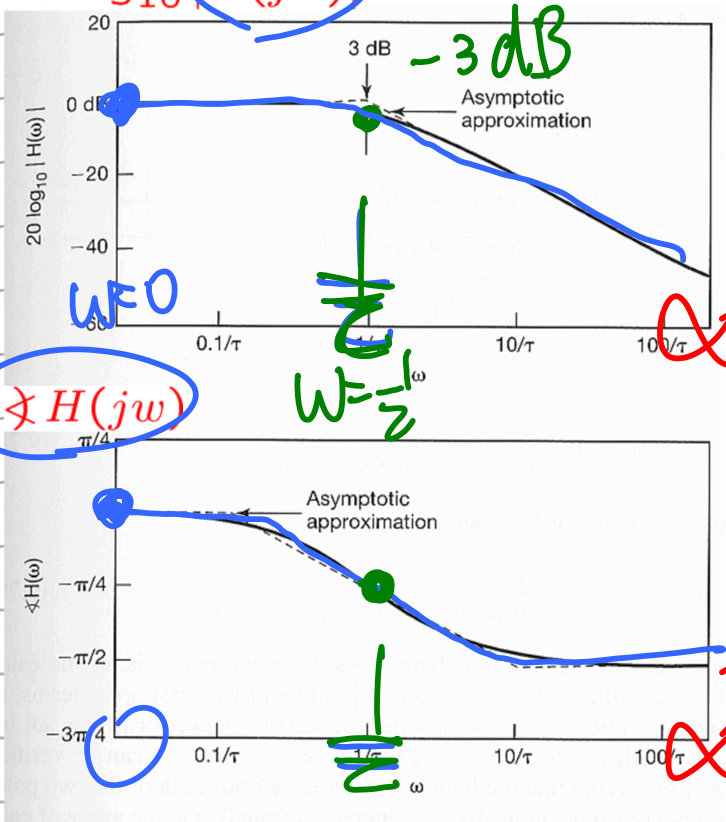
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}, \quad \mathcal{R}e\{s\} > -\frac{1}{\tau}$$

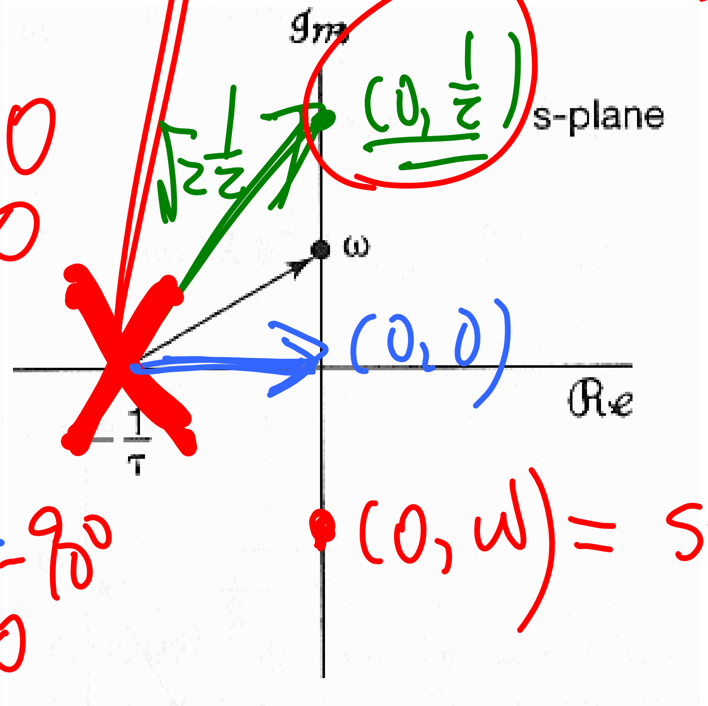
$s = jw$

$\mathcal{F}$

$20 \log_{10} |H(jw)|$

$\measuredangle H(jw)$





$(0, w) = s = jw$

Geometric Evaluation of the Fourier Transform

$$\left|H(j\,0)\right| = \left|\frac{\frac{1}{\tau}}{j\,0+\frac{1}{\tau}}\right| = \frac{\frac{1}{\tau}}{\frac{1}{\tau}} = 1$$

$$\measuredangle H(j\,0) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\,0+\frac{1}{\tau}\right) = 0^\circ - 0^\circ = 0^\circ$$

$$\left|H\left(j\,\frac{1}{\tau}\right)\right| = \left|\frac{\frac{1}{\tau}}{j\,\frac{1}{\tau}+\frac{1}{\tau}}\right| = \frac{1}{\sqrt{2}}$$

$$\measuredangle H\left(j\,\frac{1}{\tau}\right) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\,\frac{1}{\tau}+\frac{1}{\tau}\right) = 0^\circ - 45^\circ = -45^\circ$$

$$\left|H(j\infty)\right| = \left|\frac{\frac{1}{\tau}}{j\infty+\frac{1}{\tau}}\right| = \left|\frac{\text{[illegible]}}{\infty}\right| = 0$$

$$\measuredangle H(j\infty) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\infty+\frac{1}{\tau}\right) = 0^\circ - 90^\circ = -90^\circ$$

▪ Second-Order Systems:

$w_n\sqrt{1-\zeta^2}\ j$

$$H(jw) = \frac{w_n^2}{(jw)^2 + 2\zeta w_n (jw) + w_n^2}$$

$$\Rightarrow h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right] u(t)$$

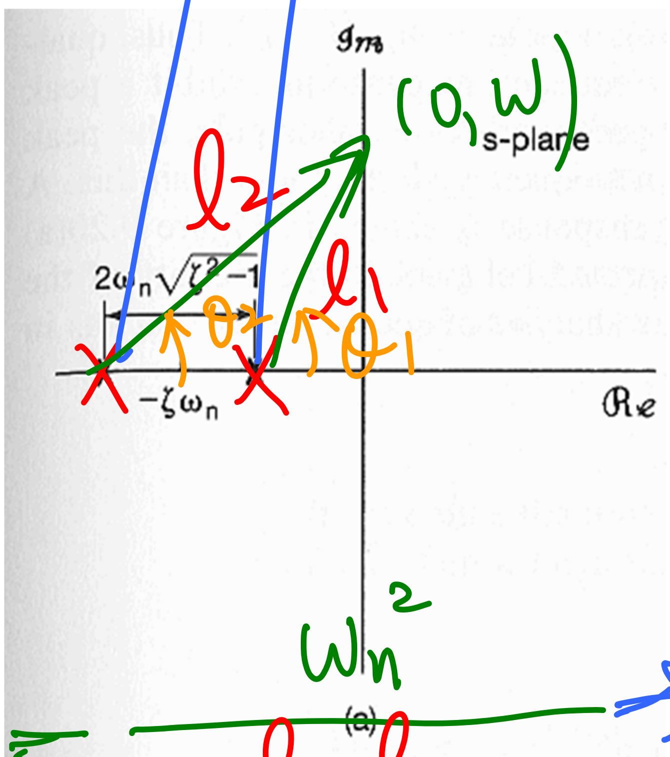
$$c_1 = -\zeta w_n + w_n\sqrt{\zeta^2 - 1}$$
$$c_2 = -\zeta w_n - w_n\sqrt{\zeta^2 - 1}$$
$$M = \frac{w_n}{2\sqrt{\zeta^2-1}}$$

$$\Rightarrow H(s) = \frac{w_n^2}{(s)^2 + 2\zeta w_n (s) + w_n^2} = \frac{w_n^2}{(s-c_1)(s-c_2)}$$

- $\zeta > 1$: c_1 & c_2 are real
- $0 < \zeta < 1$: c_1 & c_2 are complex

Pole Locations:

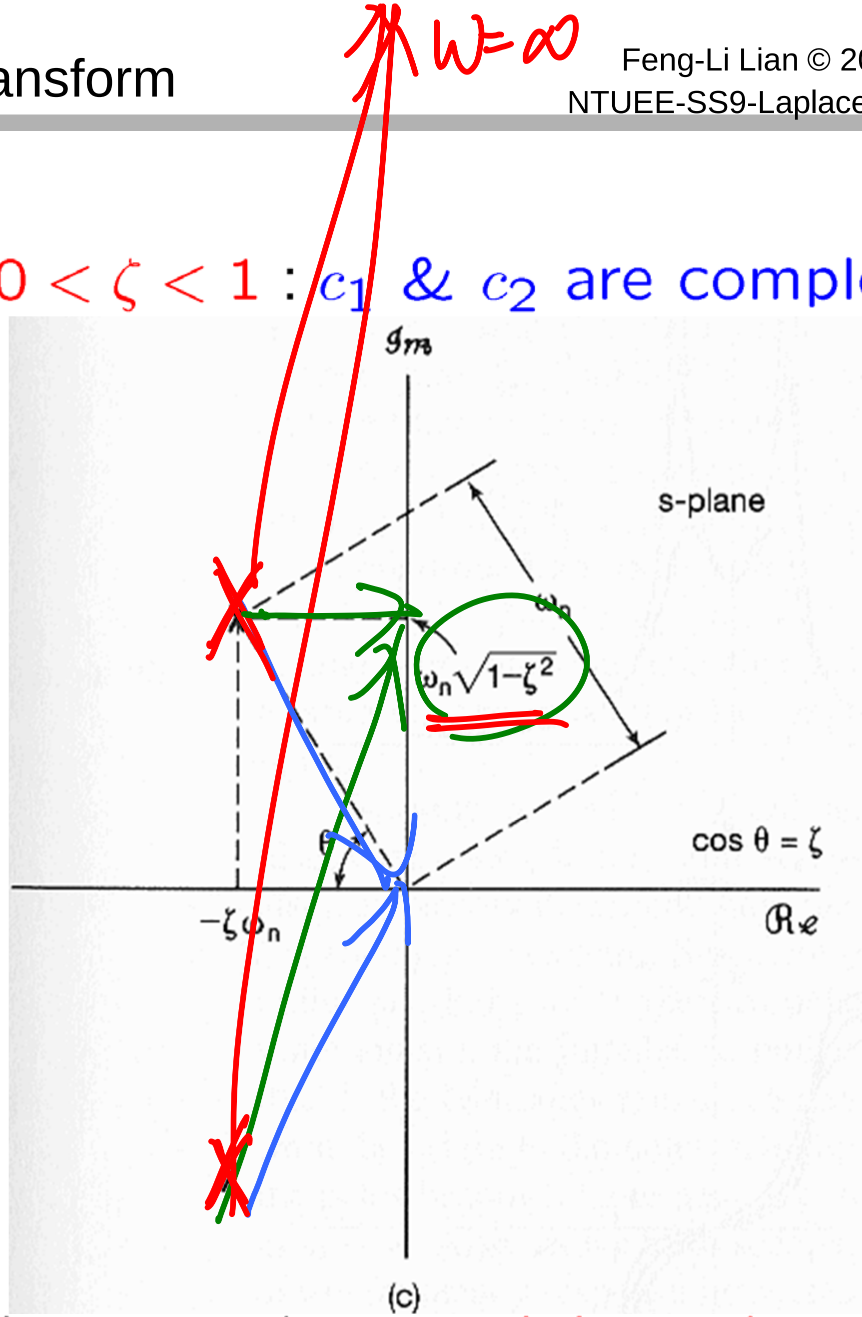
- $\zeta > 1$: c_1 & c_2 are real



$$|H| = \frac{\omega_n^2}{l_1 l_2} \Rightarrow 0$$

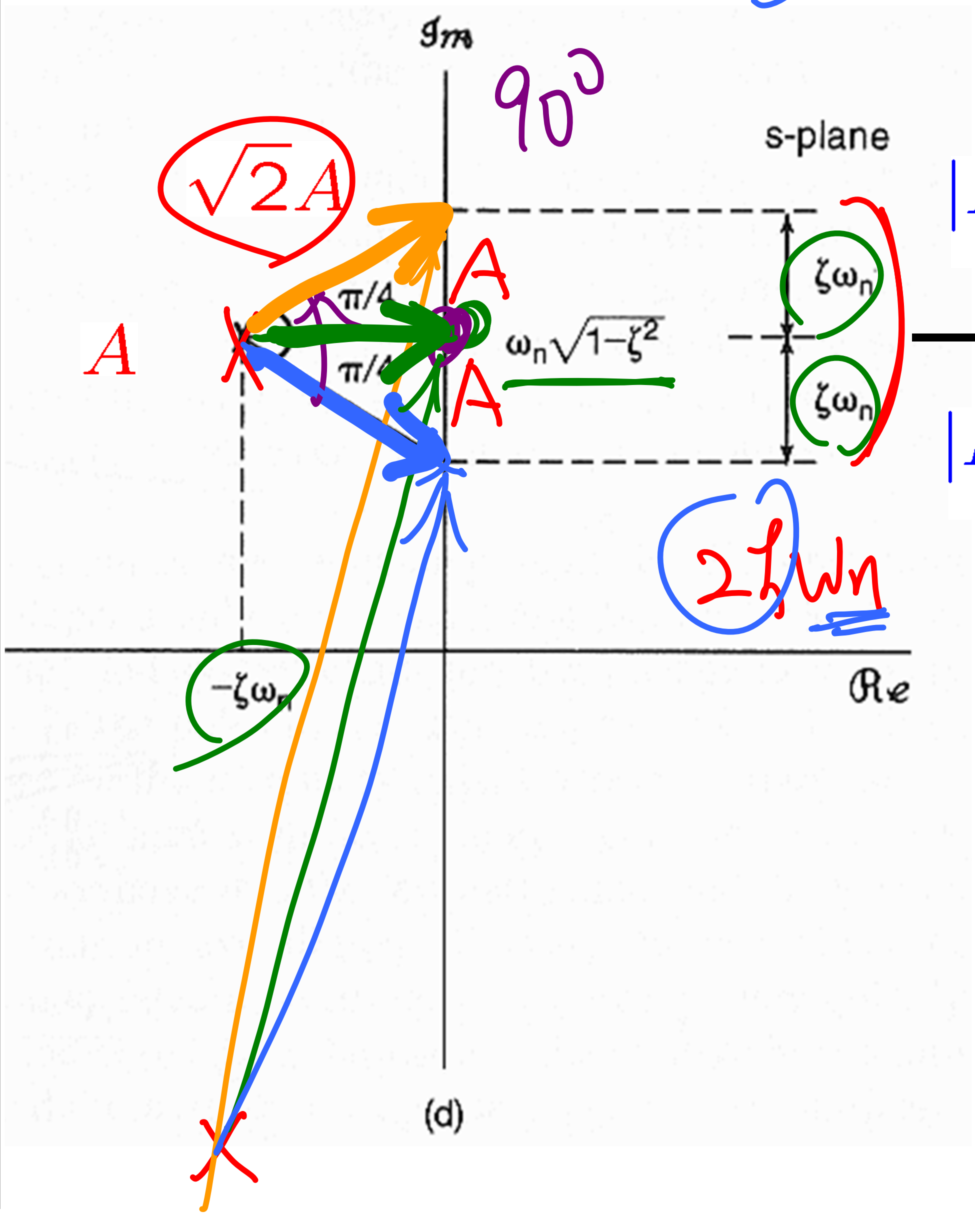
$$\measuredangle H = 0^\circ - \theta_1 - \theta_2 \Rightarrow -90^\circ - 90^\circ = -180^\circ$$

- $0 < \zeta < 1$: c_1 & c_2 are complex



one pole vector has a minimun length at $w = w_n\sqrt{1-\zeta^2}$!!!

Relative Bandwidth B:



$$\frac{|H(jw)|_{w=w_n\sqrt{1-\zeta^2}}}{|H(jw)|_{w=w_n\sqrt{1-\zeta^2}\pm\zeta w_n}} \approx \text{ or } \leq \frac{\sqrt{2}}{1}$$

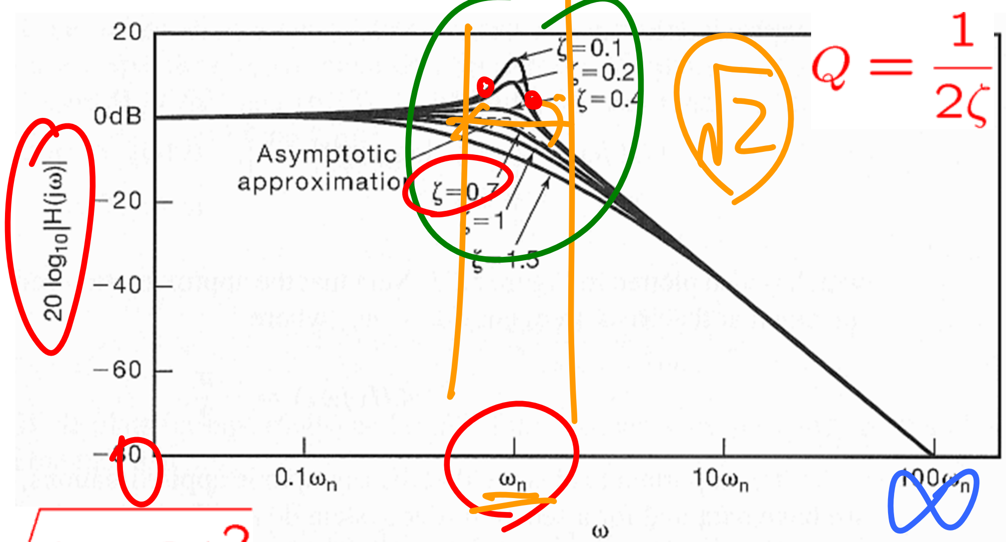
$\frac{1}{A}$ $\sqrt{2}A$

$\Rightarrow \; B = 2\zeta$

$\Rightarrow \; \Delta \measuredangle H(jw) = \frac{\pi}{2}$

▪ Second-Order CT Systems: $H(jw) = \dfrac{1}{(jw/w_n)^2 + 2\zeta(jw/w_n) + 1}$

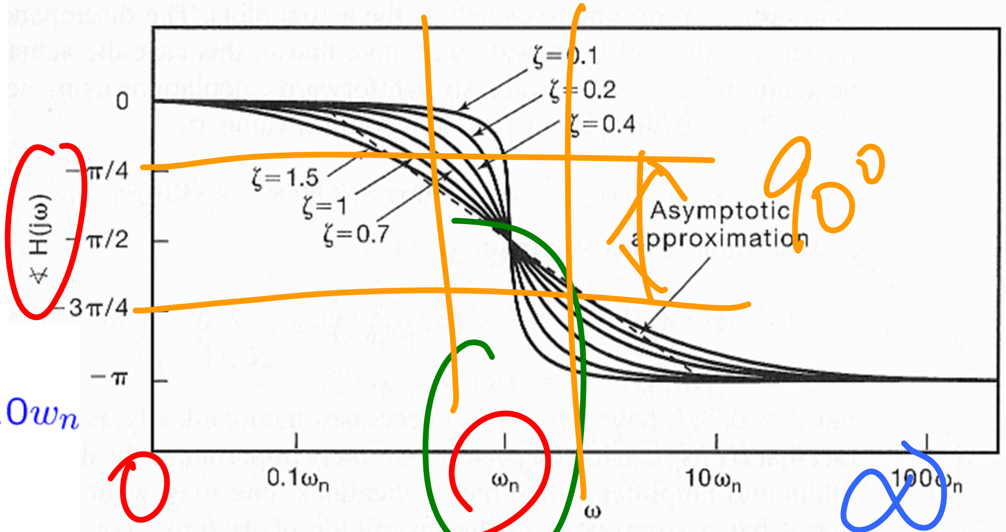
$$20\log_{10}|H(jw)| = \begin{cases} 0 & w << w_n \\ -20\log_{10}(2\zeta) & w = w_n \\ -40\log_{10}(w) + 40\log_{10}(w_n) & w >> w_n \end{cases}$$



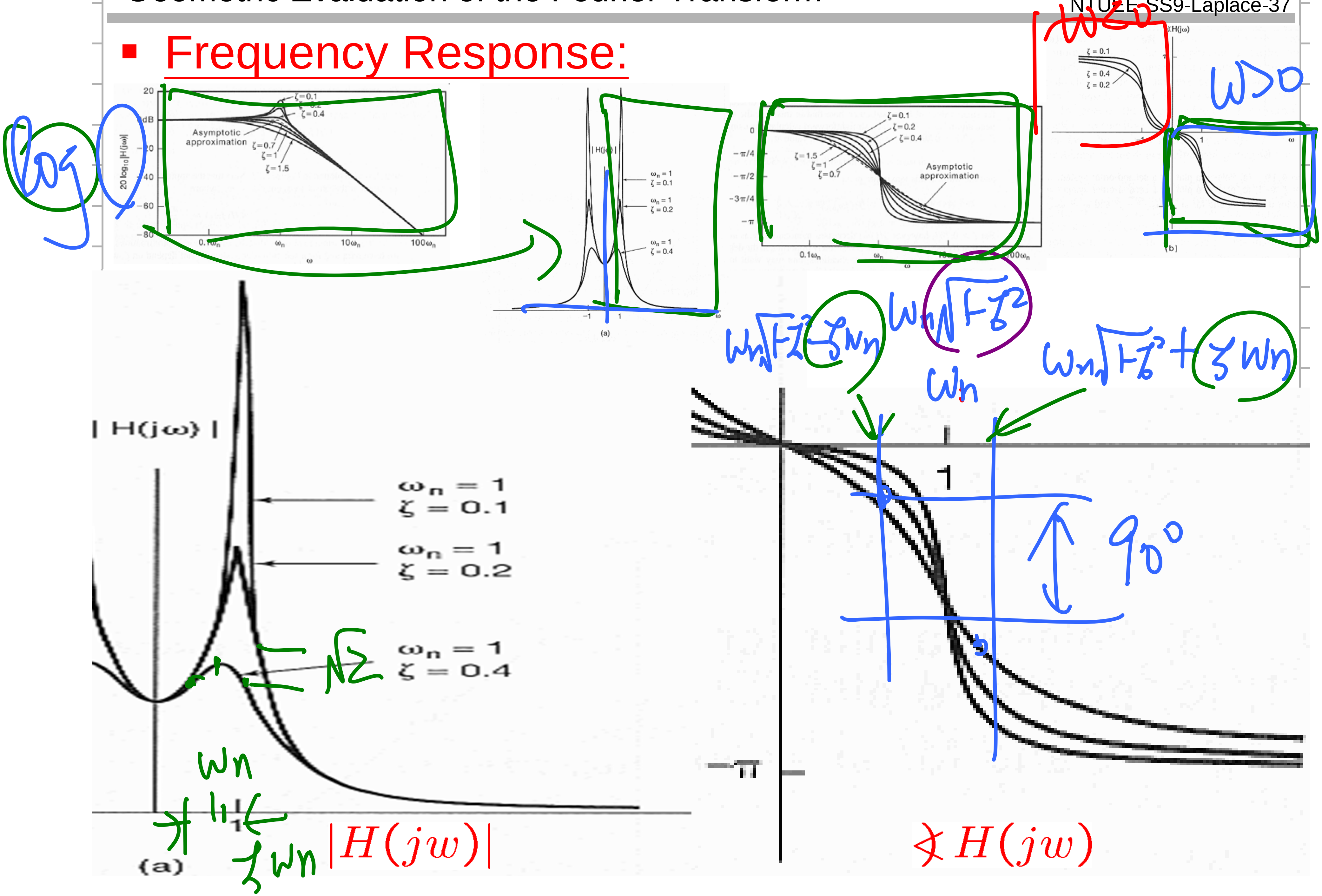
$Q = \dfrac{1}{2\zeta}$

- For $\zeta < \sqrt{2}/2$ $\quad w_{\max} = w_n\sqrt{1 - 2\zeta^2}$

$$\sphericalangle H(jw) = \begin{cases} 0 & w \le 0.1w_n \\ -(\pi/2)[\log_{10}(w/w_n) + 1] & 0.1w_n \le w \le 10w_n \\ -\pi/2 & w = w_n \\ -\pi & w \ge 10w_n \end{cases}$$



- Frequency Response:



- The Nth-Order Systems:

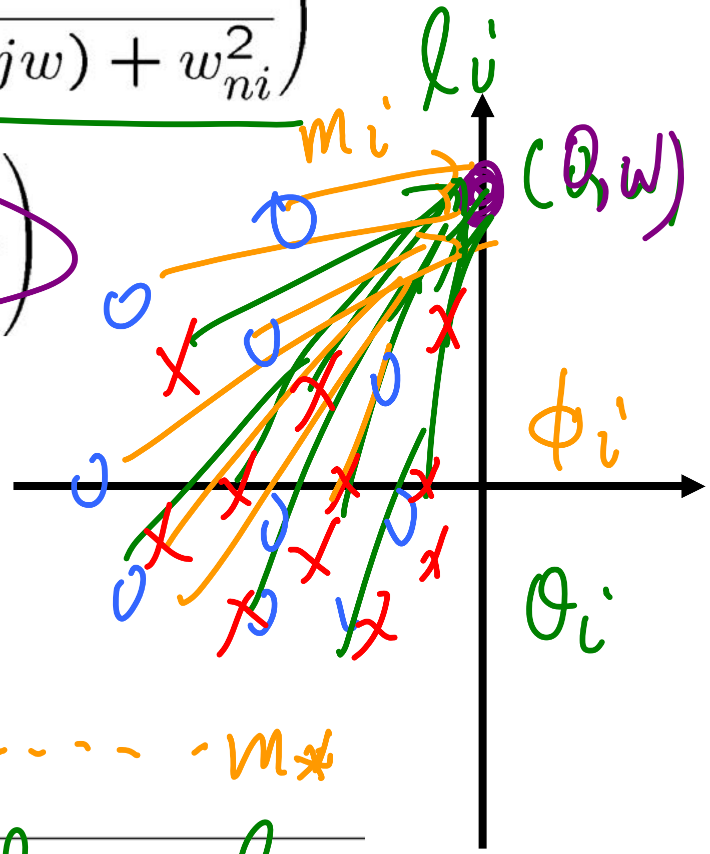
2

$$H(jw) = \left(\frac{\frac{1}{\tau}}{jw + \frac{1}{\tau}}\right) \prod_i \left(\frac{w_{ni}^2}{(jw)^2 + 2\zeta_i w_{ni}(jw) + w_{ni}^2}\right)$$

1

$$H(s) = \left(\frac{b}{s-a}\right) \left(\prod_i \frac{w_{ni}^2}{s^2 + 2\zeta_i w_{ni} s + w_{ni}^2}\right)$$

(1) (2)



$$|H(jw)| = \prod_i |H_i(jw)| = \frac{m_1 \cdot m_2 \cdots\cdots m_*}{l_1\ l_2\ l_3 \cdots l_*}$$

$$\measuredangle H(jw) = \sum_i \measuredangle H_i(jw) = (\phi_1 + \phi_2 + \cdots) - (\theta_1 + \theta_2 + \cdots\cdots)$$

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11:05am

Outline

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Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

Linearity of the Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$\int_{-\infty}^{+\infty} (a\, x_1(t) + b\, x_2(t)) e^{-st} dt \overset{\mathcal{L}}{\longleftrightarrow} a\, X_1(s) + b\, X_2(s),$$

with ROC containing $\boxed{R_1 \cap R_2}$

$$= \int_{-\infty}^{\infty} a\, x_1(t) e^{-st} dt + \int_{-\infty}^{\infty} b\, x_2(t) e^{-st} dt$$

$$= a \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + b \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$= a\, X_1(s) + b\, X_2(s)$$

$$\frac{\quad}{(s+a)(s+b)} + \frac{\quad (s+d)}{(s+c)(s+d)}$$

$$= \frac{\quad}{(s+a)(s+b)(s+c)}$$

- **Example 9.13:**

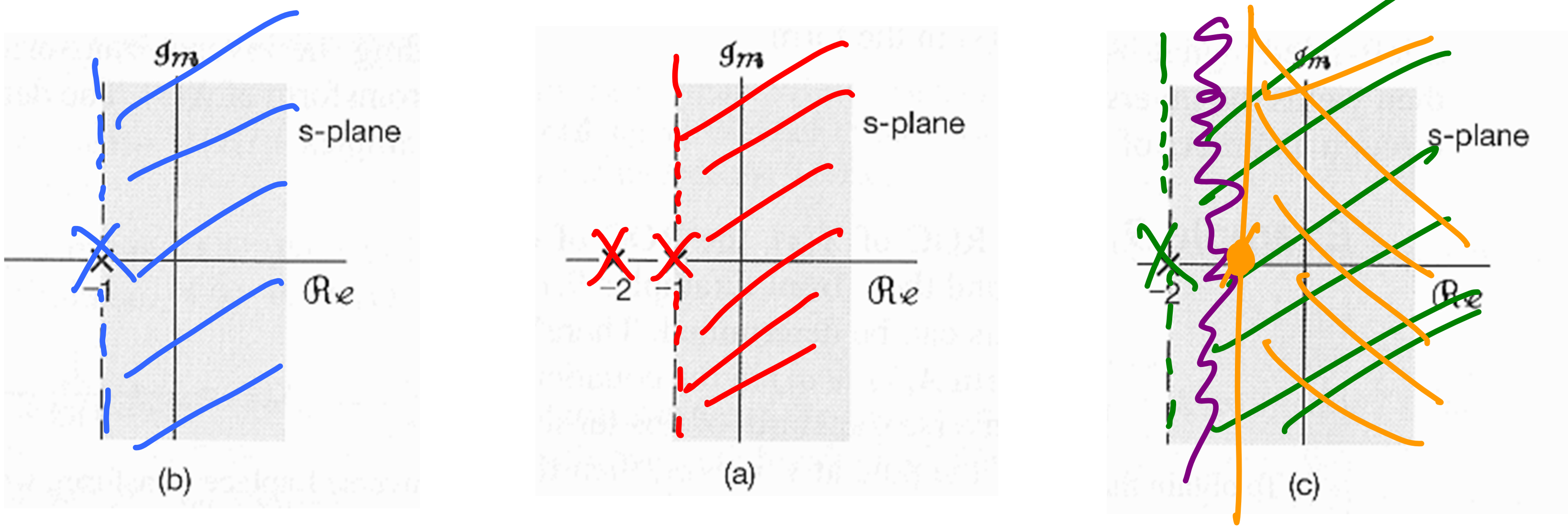
$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{(s+1)}, \quad \boxed{\mathcal{R}e\{s\} > -1}$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \boxed{\mathcal{R}e\{s\} > -1}$$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} = \frac{1}{(s+2)}$$

$\boxed{Re\{s\} > -2}$



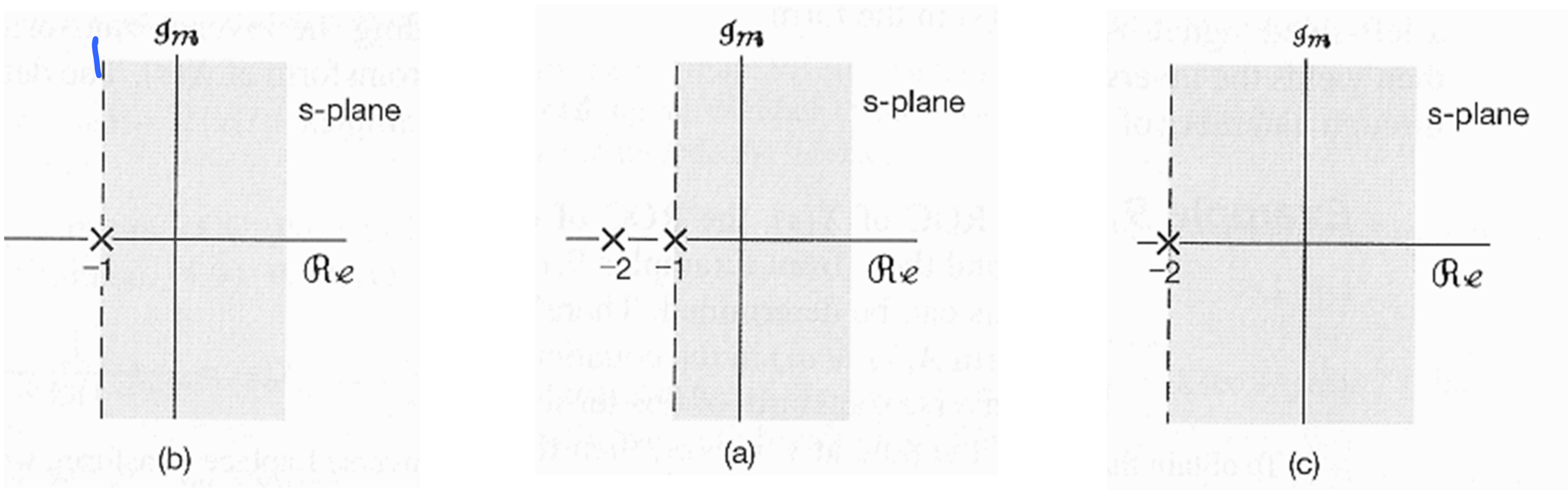
- Example 9.13:

$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathcal{R}e\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} \quad = \frac{1}{(s+2)}$$



Time Shifting:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$j\omega$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \boxed{ROC = R}$$

$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s), \quad \boxed{ROC = R}$$

e^{-st_0}

$$X_0(s) = \int_{-\infty}^{\infty} \boxed{x(t-t_0)} e^{-st} dt$$

$\theta = t - t_0 \qquad t = \theta + t_0$

$dt = d\theta$

$$= \int_{-\infty + t_0}^{\infty + t_0} x(\theta) e^{-s(\theta + t_0)} d\theta$$

$$= \int_{-\infty}^{\infty} x(\theta) e^{-s(\theta)} e^{-s(t_0)} d\theta$$

$$= e^{-s(t_0)} \boxed{\int_{-\infty}^{\infty} x(\theta) e^{-s(\theta)} d\theta} = X(s)$$

$t \qquad t \qquad t$

- Shifting in the s-Domain:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad ROC = R + \mathcal{R}e\{s_0\}$$

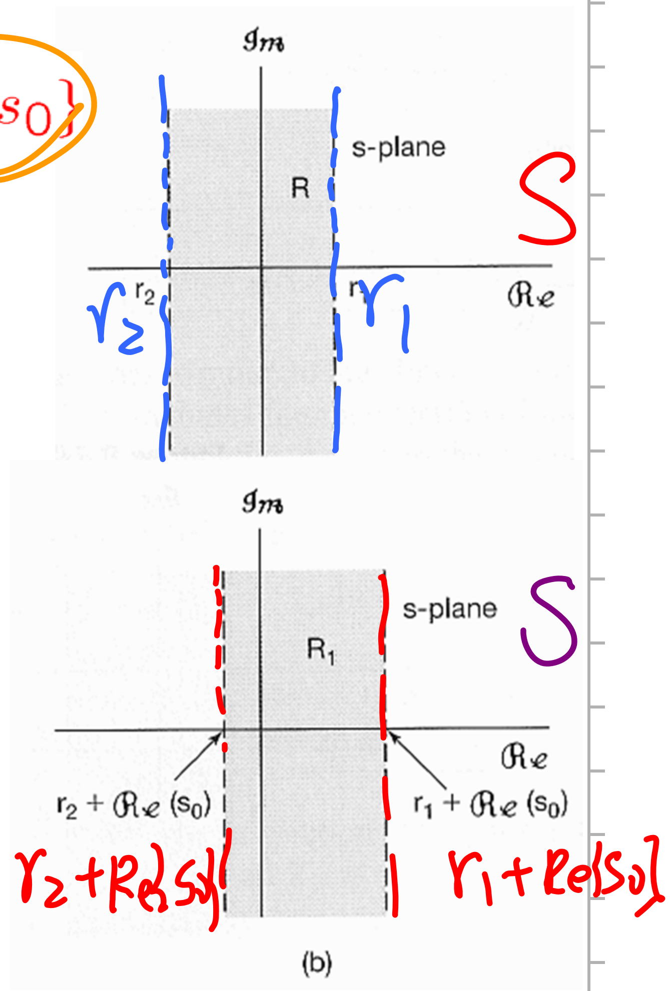
$$\boxed{X(s - s_0)} = \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-st} e^{+s_0 t} dt$$

$\mathcal{L}$

$$= \int_{-\infty}^{+\infty} \boxed{x(t) e^{s_0 t}} e^{-st} dt$$

$X(s-a)$
$\Downarrow$
$X(s-s_0=a)$
$= X(s-a+s_0)$



(b)

- Time Scaling:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad ROC = \boxed{aR}$$

$X(s = b)$

$\Rightarrow X\left(\frac{s}{a} = b\right)$

$= X(s = ab)$

$$X_a(s) = \int_{-\infty}^{\infty} x(a\,t) e^{-st} dt$$

$\theta = at \quad t = \frac{1}{a}\theta \quad dt = \frac{1}{a} d\theta$

$a > 0$

$$= \int_{-\infty}^{\infty} x(\theta) e^{-s\left(\frac{\theta}{a}\right)} \frac{1}{a} d\theta = X\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\theta) e^{-\left(\frac{s}{a}\right)\theta} d\theta = \frac{1}{a} X\left(\frac{s}{a}\right)$$

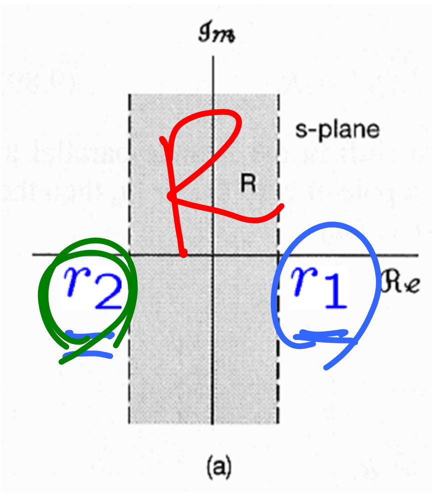
$a < 0$

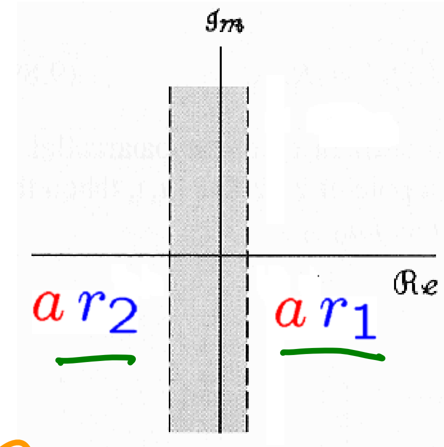
$$= \int_{+\infty}^{-\infty} x(\theta) e^{-s\left(\frac{\theta}{a}\right)} \frac{1}{a} d\theta = X\left(\frac{s}{a}\right)$$

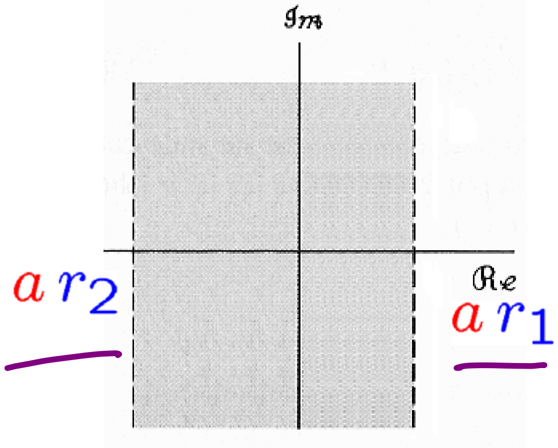
$$= \frac{-1}{a} \int_{-\infty}^{+\infty} x(\theta) e^{-\left(\frac{s}{a}\right)\theta} d\theta = \frac{-1}{a} X\left(\frac{s}{a}\right)$$

$\frac{1}{|a|} X\left(\frac{s}{a}\right)$

Properties of the Laplace Transform

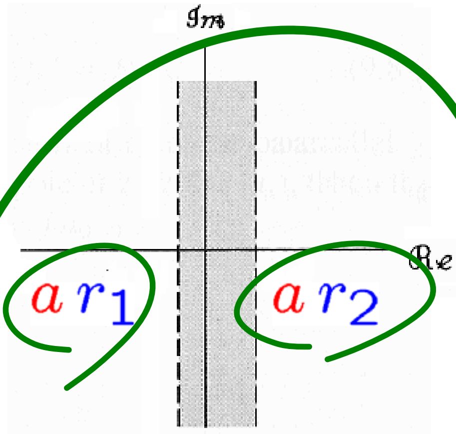


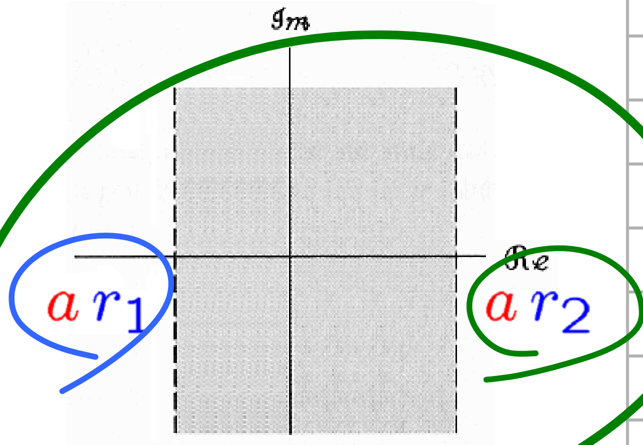




$a>0$ $\quad$ $\boxed{0 < a < 1}$ $\quad$ $\boxed{1 < a}$

$$s \longrightarrow \frac{s}{a}$$





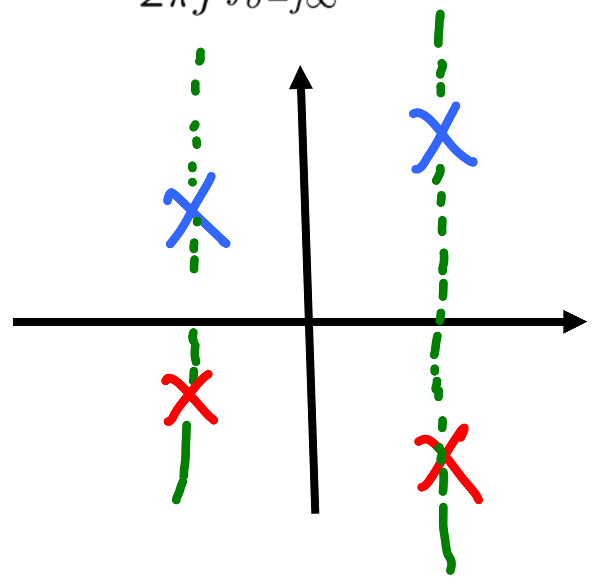
$a<0$ $\quad$ $\boxed{-1 < a < 0}$ $\quad$ $\boxed{a < -1}$

$$x(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(-s), \quad ROC = -R$$

- **Conjugation:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$



$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\boxed{x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)}, \quad ROC = R$$

$$\boxed{\left(X(s)\right)^*} = \left(\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt\right)^*$$

s^*

$$= \int_{-\infty}^{\infty} x^*(t) \left(e^{-st}\right)^* (dt)^*$$

$$= \int_{-\infty}^{\infty} \boxed{x^*(t)}\, e^{-s^* t}\, dt \qquad s^* \to s$$

$$\boxed{X^*(s^*)} = \int_{-\infty}^{\infty} \boxed{x^*(t)}\, e^{-st}\, dt$$

- **Convolution Property:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$x_1(t) * x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s), \qquad ROC \text{ containing } \boxed{R_1 \cap R_2}$$

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right) e^{-st} dt$$

$$\frac{\cancel{(s+d)}}{(s+a)(s+b)} \cdot \frac{(s+e)}{(s+c)\cancel{(s+d)}}$$

$$\int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{+\infty} x_2(t-\tau) e^{-st} dt\, d\tau$$

$$\theta = t - \tau \quad t = \theta + \tau \quad dt = d\theta$$

$$\int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{+\infty} x_2(\theta) e^{-s\theta} e^{-s\tau} d\theta\, d\tau \qquad X_2(s)$$

$$\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} \left(\int x_2(\theta) e^{-s\theta} d\theta \right) d\tau$$

$$X_1(s) = \boxed{\int_{-\infty}^{\infty} x_1(\tau) e^{-s\tau} d\tau}\; X_2(s) = X_1(s) X_2(s)$$

Differentiation in the Time & s-Domain:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\frac{d}{dt} x(t) \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \quad ROC \text{ containing } R$$

$$-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s), \quad ROC = R$$

$$\boxed{\frac{d}{dt} x(t)} = \frac{d}{dt} \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \boxed{X(s)\, s}\, e^{st} ds$$

$$\boxed{\frac{d}{ds} X(s)} = \frac{d}{ds} \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \boxed{x(t)(-t)}\, e^{-st} dt$$

$$d\left(\frac{B}{A}\right) = \frac{B'A - BA'}{A^2}$$

$$s \quad \frac{\cancel{s}}{s(s+1)(s+3)}$$

$\frac{1}{s}$

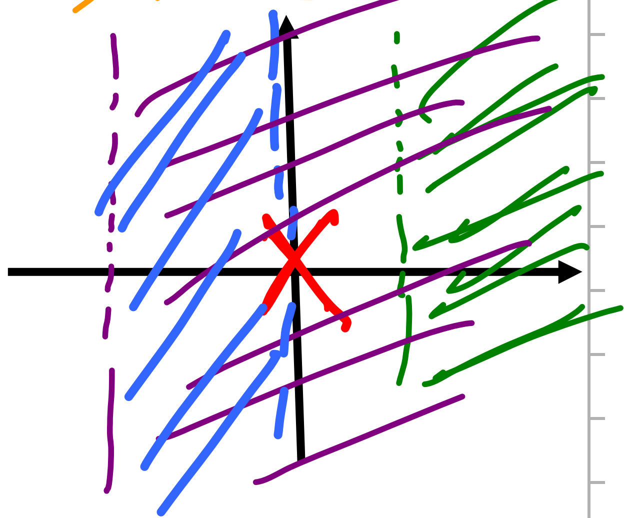
▪ Integration in the Time Domain:

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s), \qquad ROC \text{ containing } R \cap \{\mathcal{R}e\{s\} > 0\}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \Longleftrightarrow X(s)\,U(s)$$

$$U(s) = \frac{1}{s}$$



$$\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{t} x(\tau)d\tau\right)e^{-st}dt$$

$$= \cdots = \frac{1}{s}X(s)$$

$$\boxed{\int_0^\infty u'v\,dt = uv\Big|_0^\infty - \int_0^\infty uv'\,dt}$$

- **The Initial-Value Theorem:**

  If $x(t) = 0$ for $t < 0$ $\quad \Rightarrow \quad x(0^+) = \lim_{s\to\infty} sX(s)$

- **The Final-Value Theorem:**

  If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finit limit as $t \to \infty$,

  $$\Rightarrow \quad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

$$\int_0^\infty u'v\,dt = uv\Big|_0^\infty - \int_0^\infty uv'\,dt$$

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \frac{d}{dt}x(t)\; e^{-st}dt = x(t)\,e^{-st}\Big|_0^\infty - \int_0^\infty x(t)\,(-s)\,e^{-st}dt$$

$$= x(\infty)\frac{1}{e^{s\infty}} - x(0^+)\frac{1}{e^{s0}} + s\int_0^\infty x(t)\; e^{-st}dt \quad = s\,X(s) - x(0^+)$$

$$\lim_{s\to\infty} \int_0^\infty \frac{d}{dt}x(t)\; e^{-st}dt = \lim_{s\to\infty}\left\{s\,X(s) - x(0^+)\right\}$$

$$\lim_{s\to 0} \int_0^\infty \frac{d}{dt}x(t)\; e^{-st}dt = \lim_{t\to\infty} x(t) - x(0) = \lim_{s\to 0}\left\{s\,X(s) - x(0^+)\right\}$$

Properties of the Laplace Transform

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Outline

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Some Laplace Transform Pairs

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re\{s\} > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n\text{ times}}$	$\frac{1}{s^n}$	$\Re\{s\} > 0$

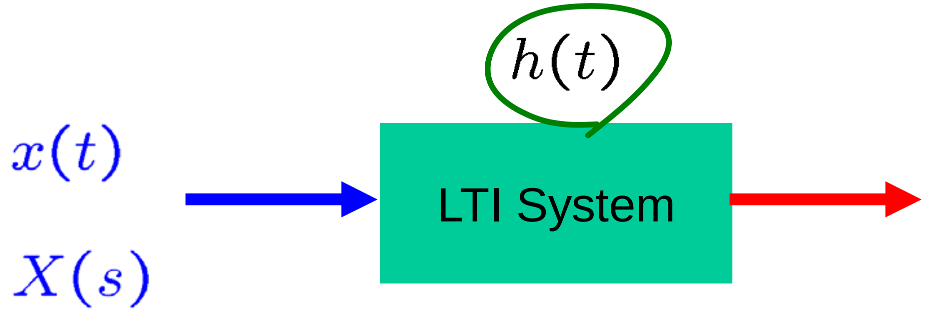
signals

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs

systems

- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Analysis & Characterization of LTI Systems:



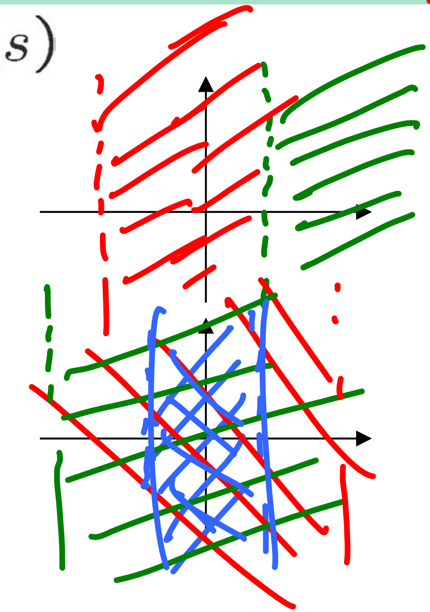
$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s)$$

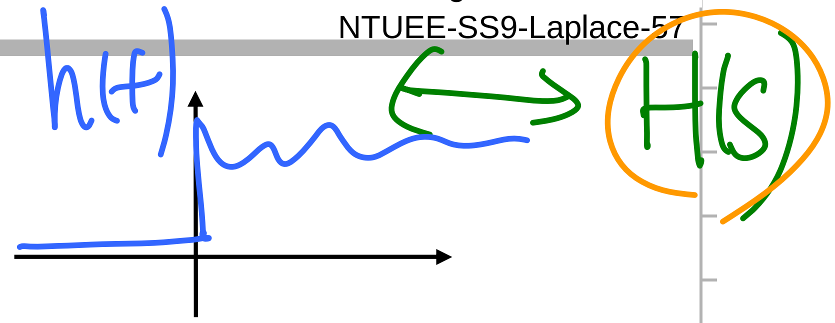
$$H(s) = \mathcal{L}\{h(t)\}$$

$H(s)$: system function

or transfer function

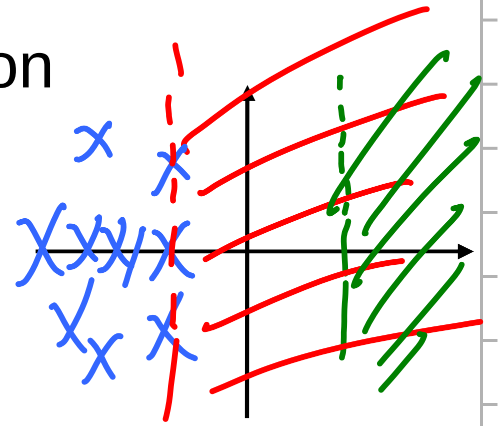
	def	Theorem	$\mathcal{L}$, S
	$\forall t$ $\forall x(t)$ $\forall y(t)$	$h(t)$	$H(s)$
Causality	$x(t) \to y(t)$	$h(t)=0$ $\forall t<0$	
Stability	$x(t) \to y(t)$	$\int \lvert h(t)\rvert dt < M$	

Causality:



- For a causal LTI system,
  h(t) = 0 for t < 0, and thus is right sided

- The ROC associated with the system function
  for a causal system is a right-half plane

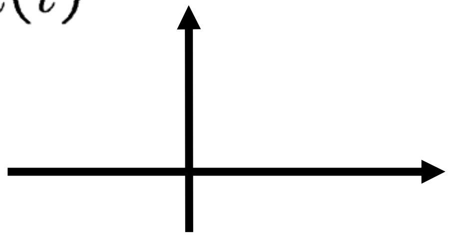
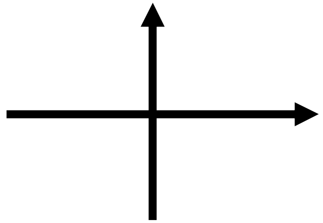


H(s)

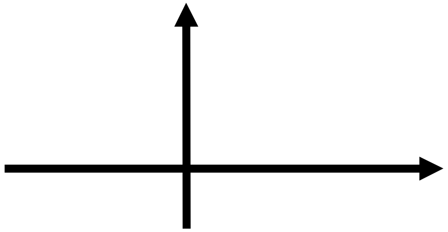
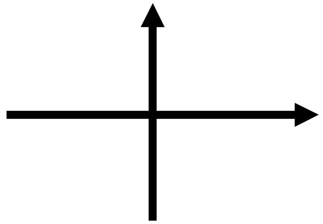
- For a system with a rational system function,
  causality of the system is equivalent to
  the ROC being
  the right-half plane to the right of the rightmost pole

■ Examples 9.17, 9.18, 9.19:

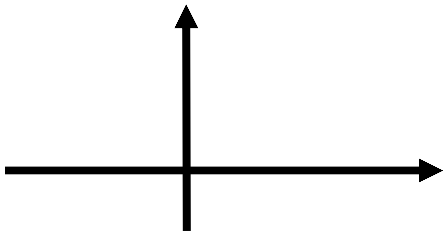
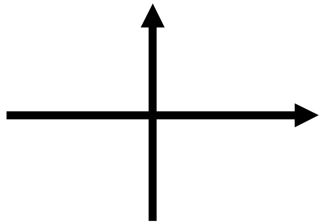
$$h(t) = e^{-t}u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$h(t) = e^{-|t|} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{-2}{s^2-1}, \quad -1 < \mathcal{R}e\{s\} < +1$$

$$h(t) = e^{-(t+1)}u(t+1) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{e^{s}}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

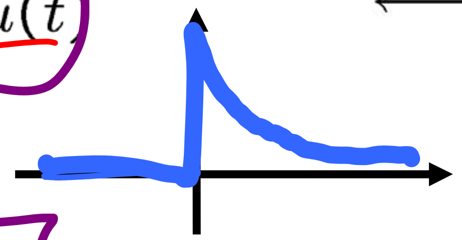
$\left\{ \begin{array}{l} h(t): \\ H(s): \\ ROC: \end{array} \right.$ causal ? rational ? right-sided ?

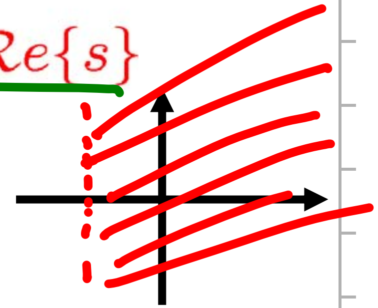
$\left\{ \begin{array}{l} h(t): \\ H(s): \\ ROC: \end{array} \right.$ causal ? rational ? right-sided ?

$\left\{ \begin{array}{l} h(t): \\ H(s): \\ ROC: \end{array} \right.$ causal ? rational ? right-sided ?

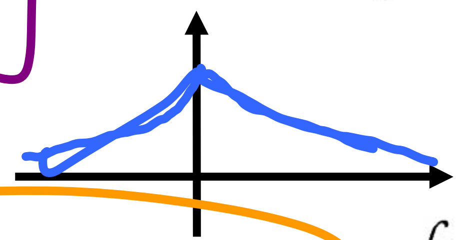
- Examples 9.17, 9.18, 9.19:

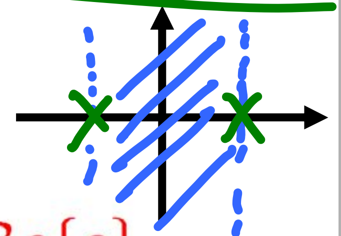
$$h(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$



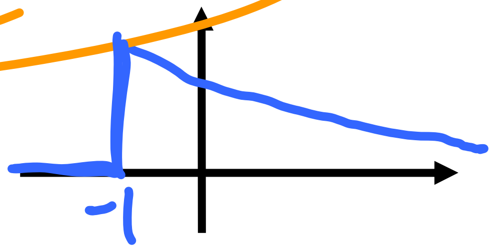


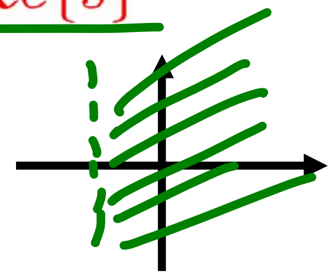
$$h(t) = e^{-|t|} \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{-2}{s^2-1}, \quad -1 < \mathcal{R}e\{s\} < +1$$





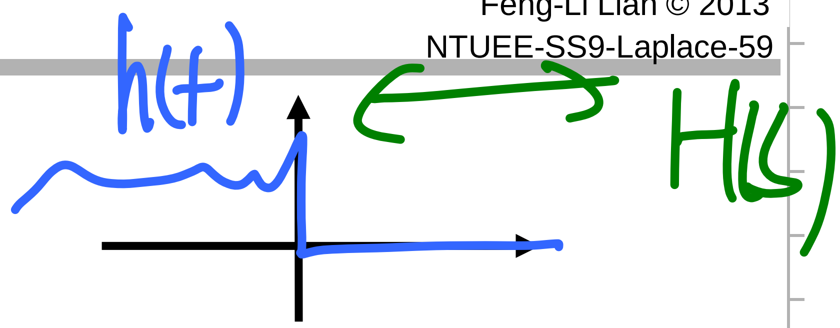
$$h(t) = e^{-(t+1)}u(t+1) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{e^s}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$





$$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{rational} \\ ROC: & \text{right-sided} \end{cases} \qquad \begin{cases} h(t): & \text{not causal} \\ H(s): & \text{rational} \\ ROC: & \text{not right-sided} \end{cases} \qquad \begin{cases} h(t): & \text{not causal} \\ H(s): & \text{not rational} \\ ROC: & \text{right-sided} \end{cases}$$

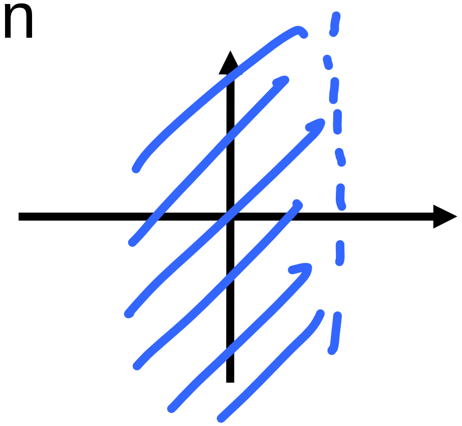
■ Anti-causality:



- For a anti-causal LTI system,
  $h(t) = 0$ for $t > 0$, and thus is left sided

H(s)

- The ROC associated with the system function for a anti-causal system is a left-half plane



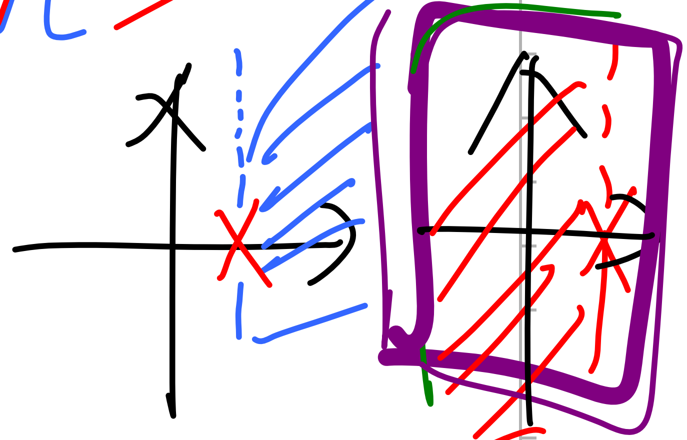
H(s)

- For a system with a rational system function, anti-causality of the system is equivalent to the ROC being the left-half plane to the left of the leftmost pole

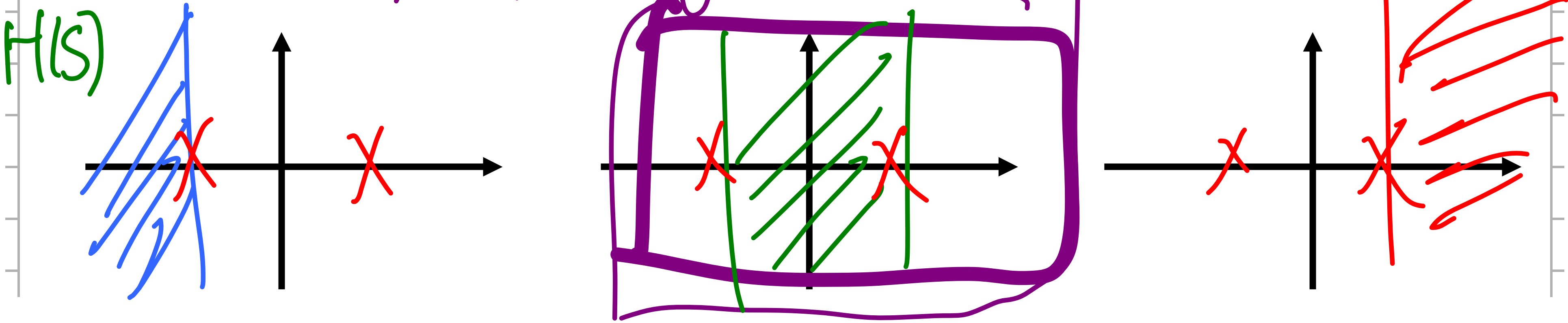
Stability:

- An LTI system is stable
  if and only if $\Longleftrightarrow$
  the ROC of its system function H(s)
  includes the $\boxed{\text{entire } j\omega\text{-axis}}$ [i.e., Re{s} = 0]

1. stable: $\forall\, |x(t)| < A \Rightarrow \forall\, |y(t)| < B \;\; \forall t$
2. $h(t)$: $\int_{-\infty}^{+\infty} |h(t)|\, dt < M$
3. $h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) \leftarrow$ ROC
   $\overset{\mathcal{F}}{\longleftrightarrow} H(j\omega)$
4. H(s) includes $j\omega$-axis



H(S)



■ Stability:

- An LTI system is stable
  if and only if $\Longleftrightarrow$
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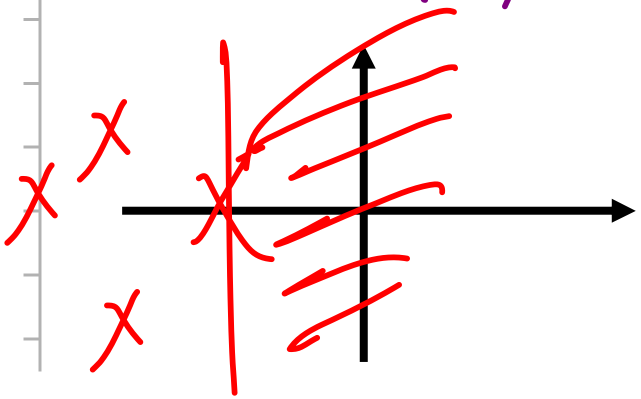
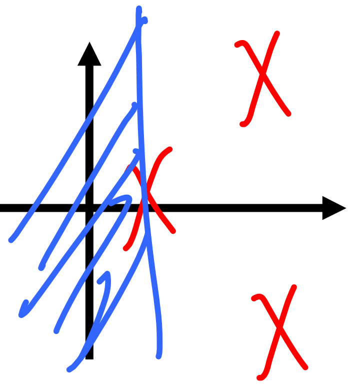
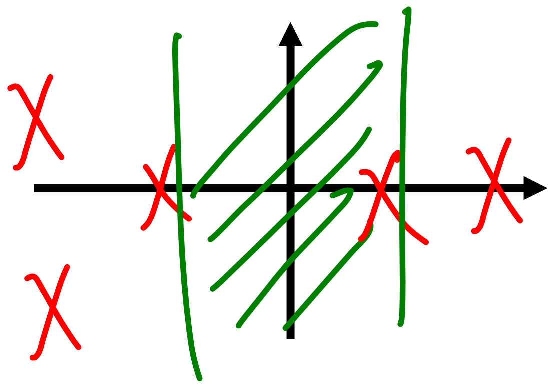
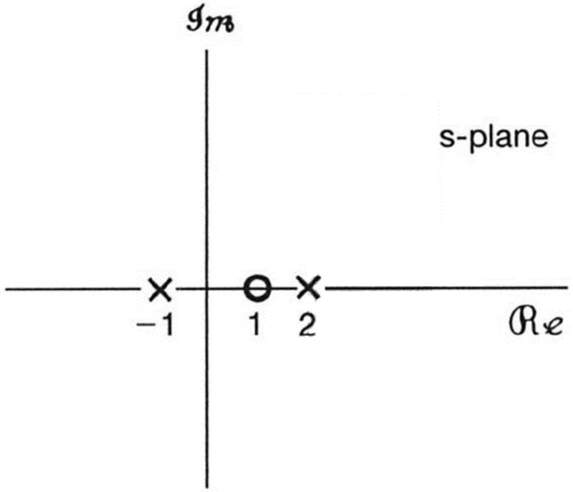
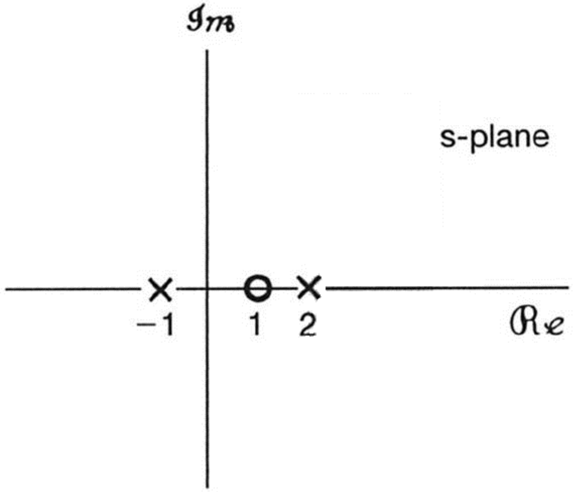
2. $h(t)$: $\int_{-\infty}^{+\infty} |h(t)|\,dt < M$
3. $h(t) \xrightarrow{\mathcal{L}} H(s) \leftarrow$ ROC
   $h(t) \xrightarrow{\mathcal{F}} H(j\omega)$
4. H(s) includes jω-axis

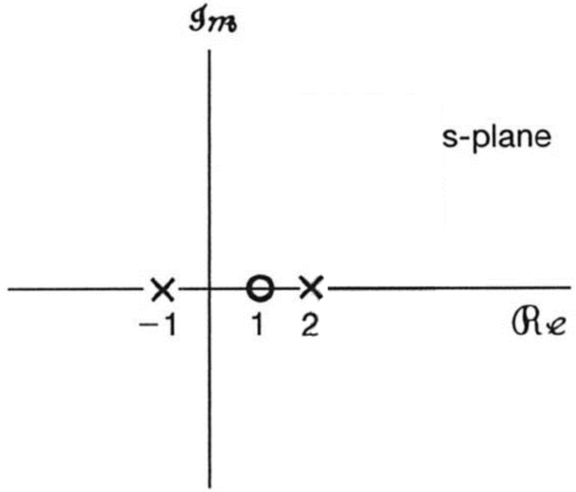




- Example 9.20:

$$H(s) = \frac{s-1}{(s+1)(s-2)} \qquad = \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s-2}$$







$$h(t) = \frac{2}{3}e^{\;\;t}u(\;\;t) \quad \frac{1}{3}e^{\;\;t}u(\;\;t)$$

$$h(t) = \frac{2}{3}e^{\;\;t}u(\;\;t) \quad \frac{1}{3}e^{\;\;t}u(\;\;t)$$

$$h(t) = \frac{2}{3}e^{\;\;t}u(\;\;t) \quad \frac{1}{3}e^{\;\;t}u(\;\;t)$$

causal ?

stable ?

causal ?

stable ?

causal ?

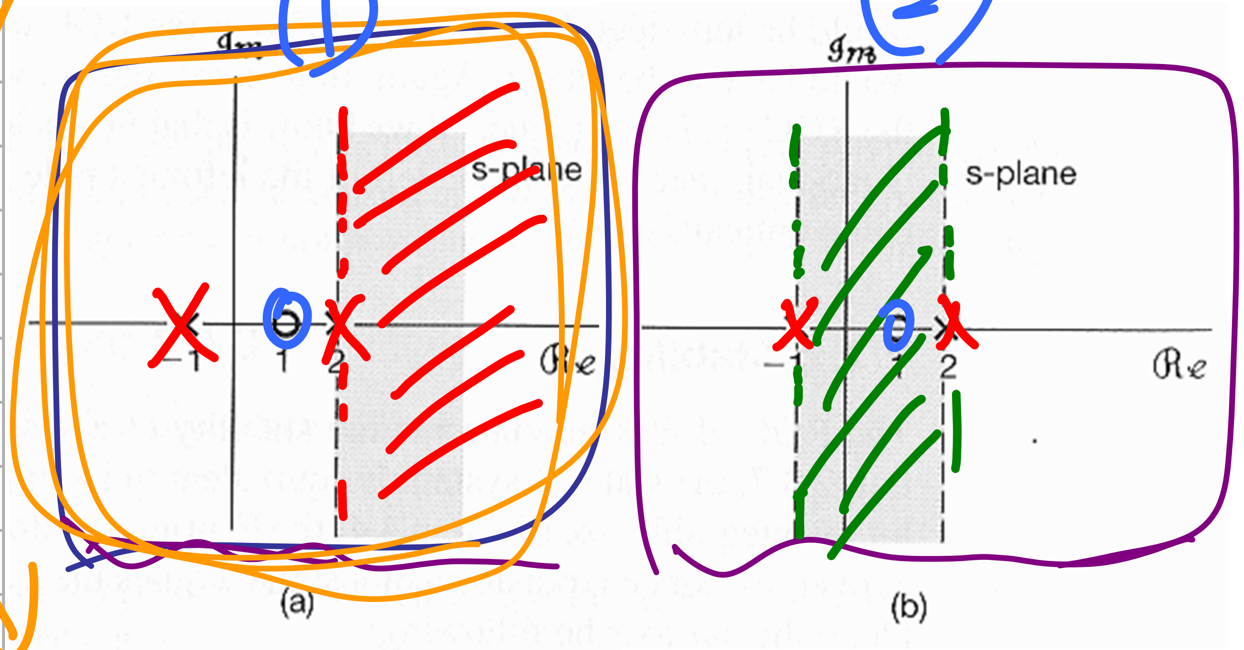
stable ?

- Example 9.20:

$$H(s) = \boxed{\frac{s-1}{(s+1)(s-2)}} = \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s-2}$$

ROC ① ② ③

$\mathcal{L}^{-1}$



(a) (b) (c)

$$\boxed{h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)}$$

causal, (unstable)

$$\boxed{h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)}$$

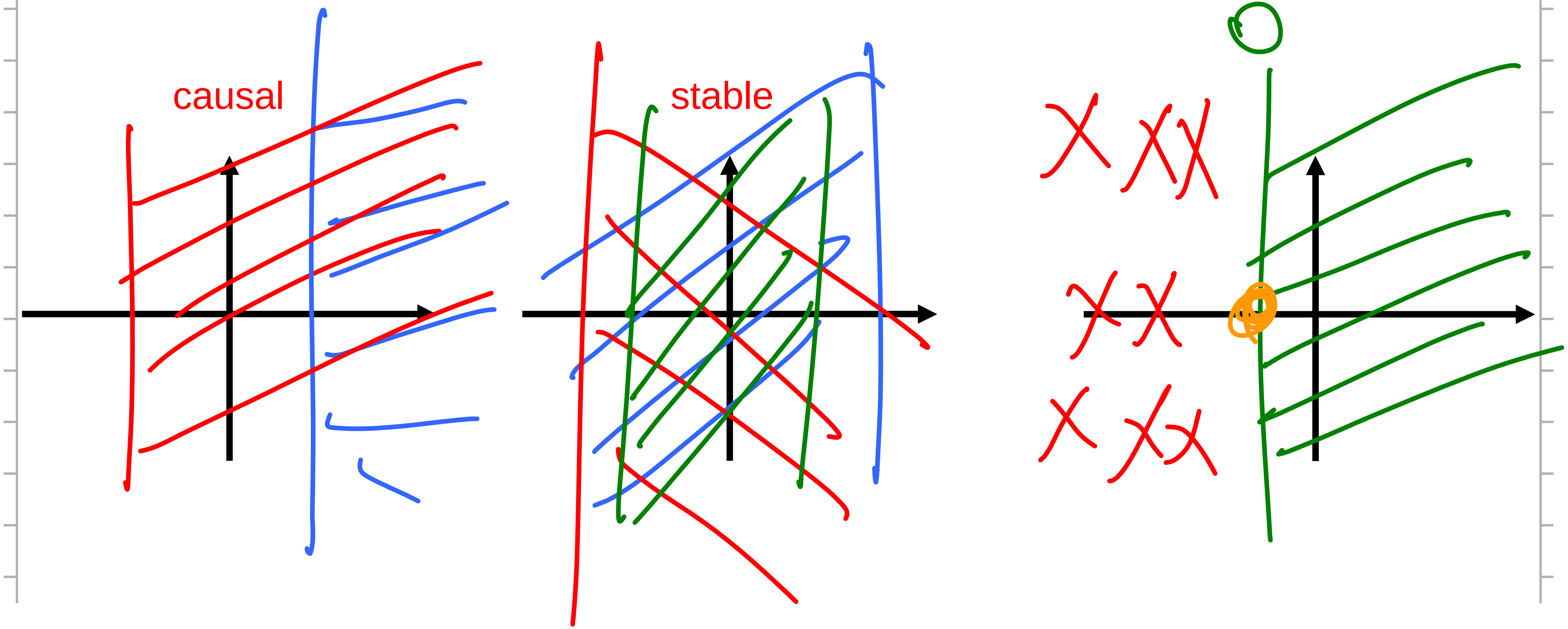
stable (not causal)

$$\boxed{h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)}$$

unstable, anticausal

Stability:

- A causal system with rational system function H(s) is stable if and only if all of the poles of H(s) lie in the left-half of s-plane, i.e., all of the poles have negative real parts

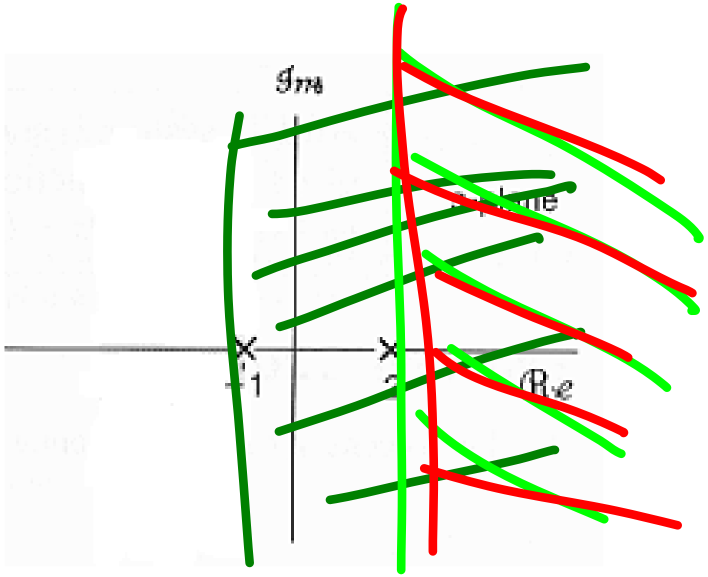


Examples 9.17, 9.21:

$$h(t) = e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$h(t) = e^{2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s-2}, \quad 2 < \mathcal{R}e\{s\}$$

$$\begin{cases} h(t): \\ H(s): \end{cases} \qquad \begin{cases} h(t): \\ H(s): \end{cases}$$



- Examples 9.17, 9.21:

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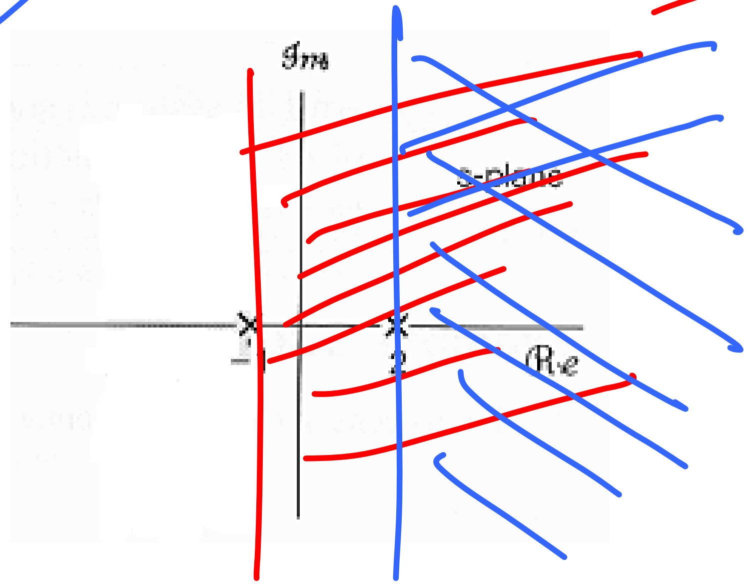
$\begin{cases} h(t): & \text{causal} \\ H(s): & \text{stable, rational} \end{cases}$ $\qquad$ $\begin{cases} h(t): & \text{causal} \\ H(s): & \text{unstable, rational} \end{cases}$



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- LTI Systems by Linear Constant-Coef Differential Equations:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

( ) (s) = 0

poles

( ) X(s) = 0

zeros

h(t)

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(s) = X(s)H(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\mathcal{L}\left\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{L}\left\{ \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^{N} a_k \, \mathcal{L}\left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^{M} b_k \, \mathcal{L}\left\{ \frac{d^k x(t)}{dt^k} \right\}$$

$(j\omega)^k$

$$\sum_{k=0}^{N} a_k \, s^k \, Y(s) = \sum_{k=0}^{M} b_k \, s^k \, X(s)$$

$$Y(s) \left[\sum_{k=0}^{N} a_k \, s^k \right] = X(s) \left[\sum_{k=0}^{M} b_k \, s^k \right]$$

zeros

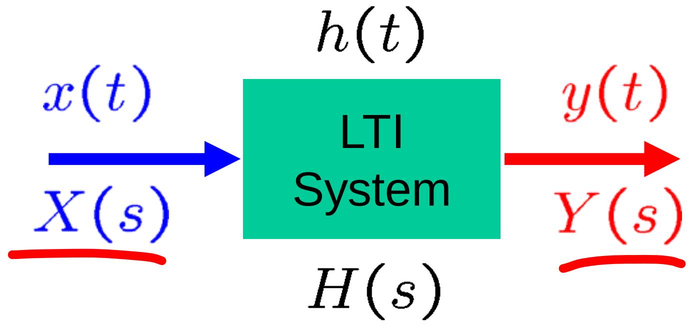
$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} \cancel{=} \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + \cdots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0}$$

$= 0$

$\leftarrow 0$

poles

Example 9.23:



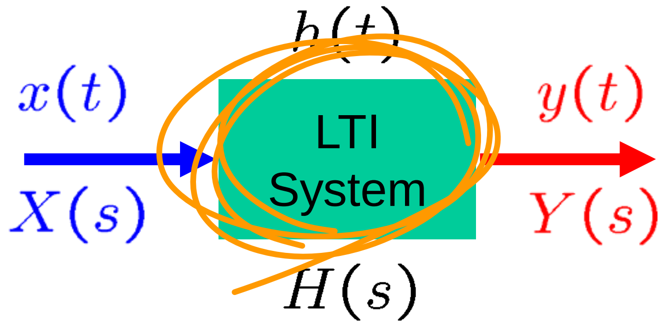
$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$\Rightarrow \quad sY(s) + 3Y(s) = X(s) \qquad\qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow \quad (s+3)\,Y(s) = X(s) \qquad\qquad \Rightarrow \quad H(s) = \frac{1}{s+3}$$

- If causal, $\Rightarrow \mathcal{R}\{s\} > -3, \quad \Rightarrow \quad h(t) = e^{-3t}\,u(t)$
- If anti-causal, $\Rightarrow \mathcal{R}\{s\} < -3, \quad \Rightarrow \quad h(t) = -\,e^{-3t}\,u(-t)$

- Example 9.23:



$$\mathcal{L}\left\{\frac{dy(t)}{dt} + 3y(t)\right\} = \mathcal{L}\{x(t)\}$$

$$\Rightarrow \; sY(s) + 3Y(s) = X(s) \qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow \; (s+3)\,Y(s) = X(s) \qquad \Rightarrow \; H(s) = \boxed{\frac{1}{s+3}}$$

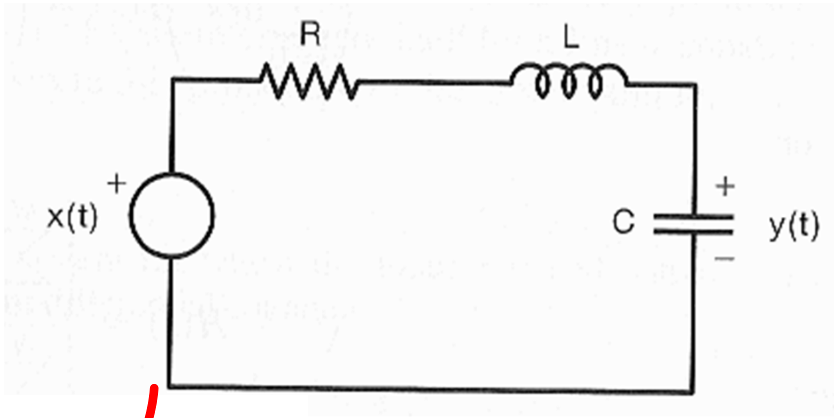
RoC

$s=-1$

$\downarrow \mathcal{L}^{-1}$

- If causal, $\;\Rightarrow\; \boxed{\mathcal{R}\{s\} > -3,} \;\Rightarrow\; h(t) = e^{-3t}u(t)$
- If anti-causal, $\;\Rightarrow\; \mathcal{R}\{s\} < -3, \;\Rightarrow\; h(t) = -e^{-3t}u(-t)$

- **Example 9.24:**



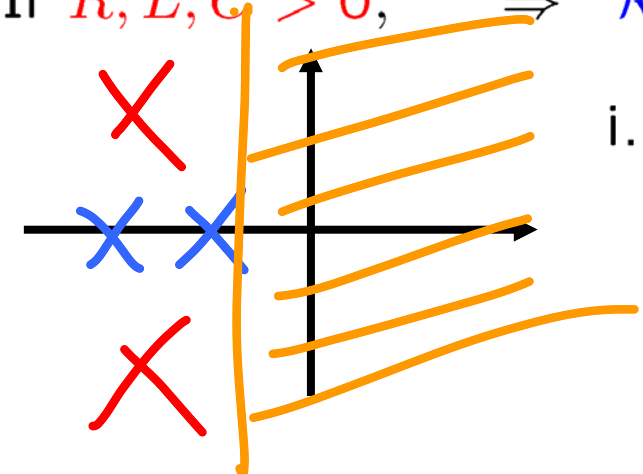
$$\cancel{LC}\frac{d^2y(t)}{dt^2} + \frac{R\cancel{C}}{L\cancel{C}}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

$$\Rightarrow \quad H(s) = \frac{\left(\frac{1}{LC}\right)}{\underbrace{s^2}_{A} + \underbrace{\left(\frac{R}{L}\right)}_{B} s + \underbrace{\left(\frac{1}{LC}\right)}_{C}} \qquad = \frac{\left(\frac{1}{LC}\right)}{(s-a)(s-b)}$$

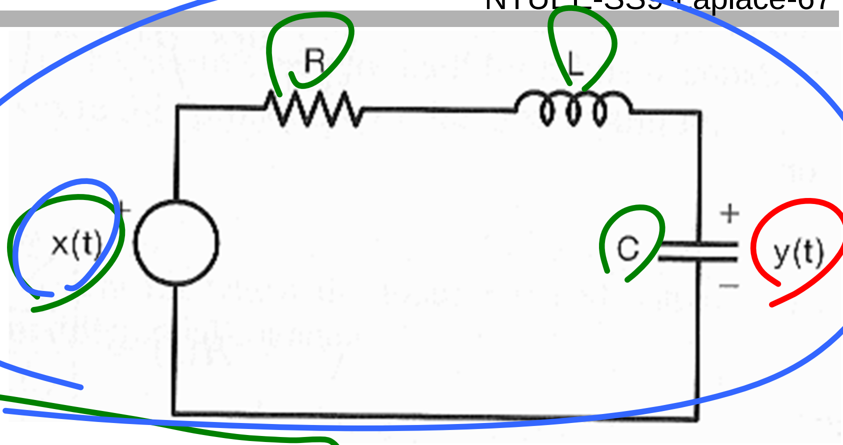
$$\frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

- If $R, L, C > 0$, $\quad \Rightarrow \quad \mathcal{R}e\{a\}, \mathcal{R}e\{b\} < 0$

  i.e., poles with negative real parts



Example 9.24:

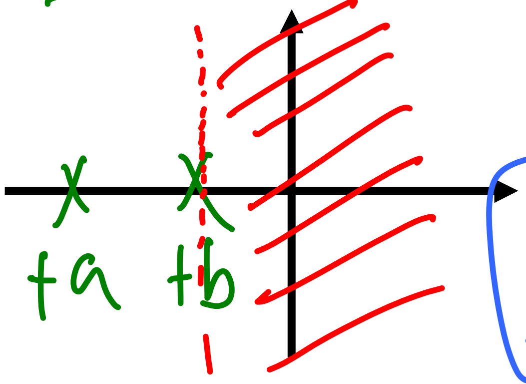


$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + 0\,y(t) = 1\,x(t)$$

$$\Rightarrow \quad H(s) = \frac{\left(\frac{1}{LC}\right)}{1\,s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right)} = \frac{\left(\frac{1}{LC}\right)}{(s-a)(s-b)}$$

- If $R, L, C > 0$, $\Rightarrow$ $\mathcal{R}e\{a\}, \mathcal{R}e\{b\} < 0$

  i.e., poles with negative real parts



causal $\Rightarrow$ stable

- Example 9.25:

$(s+2) - (s+1)$

$\frac{1}{s+1} - \frac{1}{s+2}$

?

$x(t) = e^{-3t}u(t)$ → LTI System → $y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$

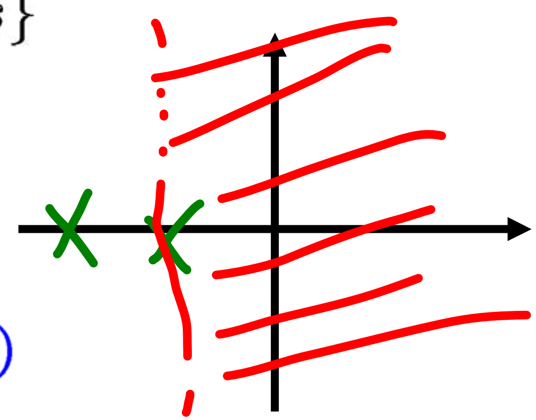
$$X(s) = \frac{1}{s+3}, \quad -3 < \mathcal{R}e\{s\}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad -1 < \mathcal{R}e\{s\}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

$$\text{ROC}: \ -1 < \mathcal{R}e\{s\}$$

$\Rightarrow$



$$\Rightarrow \frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t)$$

- Example 9.25:

?

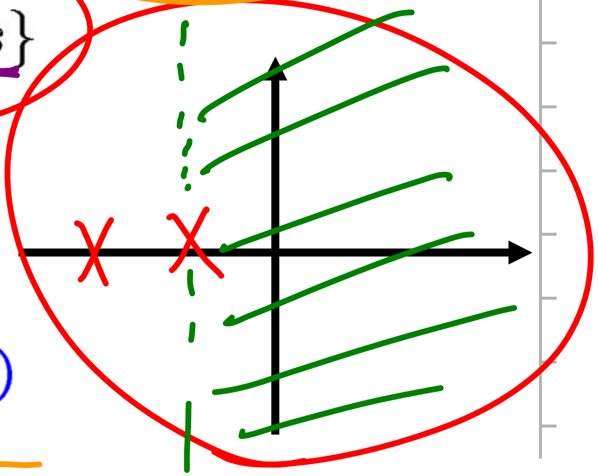
$x(t) = e^{-3t}u(t)$ → LTI System → $y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$

$$X(s) = \frac{1}{s+3}, \quad -3 < \mathcal{R}e\{s\} \qquad\qquad Y(s) = \frac{1}{(s+1)(s+2)}, \quad -1 < \mathcal{R}e\{s\}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

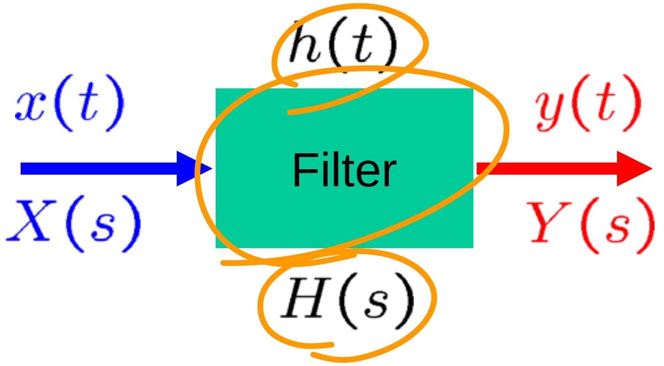
$$\text{ROC}: \; -1 < \mathcal{R}e\{s\}$$

$\Rightarrow$ casual, stable



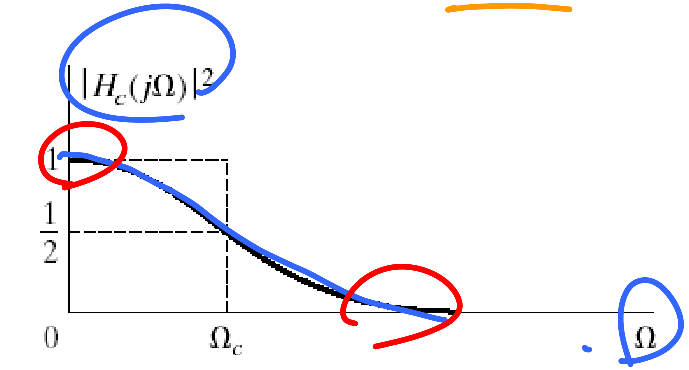
$$\Rightarrow \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

Continuous-Time Filters

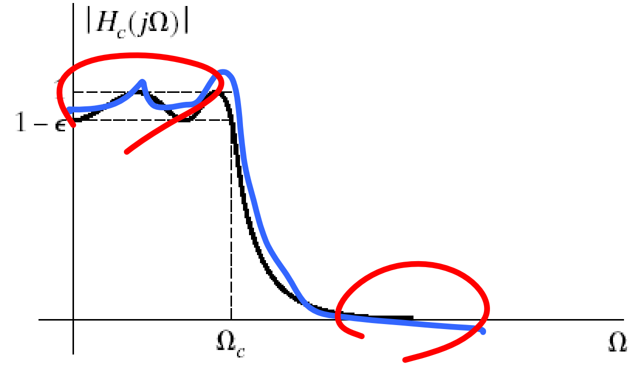


$H(j\omega)$

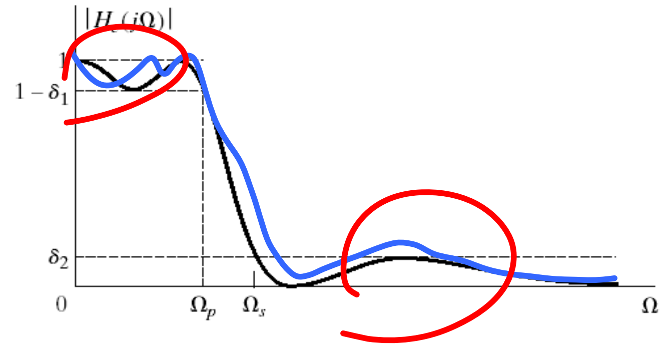
- Butterworth Lowpass Filters:



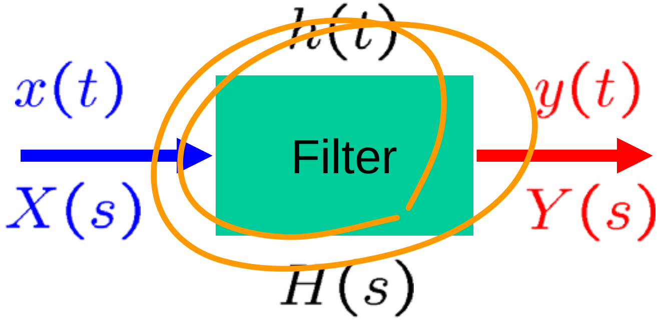
- Chebyshev Filters:



- Elliptic Filters:



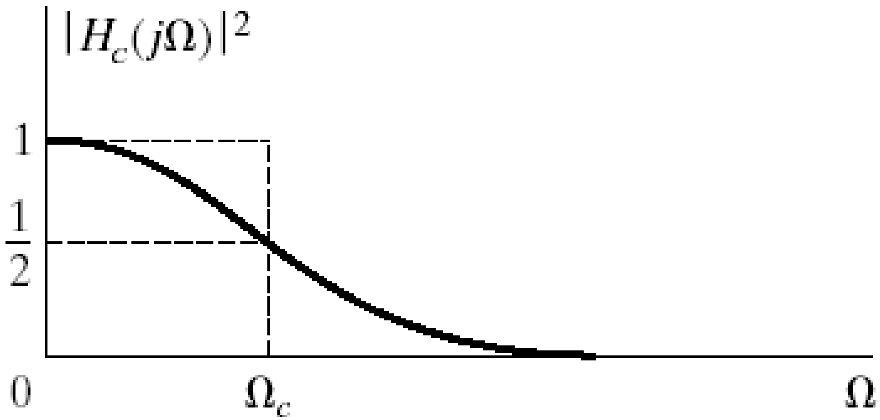
Butterworth Lowpass Filters:

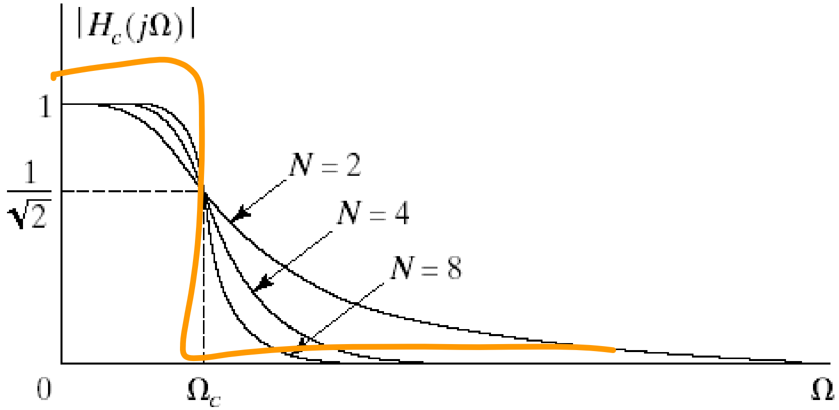


N-th order

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$|H_c(s)|^2$ $1+\left(\frac{s}{j\Omega_c}\right)^{2N}$





An Nth-Order Lowpass Butterworth Filters:

$$|B(jw)|^2 = \frac{1}{1+(jw/jw_c)^{2N}}$$

h(t)

- If impulse response is real, $\Rightarrow$ $B^*(jw) = B(-jw)$

$$|B(jw)|^2 = B(jw)\,B^*(jw) = B(jw)\,B(-jw)$$

$$\Rightarrow B(jw)\,B(-jw) = \frac{1}{1+(jw/jw_c)^{2N}}$$

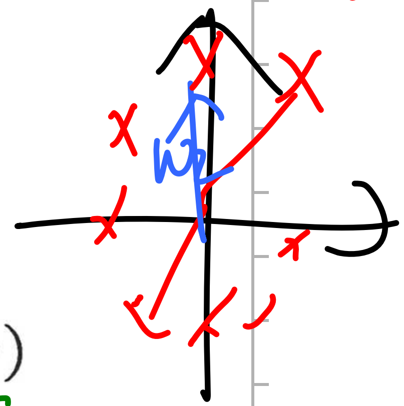
$$\Rightarrow B(s)\,B(-s) = \frac{1}{1+(s/jw_c)^{2N}} \quad \text{at } s_p = \left((-1)^{1/2N}(jw_c)\right) = -1$$

2N

$$\Rightarrow s_p = w_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \frac{\pi}{2}\right]\right)$$

$$\Rightarrow \begin{cases} |s_p| = w_c \\ \measuredangle s_p = \frac{\pi(2k+1)}{2N} + \frac{\pi}{2} \end{cases} \qquad k = 0, 1, 2, ..., 2N-1$$

2N



▪ An Nth-Order Lowpass Butterworth Filters:

$$B(s)\,B(-s) = \frac{1}{1+(s/jw_c)^{2N}} \qquad s_p = w_c \exp\left(j\left[\frac{\pi(2k+1)}{2N} + \frac{\pi}{2}\right]\right)$$

$N + N = 2N$

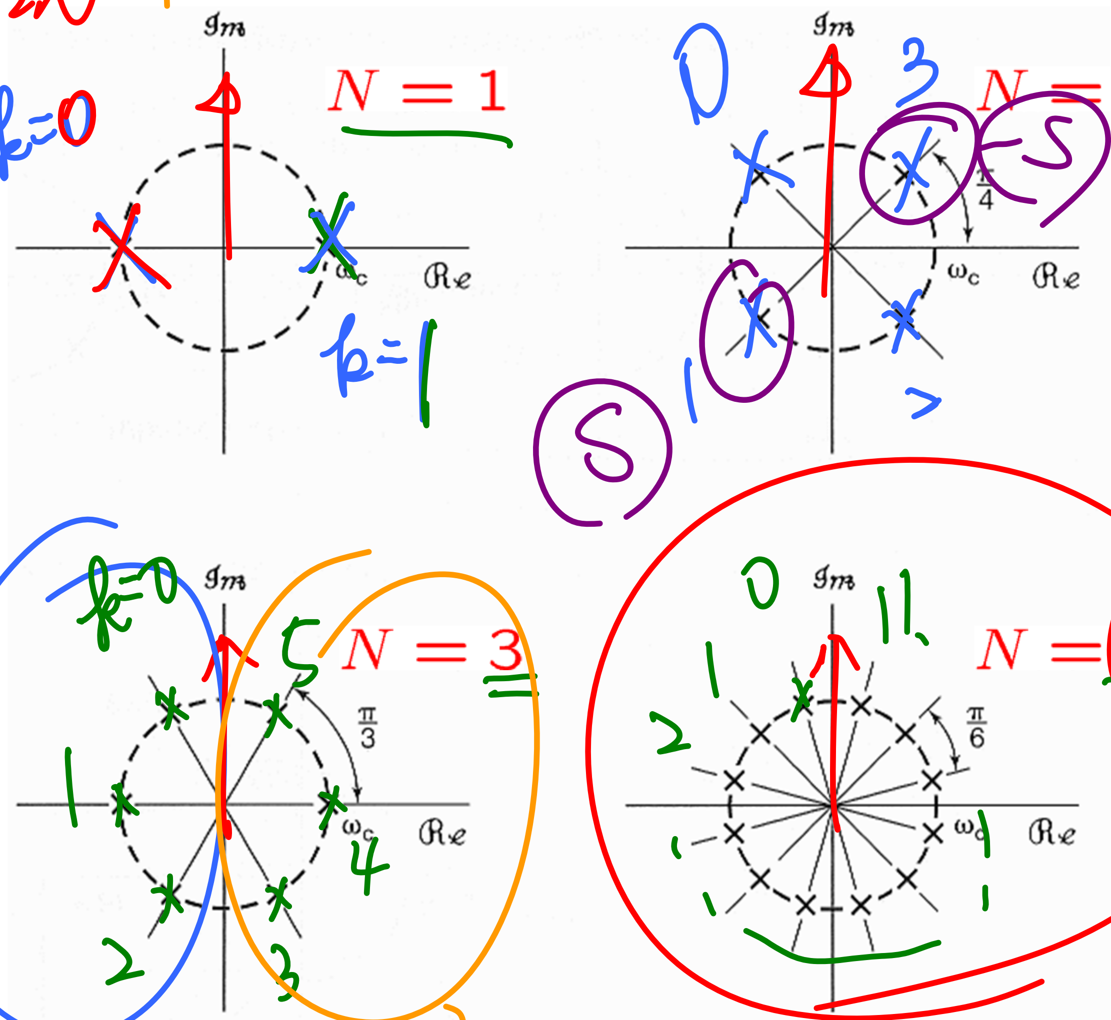
$6 + 6$ $k=0$

$k=1$

$s \to -s$

$= 2 \times 6 = 12$

6 6

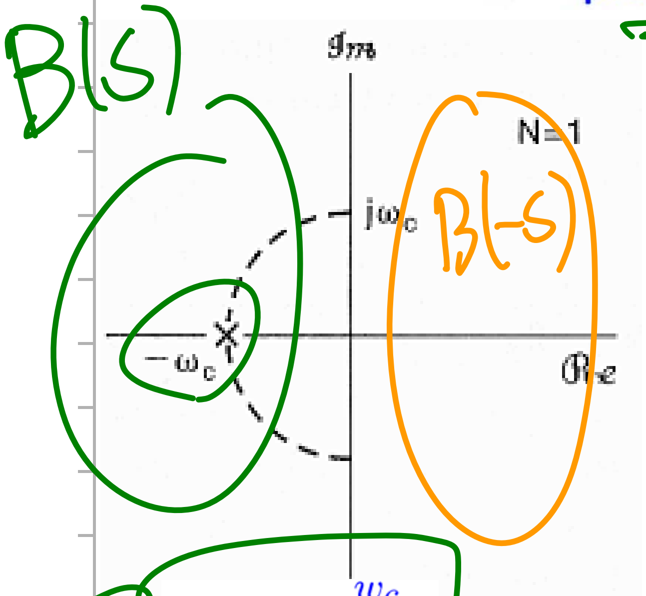


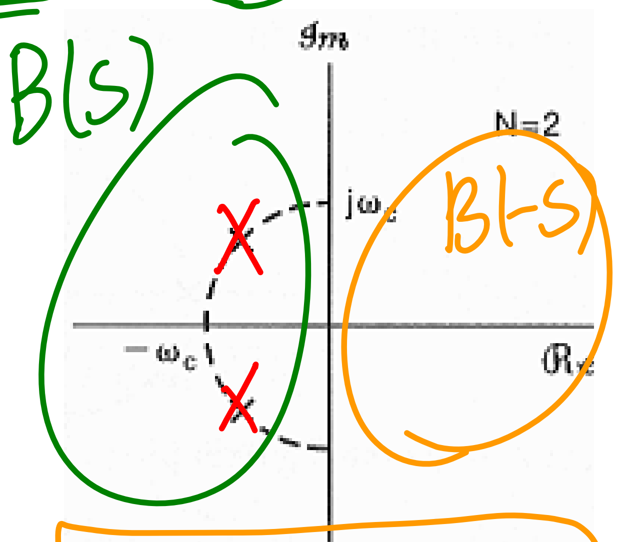
B(s) B(-s)

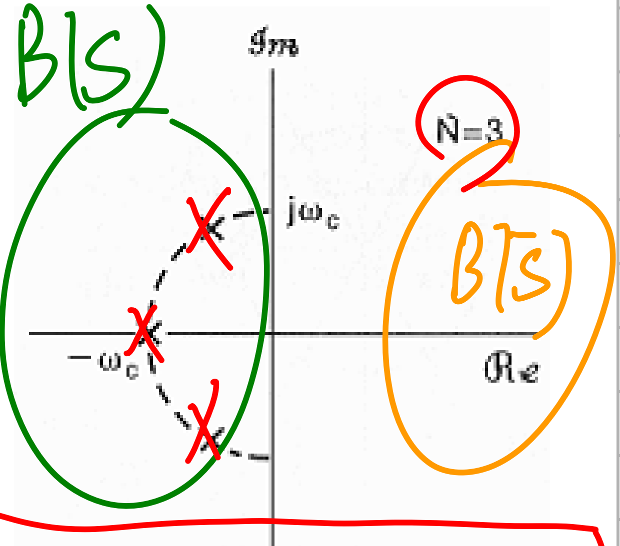
An Nth-Order Lowpass Butterworth Filters:

- Both $s = s_p$ and $s = -s_p$ are poles of $B(s)B(-s)$
- If the system is stable & causal

$\Rightarrow$ poles of $B(s)$ are in the left-hand plane







$$B(s) = \frac{w_c}{s + w_c}$$

$$B(-s) = \frac{w_c}{-s + w_c}$$

$$\frac{dy(t)}{dt} + w_c\, y(t) = w_c\, x(t)$$

$$B(s) = \frac{w_c^2}{s^2 + \sqrt{2} w_c s + w_c^2}$$

$$B(-s) = \frac{w_c^2}{s^2 - \sqrt{2} w_c s + w_c^2}$$

$$\frac{d^2 y(t)}{dt^2} + \sqrt{2} w_c \frac{dy(t)}{dt} + w_c^2\, y(t) = w_c^2\, x(t)$$

$$B(s) = \frac{w_c^3}{s^3 + 2w_c s^2 + 2w_c^2 s + w_c^3}$$

$$B(-s) = \frac{w_c^3}{-s^3 + 2w_c s^2 - 2w_c^2 s + w_c^3}$$

- An Nth-Order Lowpass Butterworth Filters:

$$B(s) = \frac{w_c}{s + w_c}$$

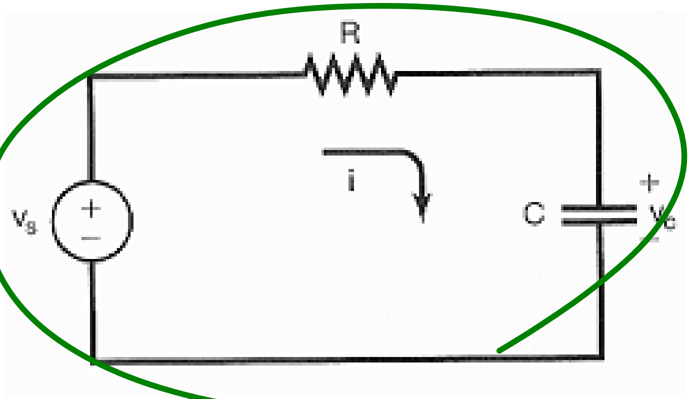
$$B(s) = \frac{w_c^2}{s^2 + \sqrt{2} w_c s + w_c^2}$$

$$B(s) = \frac{w_c^3}{s^3 + 2w_c s^2 + 2w_c^2 s + w_c^3}$$

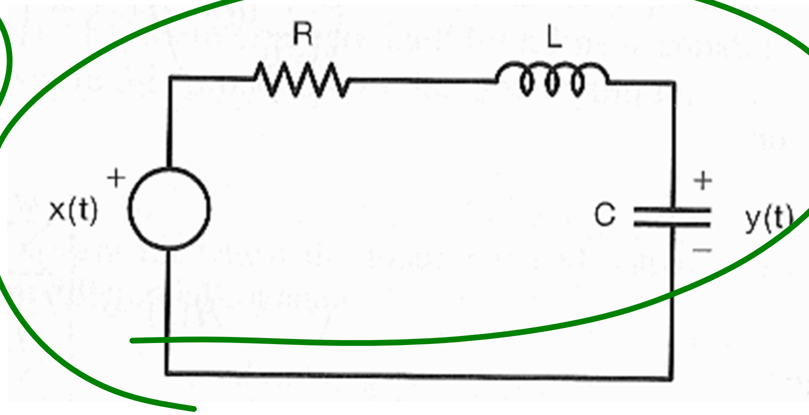
$$\frac{dy(t)}{dt} + w_c\, y(t) = w_c\, x(t)$$

$$\frac{d^2y(t)}{dt^2} + \sqrt{2} w_c \frac{dy(t)}{dt} + w_c^2\, y(t) = w_c^2\, x(t)$$

1



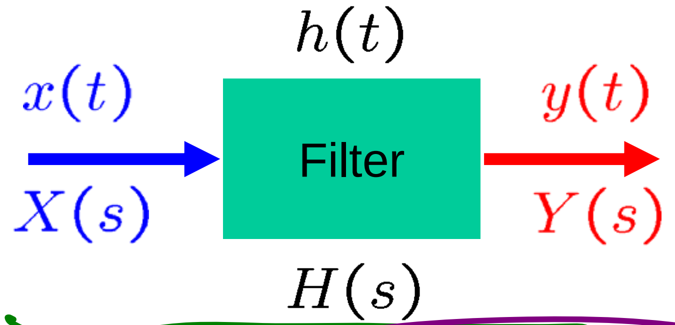
2



$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

$$LC \frac{d^2y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

- **Chebyshev Filters:**



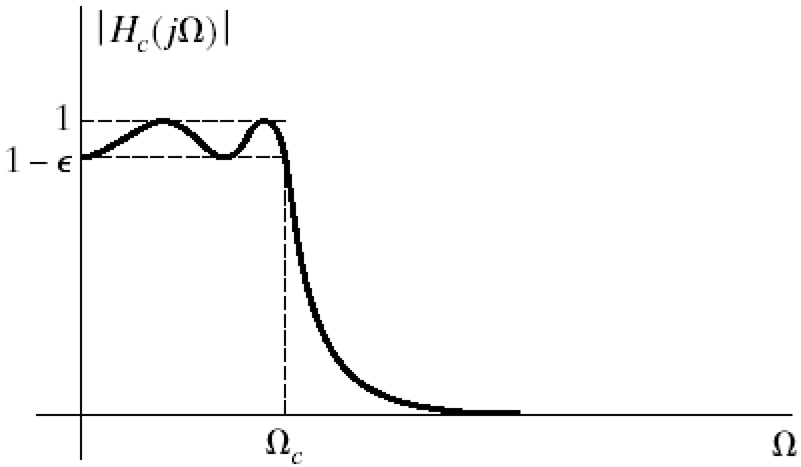
$$V_N(x) = \cos\left(N \cos^{-1} x\right)$$

Nth-order Chebyshev polynomial

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\epsilon^2 V_N^2\left(\frac{\Omega}{\Omega_c}\right)} = 0$$

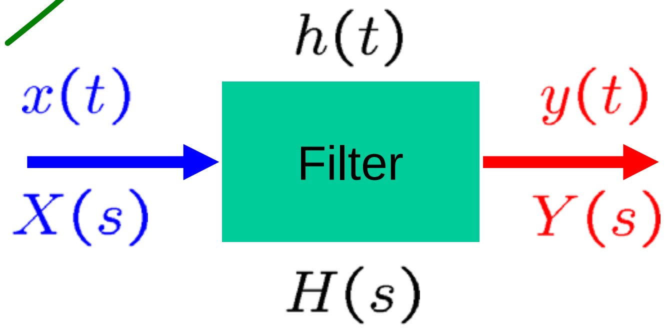
poles

$$\begin{aligned}
V_0(x) &= 1\\
V_1(x) &= x\\
V_2(x) &= 2x^2 - 1\\
V_3(x) &= 4x^3 - 3x\\
\cdots & \quad \cdots
\end{aligned}$$



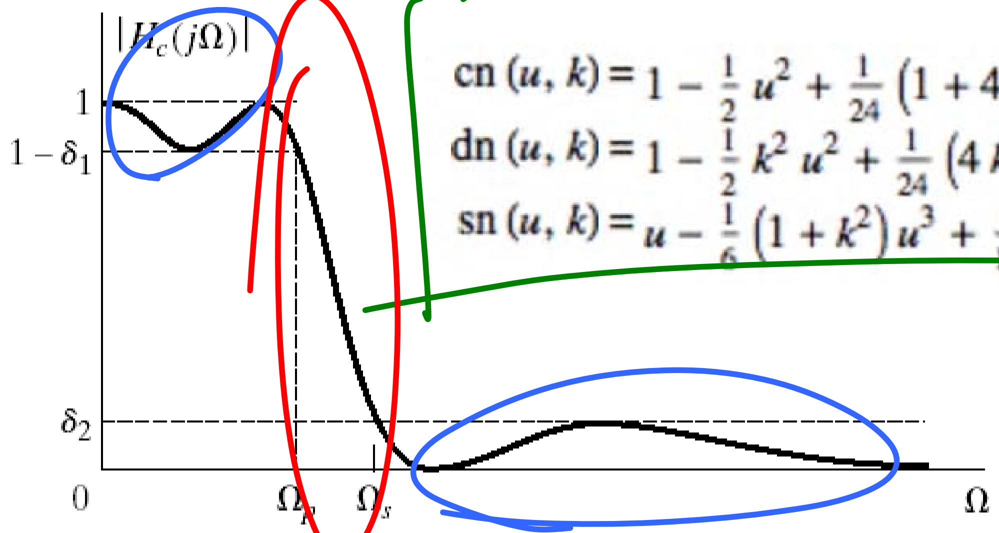
$$\begin{aligned}
\cos(0\theta) &= 1\\
\cos(1\theta) &= \cos(\theta)\\
\cos(2\theta) &= 2\cos(\theta)\cos(\theta) - \cos(0\theta)\\
&= 2\cos^2(\theta) - 1\\
\cos(3\theta) &= 2\cos(\theta)\cos(2\theta) - \cos(\theta)\\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

- **Elliptic Filters:**



$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1 + \epsilon^2 U_N^2(\Omega)}$$

$U_N(x)$: Jacabian elliptic function

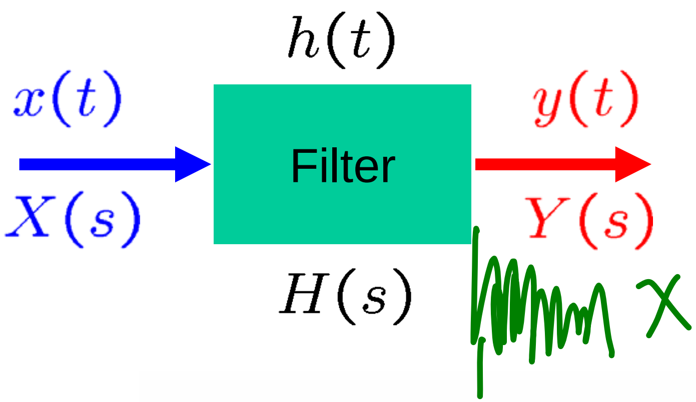


$$\mathrm{cn}(u,k) = 1 - \frac{1}{2}u^2 + \frac{1}{24}\left(1 + 4k^2\right)u^4 - \frac{1}{720}\left(1 + 44k^2 + 16k^4\right)u^6 + \ldots$$

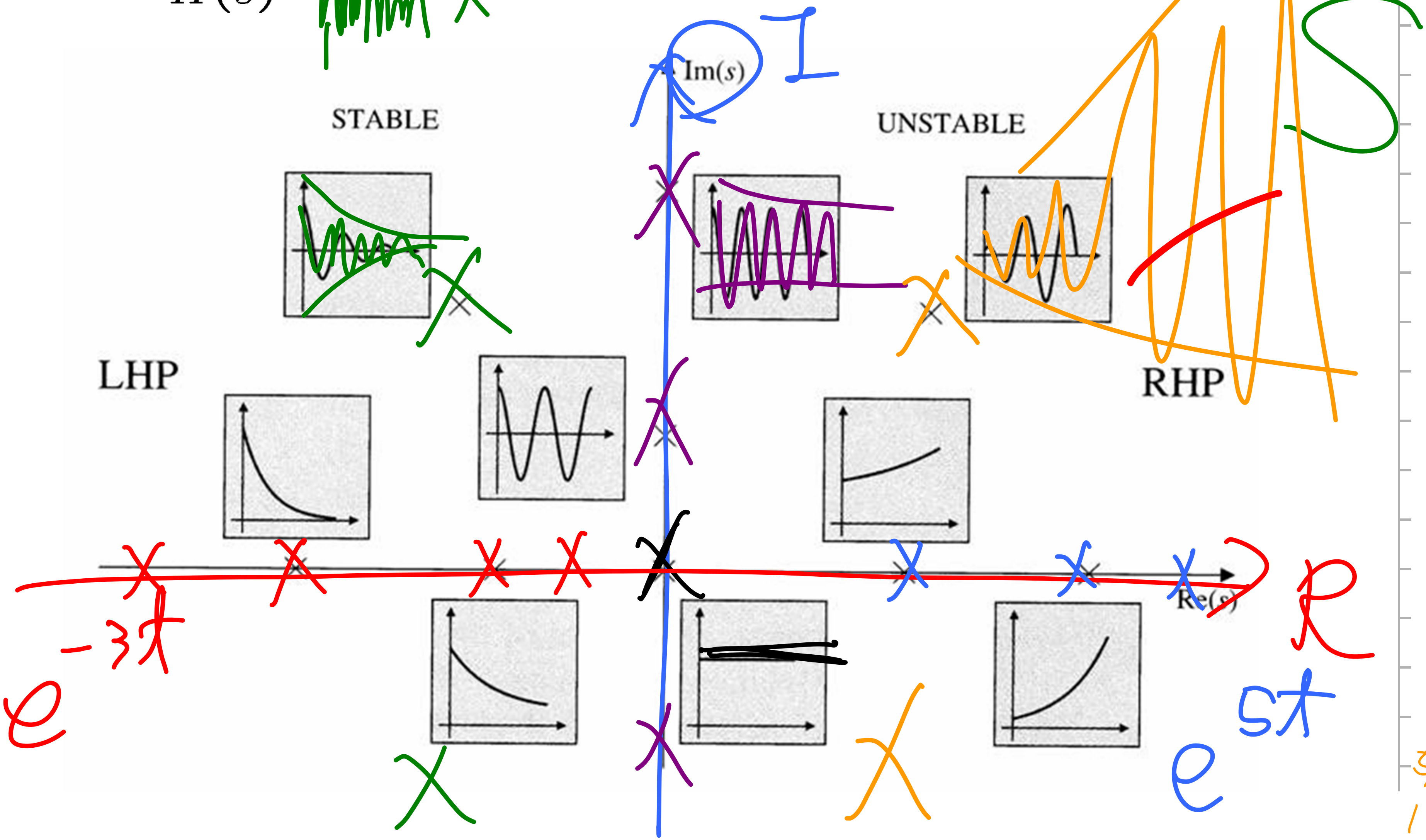
$$\mathrm{dn}(u,k) = 1 - \frac{1}{2}k^2u^2 + \frac{1}{24}\left(4k^2 + k^4\right)u^4 - \frac{1}{720}\left(16k^2 + 44k^4 + k^6\right)u^6 + \ldots$$

$$\mathrm{sn}(u,k) = u - \frac{1}{6}\left(1 + k^2\right)u^3 + \frac{1}{120}\left(1 + 14k^2 + k^4\right)u^5 + \ldots$$

System Characteristics and Pole Location



$$H(s) = \frac{s + c}{(s+a)(s+b)}$$



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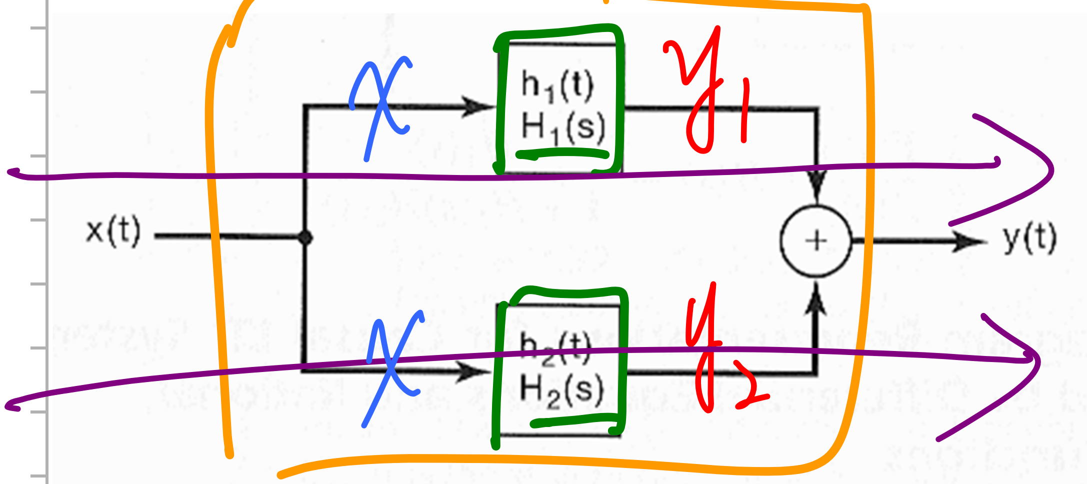
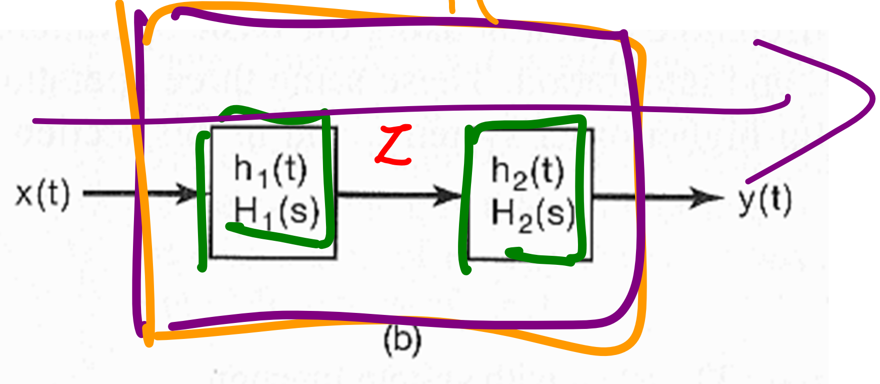
- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

System Function Blocks:

- parallel interconnection



- series interconnection



$$h(t) = h_1(t) + h_2(t)$$

$$H(s) = H_1(s) + H_2(s)$$

$y = y_1 + y_2$

$= x * h_1 + x * h_2$

$= x * (h_1 + h_2)$

$= x * h$

$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s)\, H_2(s)$$

$y = z * h_2 = (x * h_1) * h_2$

$z = x * h_1$

$= x * h_1 * h_2$

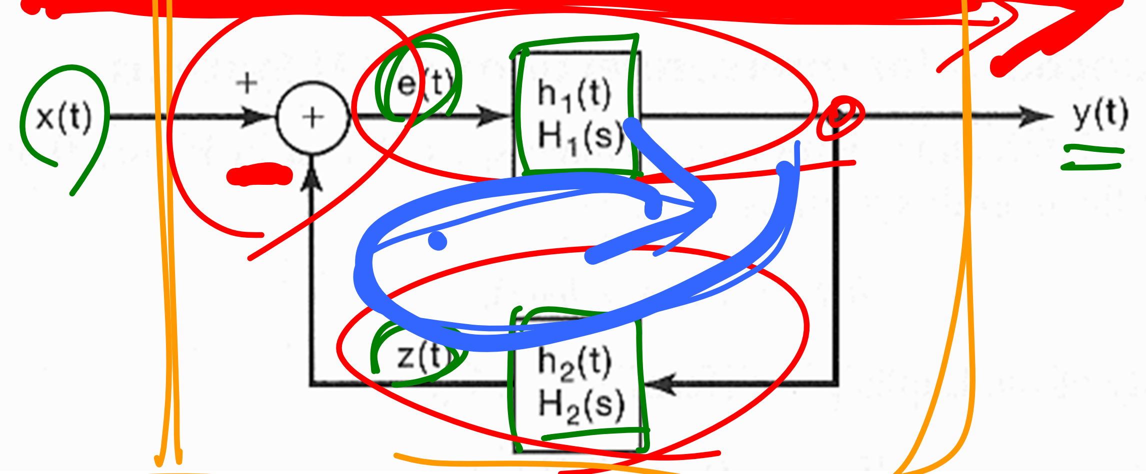
$H_1 \cdot H_2$

h

System Function Blocks:

- feedback interconnection

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$



$$Y = H_1 E = H_1(X - Z) = H_1(X - H_2 Y)$$

$$= H_1 X - H_1 H_2 Y$$

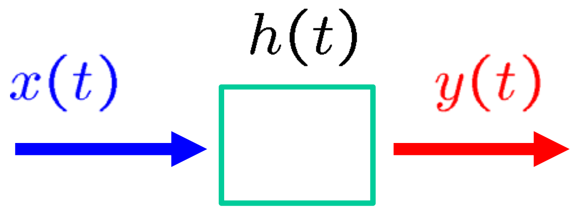
$$Z = H_2 Y$$

$$E = X - Z$$

$$H = \frac{Y}{X} = \frac{H_1}{1 + H_1 H_2}$$

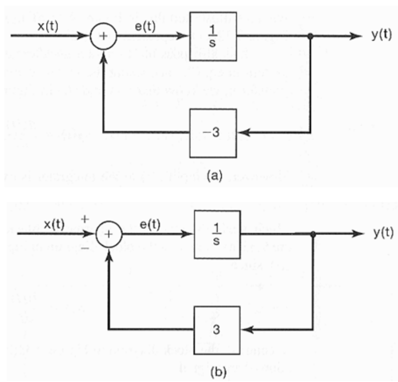
- Example 9.28:
  - Consider a causal LTI system with system function

$$H(s) = \frac{1}{s+3} \quad \Rightarrow Y(s) = \text{———} X(s)$$



$$\Rightarrow \frac{d}{dt} y(t) + \quad y(t) = x(t)$$

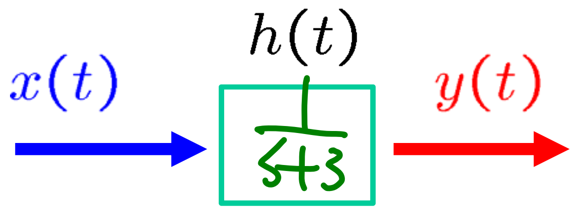
$$\Rightarrow \frac{d}{dt} y(t) = x(t) - \quad y(t)$$



Example 9.28:

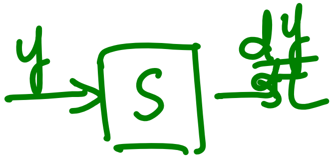
- Consider a causal LTI system with system function

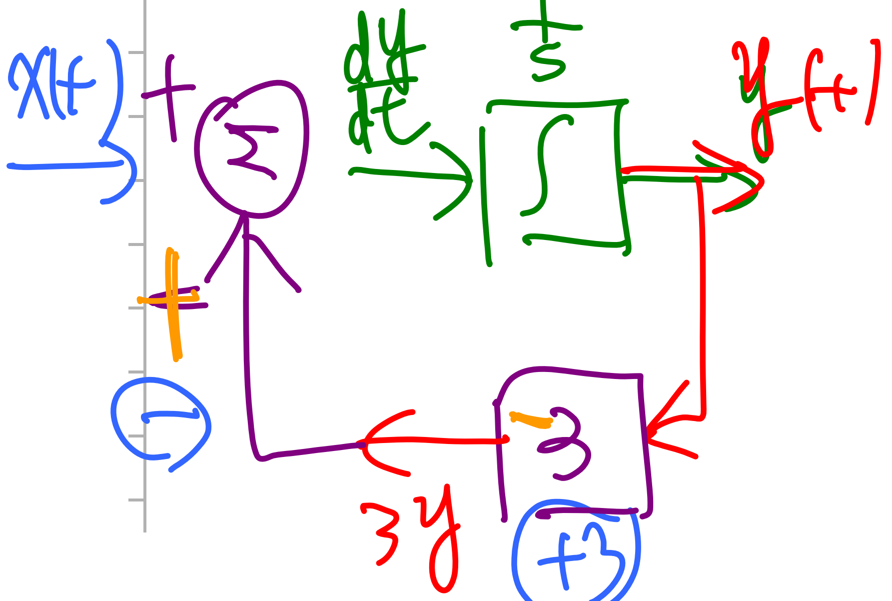
$$H(s) = \boxed{\frac{1}{s+3}} \quad \Rightarrow \quad Y(s) = \boxed{\frac{1}{s+3}} X(s)$$

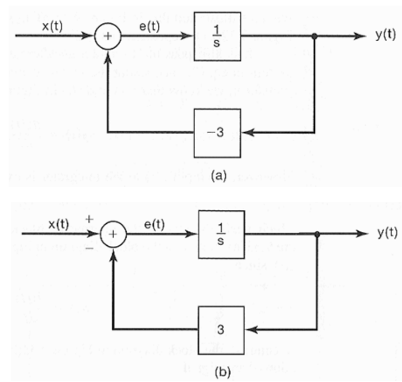


$$\Rightarrow \frac{d}{dt}y(t) + 3y(t) = x(t)$$

$$\Rightarrow \frac{d}{dt}y(t) = x(t) - 3y(t)$$





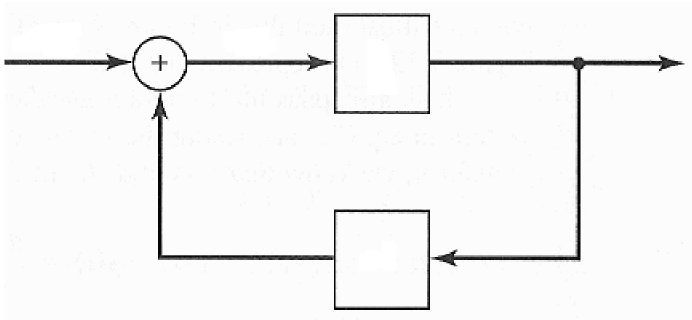


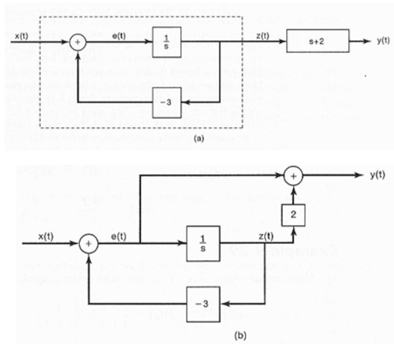
Example 9.29:

- Consider a causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{\quad}{\quad} \right) \left(\qquad \right)$$

$$\Rightarrow \quad Z(s) \triangleq \frac{\quad}{\quad} X(s) \quad \& \quad Y(s) = \left(\qquad \right) Z(s)$$

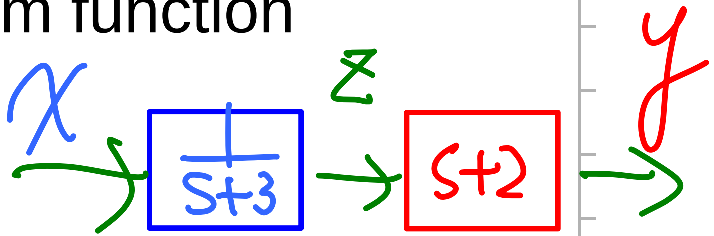




Example 9.29:

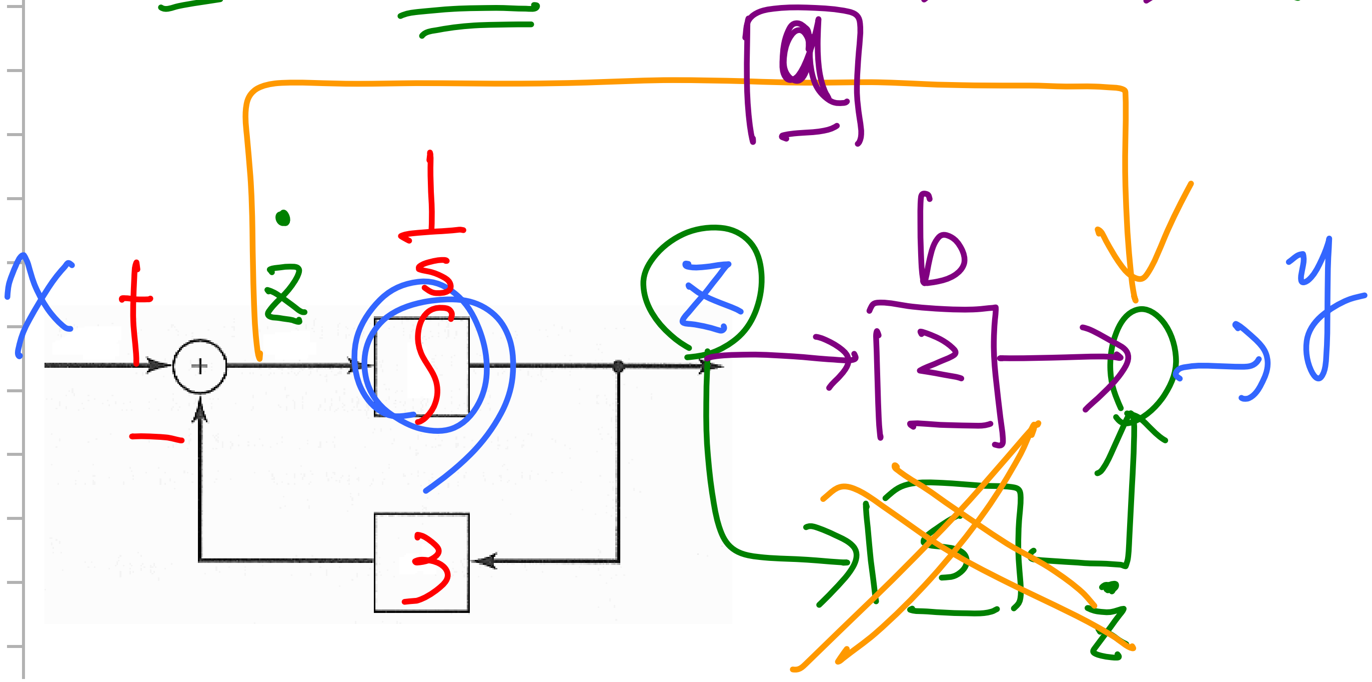
- Consider a causal LTI system with system function

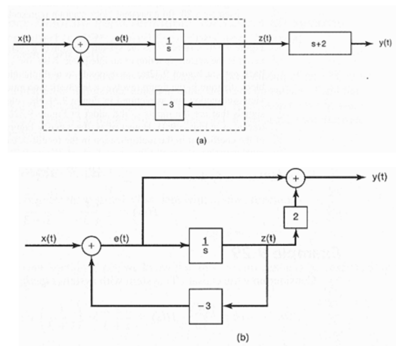
$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2)$$



$(as+b)$

$$\Rightarrow \quad Z(s) \triangleq \frac{1}{s+3}X(s) \quad \& \quad Y(s) = (s+2)Z(s)$$





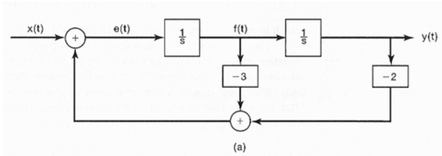
Example 9.30:

- Consider a causal LTI system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{\quad}$$

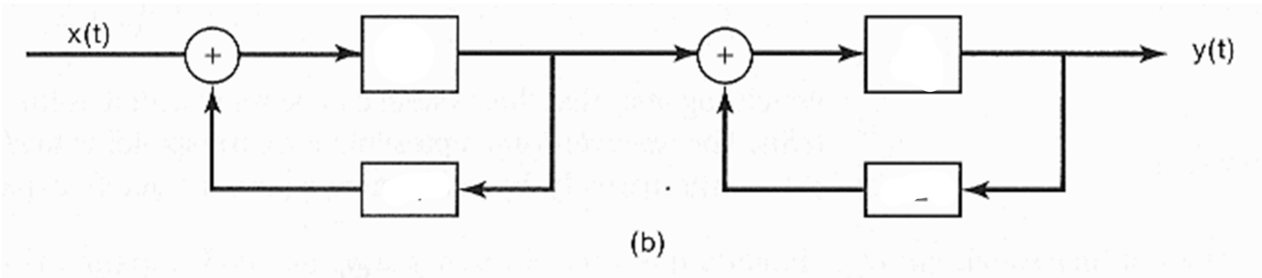
$$\Rightarrow s^2 Y + sY + Y = X$$

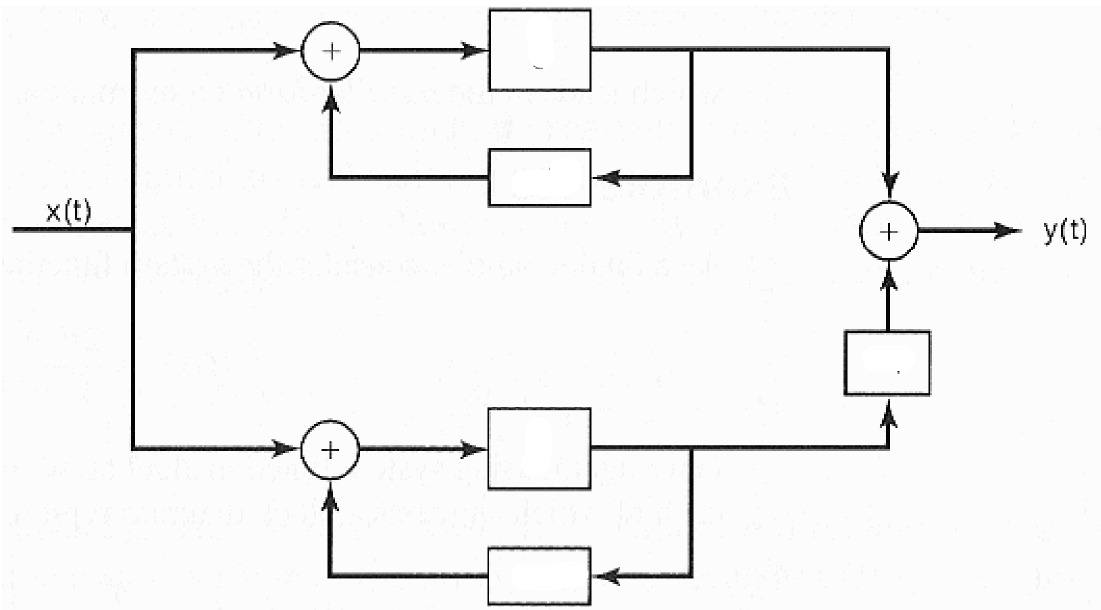
$$\Rightarrow \begin{cases} sY = \\ s^2 Y = \quad = \end{cases} \qquad \Rightarrow E = s^2 Y =$$



- Example 9.30:

$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\quad\right)\left(\quad\right) = \left(\quad\right) + \left(\quad\right)$$





Example 9.30:

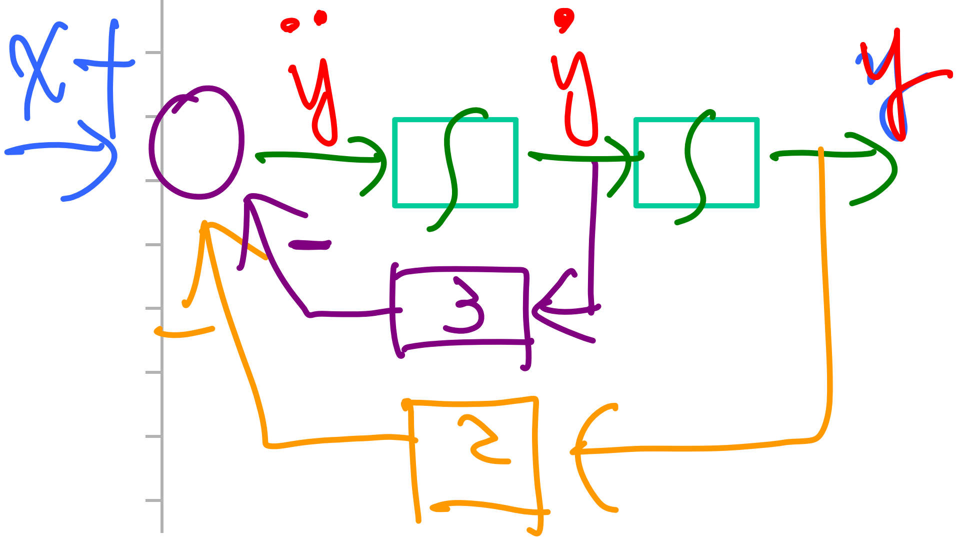
- Consider a causal LTI system with system function

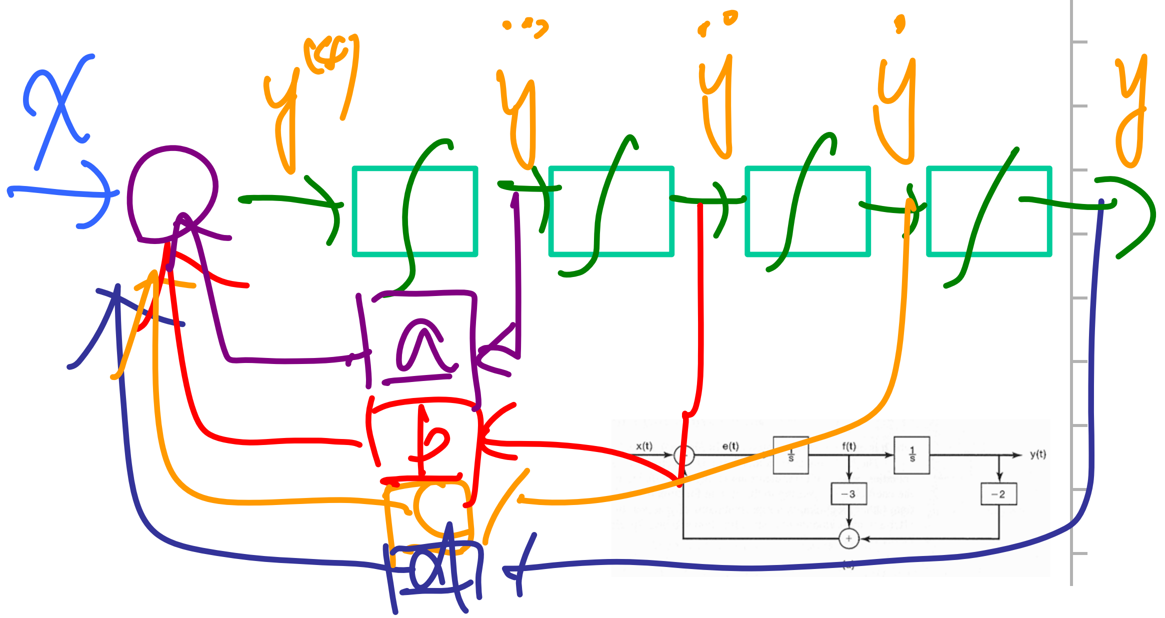
$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

$$\Rightarrow s^2Y + 3sY + 2Y = X$$

sY

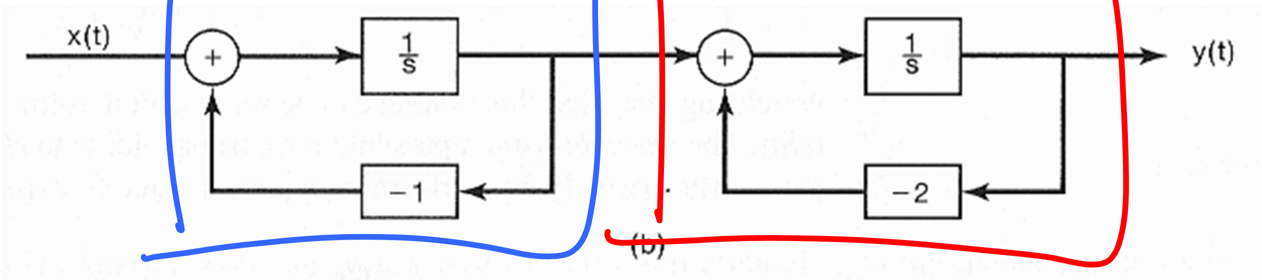
$$\Rightarrow \begin{cases} sY = F \\ s^2Y = E = sF \end{cases} \qquad \Rightarrow E = s^2Y = -3F - 2Y + X$$

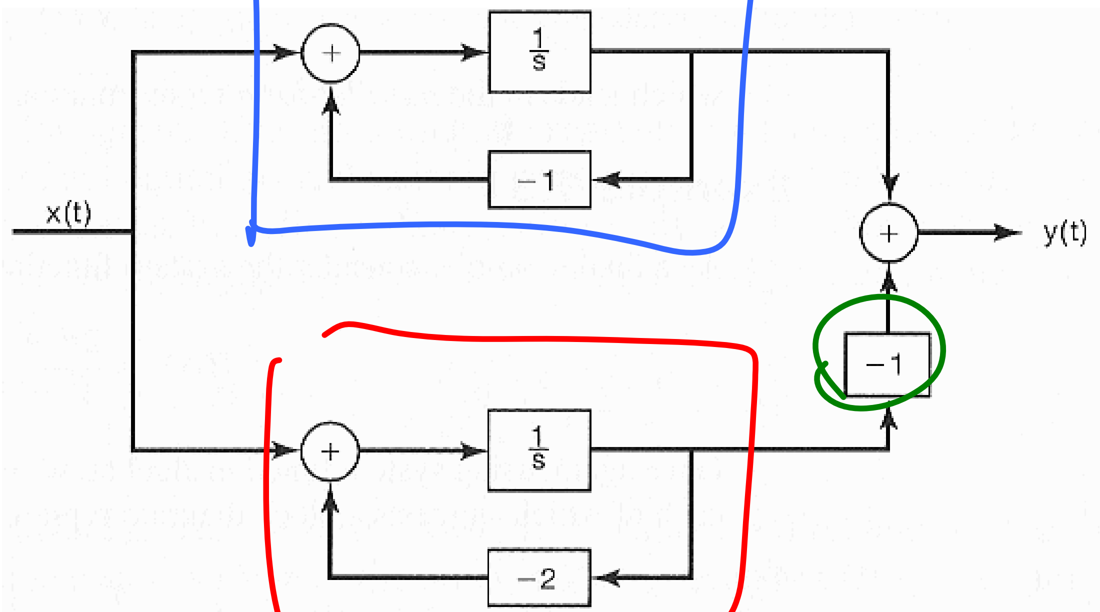




- Example 9.30:

$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right) = \left(\frac{1}{s+1}\right) + \left(\frac{-1}{s+2}\right)$$

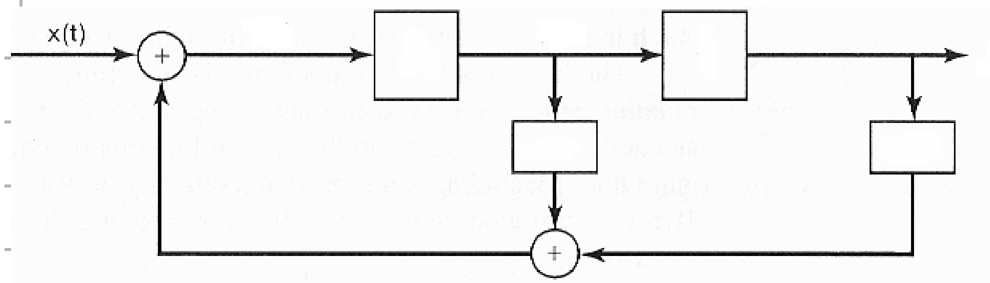


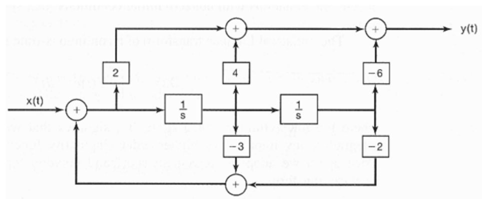


Example 9.31:

$$H(s) = \frac{2s^2+4s-6}{s^2+3s+2} = \left(\frac{1}{\quad\quad}\right)\left(\quad\quad\right)$$

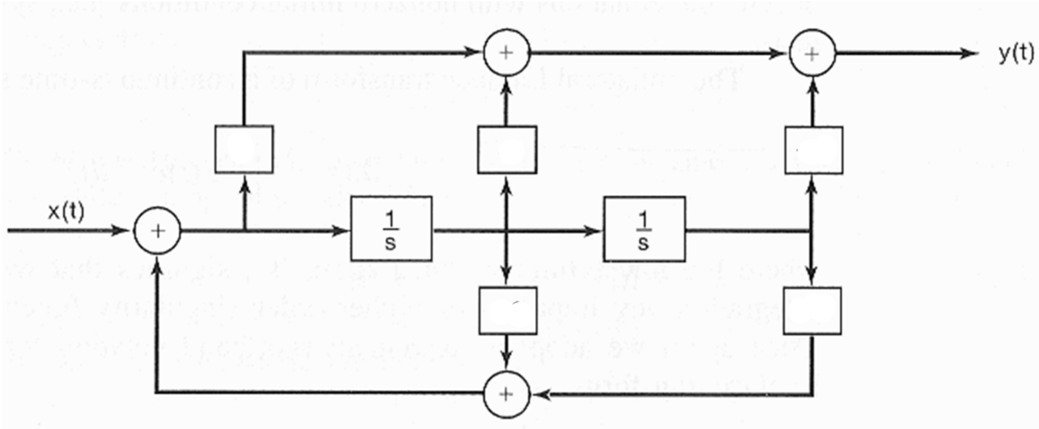
$$\Rightarrow \quad Z(s) \triangleq \frac{1}{\quad\quad} X(s) \quad \& \quad Y(s) = \left(\quad\quad\right) Z(s)$$

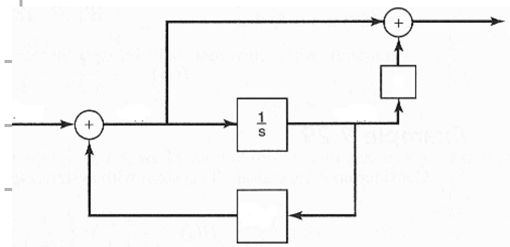


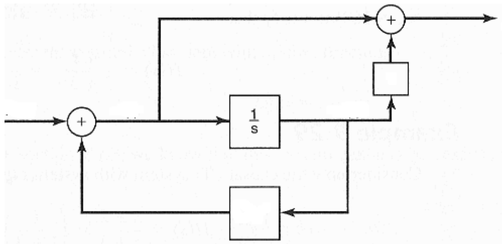


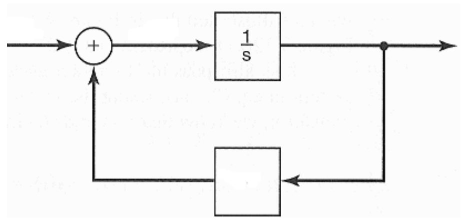
System Function Algebra & Block Diagram Representation

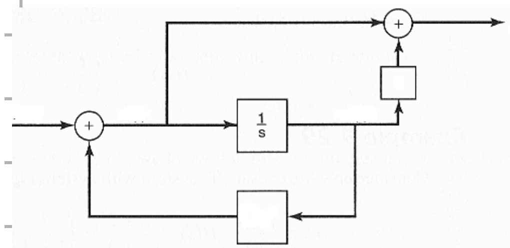
$$H(s) = \begin{cases} \dfrac{2s^2+4s-6}{s^2+3s+2} \\ \left(\dfrac{2(s-1)}{s+2}\right)\left(\dfrac{s+3}{s+1}\right) \\ \left(\dfrac{2(s-1)}{s+1}\right)\left(\dfrac{s+3}{s+2}\right) \\ 2+\dfrac{6}{s+2}+\dfrac{-8}{s+1} \end{cases}$$

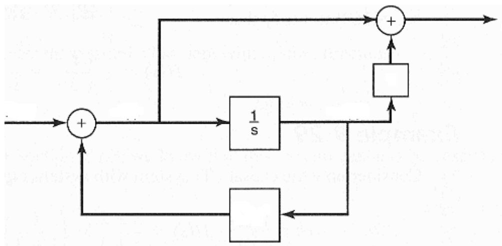


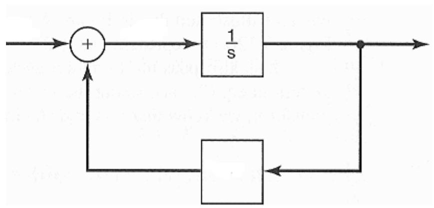








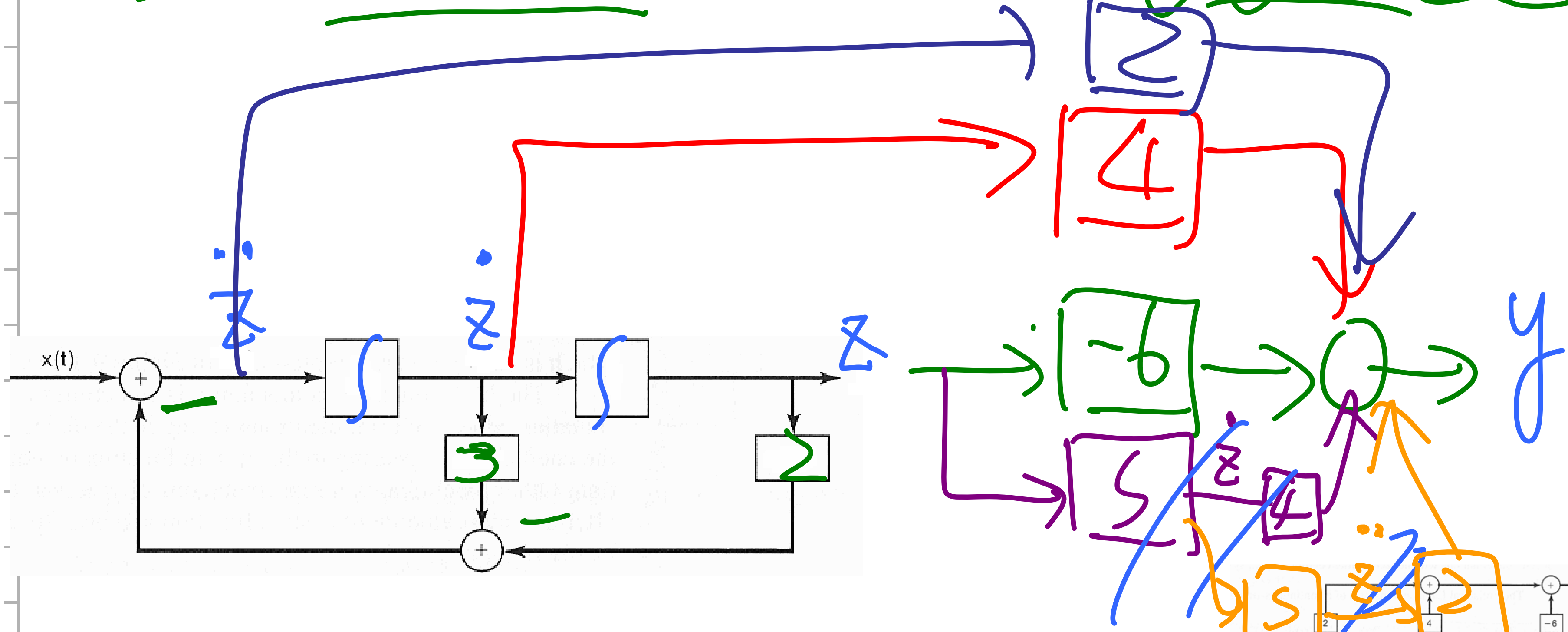


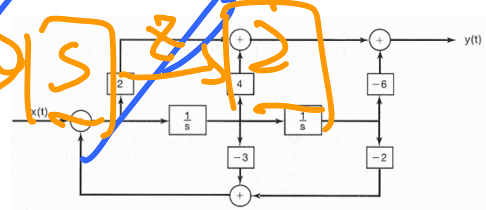


- Example 9.31:

$$H(s) = \frac{2s^2+4s-6}{s^2+3s+2} = \left(\frac{1}{s^2+3s+2}\right)(2s^2+4s-6)$$

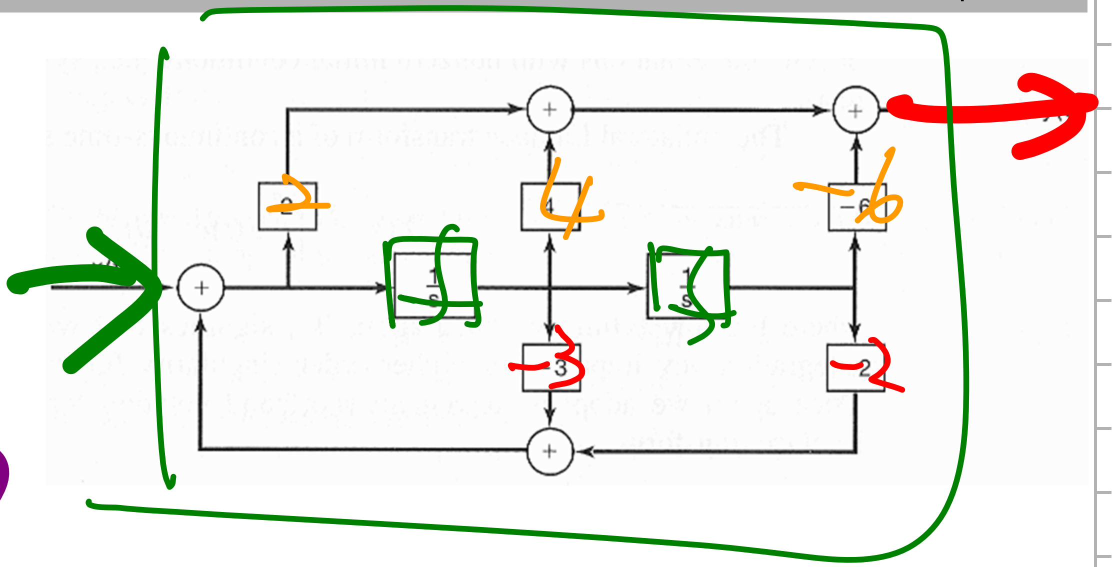
$$\Rightarrow Z(s) \triangleq \frac{1}{s^2+3s+2}X(s) \quad \& \quad Y(s) = (2s^2+4s-6)Z(s)$$

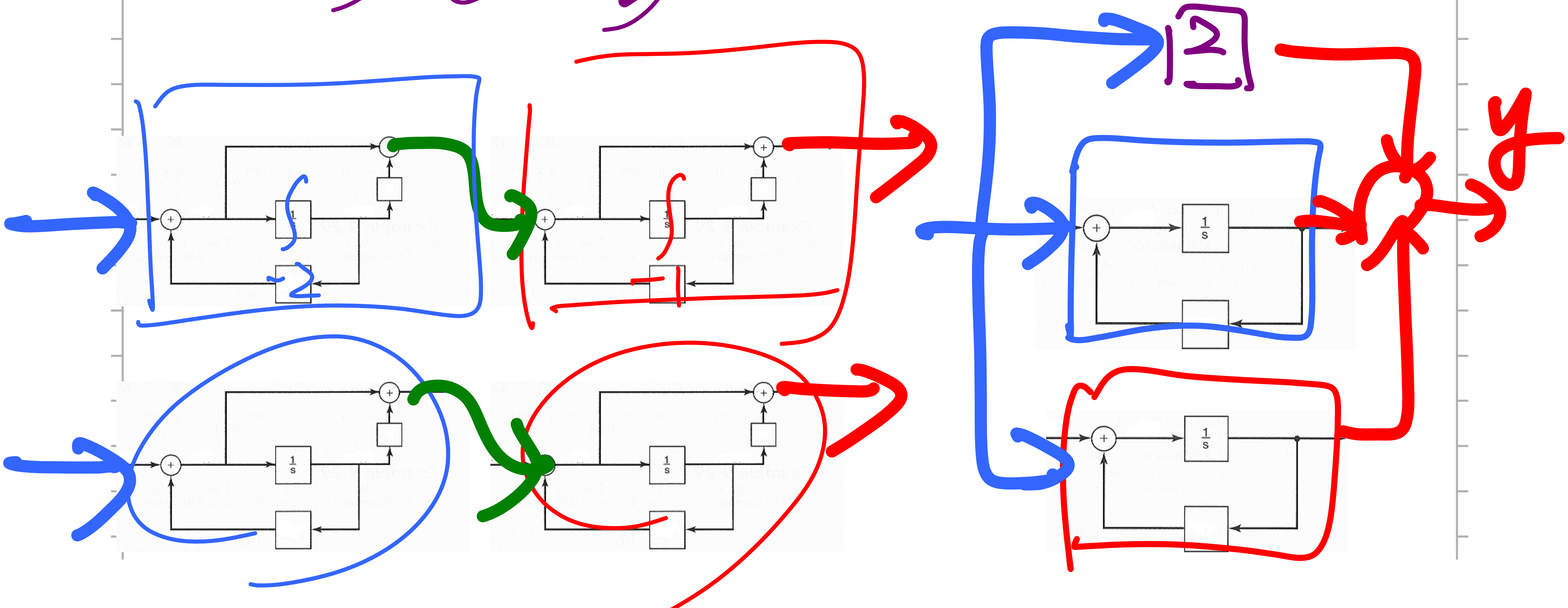




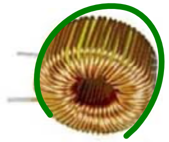
System Function Algebra & Block Diagram Representation

$$H(s) = \begin{cases} \dfrac{2s^2+4s-6}{s^2+3s+2} \\[2ex] \left(\dfrac{2(s-1)}{s+2}\right)\left(\dfrac{s+3}{s+1}\right) \\[2ex] \left(\dfrac{2(s-1)}{s+1}\right)\left(\dfrac{s+3}{s+2}\right) \\[2ex] 2+\dfrac{6}{s+2}+\dfrac{-8}{s+1} \end{cases}$$

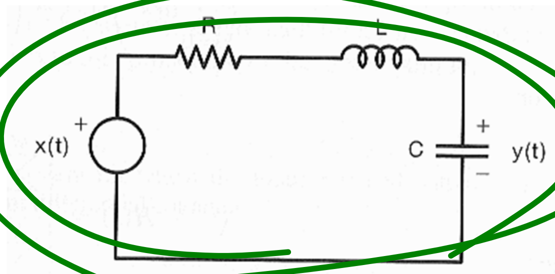




- Technology



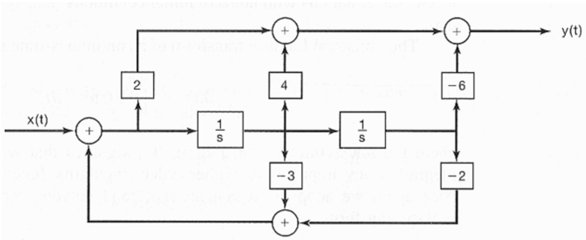
- Engineering

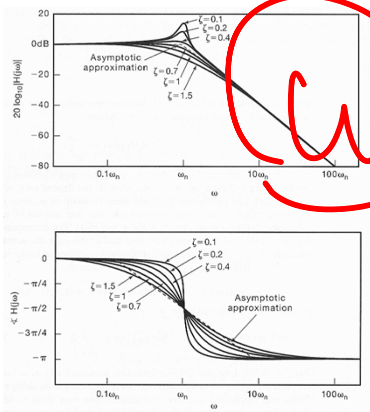


- Mathematics

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

- Graph





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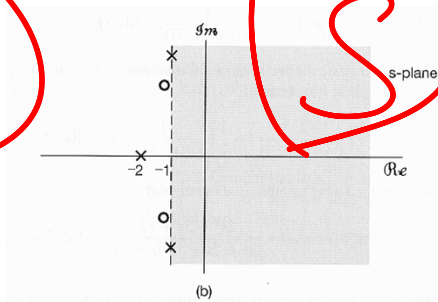


S

system

5/20/13
11:10 am

Outline

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

- The Unilateral Laplace Transform of x(t):

bilateral LT

unilateral LT
for causal system &
with nonzero initial condition

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad \mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{0} x(t) e^{-st} dt + \int_{0}^{\infty} x(t) e^{-st} dt$$

a I.C.

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$$

$$\mathcal{X}(s) = \mathcal{UL}\{x(t)\}$$

$$x(t) = \mathcal{UL}^{-1}\{\mathcal{X}(s)\}$$

ROC : a right-half plane

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$x^*(s)$
$\boxed{\text{Convolution (assuming that } x_1(t) \text{ and } x_2(t) \text{ are identically zero for } t < 0)}$	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$\boxed{s\mathfrak{X}(s) - x(0^-)}$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathfrak{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathfrak{X}(s)$$

$x_1(t) = x_2(t) \equiv 0$

$t < 0$

$sX(s) - x(0^-)$

- **Differentiation Property:**

$$\boxed{\int_0^\infty u'v\,dt = uv\Big|_0^\infty - \int_0^\infty uv'\,dt}$$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty}\frac{dx(t)}{dt}e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty}x(t)e^{-st}dt$$

$$= s\mathcal{X}(s) - x(0^-)$$

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_{0^-}^{\infty}\frac{d^2x(t)}{dt^2}e^{-st}dt = s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

- Example 9.38:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2\,y(t) = x(t) \qquad \begin{cases} y(0^-) = \beta \\ y'(0^-) = \gamma \end{cases}$$

$s^2 Y(s) \quad 3sY(s) \quad 2Y(s)$

$$x(t) = \alpha\,u(t)$$

$$\Rightarrow \left[s^2\mathcal{Y}(s) - \beta s - \gamma\right] + 3\left[s\mathcal{Y}(s) - \beta\right] + 2\left[\mathcal{Y}(s)\right] = \frac{\alpha}{s}$$

$$\Rightarrow \mathcal{Y}(s) = \underbrace{\frac{\alpha}{(s+1)(s+2)}}_{\text{zero-state response}} + \underbrace{\frac{\beta}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{zero-input response}}$$

input only response

State only response

Example 9.38:

- If $\alpha = 2, \quad \beta = 3, \quad \gamma = -5$

$$\Rightarrow \mathcal{Y}(s) = -$$

$$\Rightarrow y(t) = \left[\; - \; + \; \right] u(t), \quad \text{for } t > 0$$

- Example 9.38:

$$\mathcal{UL}\left\{\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right\} = \left\{x(t)\right\} \qquad \begin{cases} y(0^-) = \beta \\ y'(0^-) = \gamma \end{cases}$$

$$x(t) = \alpha u(t)$$

$$\Rightarrow \left[s^2\mathcal{Y}(s) - \beta s - \gamma\right] + 3\left[s\mathcal{Y}(s) - \beta\right] + 2\left[\mathcal{Y}(s)\right] = \frac{\alpha}{s}$$

$$\Rightarrow \mathcal{Y}(s) = \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\text{zero-state response}} + \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{zero-input response}}$$

Input-only response

state-only response

Example 9.38:

- If $\alpha = 2, \quad \beta = 3, \quad \gamma = -5$

$$\Rightarrow \mathcal{Y}(s) = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

$\Big\Downarrow \; \mathcal{UL}^{-1}$

$$\Rightarrow y(t) = \left[1 - e^{-t} + 3e^{-2t} \right] u(t), \quad \text{for } t > 0$$

Summary of Fourier Transform and Laplace Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

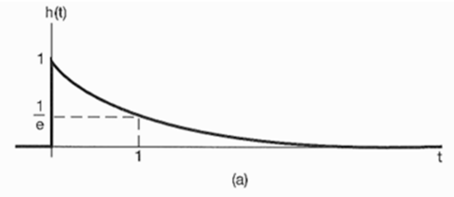
$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

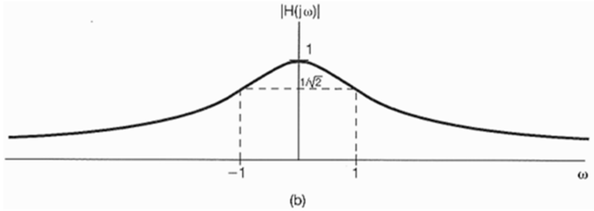
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \mathcal{F}\left\{x(t) e^{-\sigma t}\right\}$$

$$X(jw) = \mathcal{F}\left\{x(t)\right\} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = X(s)\Big|_{s=jw}$$

Summary of Fourier Transform and Laplace Transform





$$h(t) = e^{-at}u(t), \quad a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(jw) = \frac{1}{jw + a}$$

$$h(t) = e^{-at}u(t), \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{1}{s + a},$$

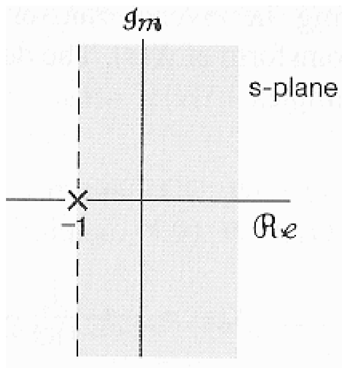
$$\mathcal{R}e\{s\} > -a$$

definition
theorem
property

Causality

Stability

ROC



- **Example 9.26:** $x(t) = 1 \rightarrow y(t) = 0$

non causal
~~un~~stable

$$H(s) = \frac{4s}{(s+2)(s-4)} = \frac{4/3}{s+2} + \frac{8/3}{s-4}$$

$$h(t) = \frac{4}{3} e^{-2t} u(t) - \frac{8}{3} e^{4t} u(-t)$$

$$Y(s) = H(s)X(s) = H(s) \; 2\pi j \, \delta(s) = H(0) = 0$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) \, x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{4}{3} e^{-2\tau} u(\tau) - \frac{8}{3} e^{4\tau} u(-\tau) \right\} x(t-\tau) d\tau$$

$$= \int_{0}^{\infty} \frac{4}{3} e^{-2\tau} d\tau - \int_{-\infty}^{0} \frac{8}{3} e^{4\tau} d\tau$$

$$= \frac{4}{3(-2)} e^{-2\tau} \Big|_{0}^{\infty} - \frac{8}{3(4)} e^{4\tau} \Big|_{-\infty}^{0} = \frac{-2}{3}(0-1) - \frac{2}{3}(1-0)$$

Problem 9.45 (p.733)

9.45. Consider the LTI system shown in Figure P9.45(a) for which we are given the following information:

$$X(s) = \frac{s+2}{s-2},$$

$$x(t) = 0, \qquad t > 0,$$

and

$$y(t) = -\frac{2}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(t). \qquad \text{[See Figure P9.45(b).]}$$

(a) Determine $H(s)$ and its region of convergence.

(b) Determine $h(t)$.

(c) Using the system function $H(s)$ found in part (a), determine the output $y(t)$ if the input is

$$x(t) = e^{3t}, \qquad -\infty < t < +\infty.$$

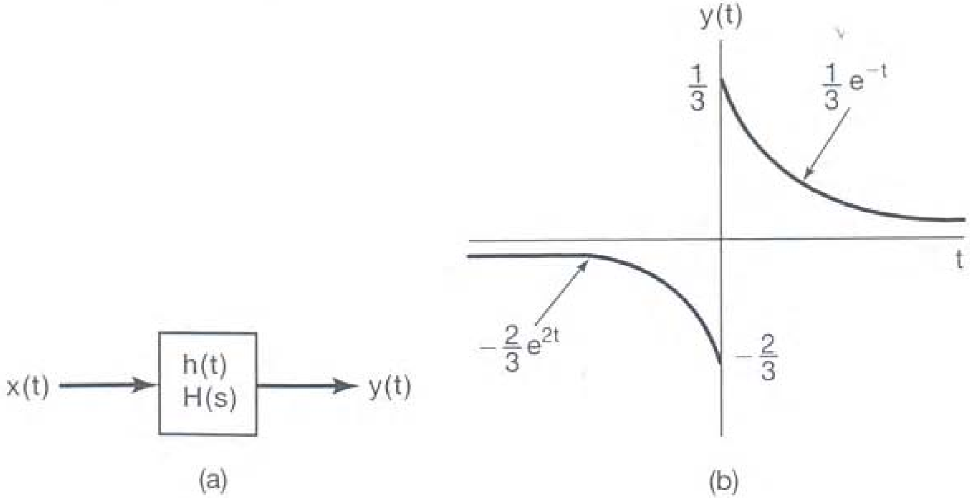


Figure P9.45

Final Exam in 2012

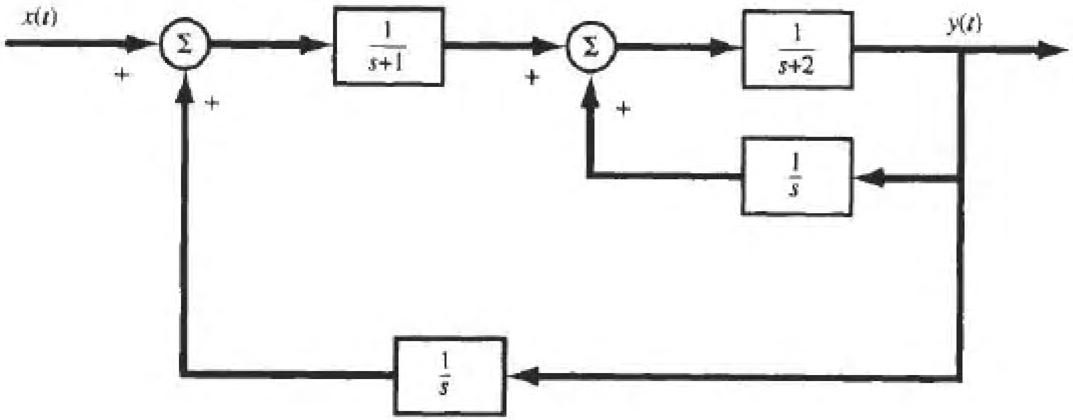
3. (10%) The system with impulse response $h(t)$ is causal and stable and has a rational system function $H(s)$. Identify the conditions on the system function $H(s)$ so that each of the following systems with impulse response $g(t)$ is stable and causal:

(a) $g(t) = \frac{d}{dt} h(t)$

(b) $g(t) = \int_{-\infty}^{t} h(\tau) d\tau$

Final Exam in 2012

4. (10%) Determine the overall system function $H(s)$ for the following system:



Final Exam in 2011

3. [18] Suppose the system function of a system is

$$H(s) = \frac{10(1-s)}{(1+s)(10+s)}$$

(a) Draw the block diagram of the system in direct, cascade, and parallel forms. [6]
(b) Sketch the Bode plot for $H(j\omega)$. [6]
(c) Use pole-zero plot to determine the magnitude and phase of $H(j\omega)$ graphically. [6]

- The Laplace Transform
- The ROC for LT
- The Inverse LT
- Geometric Evaluation of the FT
- Properties of the LT
  - Linearity
  - Time Shifting
  - Shifting in the s-Domain
  - Time Scaling
  - Conjugation
  - Convolution
  - Differentiation in the Time Domain
  - Differentiation in the s-Domain
  - Integration in the Time Domain
  - Initial- and Final-Value Theorems
- Some LT Pairs
- Analysis & Charac. of LTI Systems Using the LT
- System Function Algebra, Block Diagram Repre.
- The Unilateral LT

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS — **CT**, **DT** (Chap 3)

Aperiodic

FT
- **CT** (Chap 4)
- **DT** (Chap 5)

Unbounded/Non-convergent

LT — **CT** (Chap 9)

zT — **DT** (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)