Spring 2015

信號與系統
Signals and Systems

Chapter SS-9
The Laplace Transform

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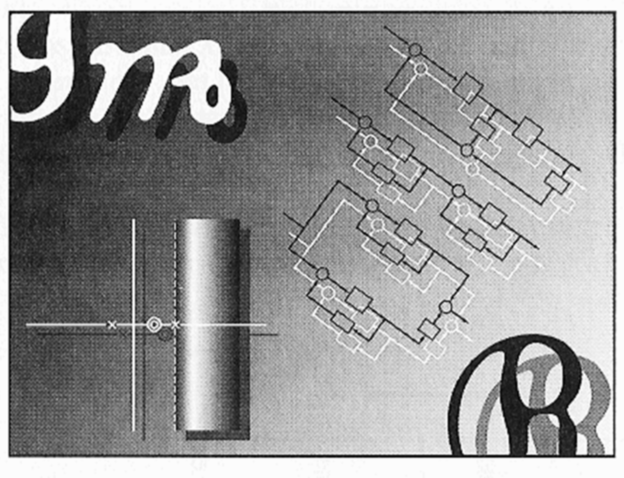
NTU-EE

Feb15 – Jun15



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS
CT
DT
(Chap 3)

Aperiodic

FT
CT (Chap 4)
DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6)

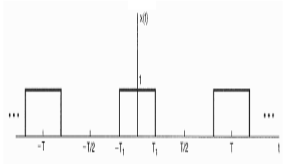
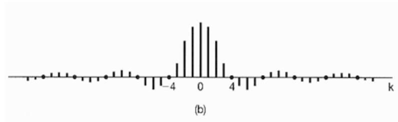
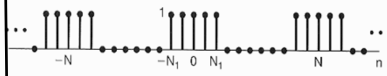
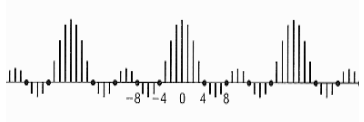
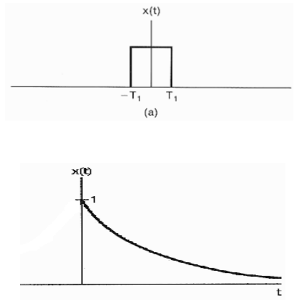
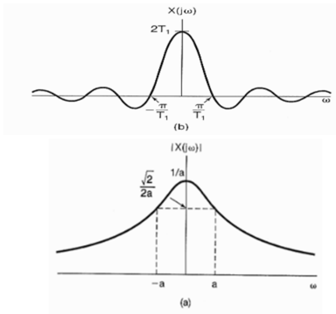
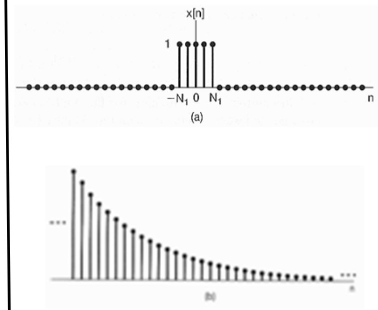
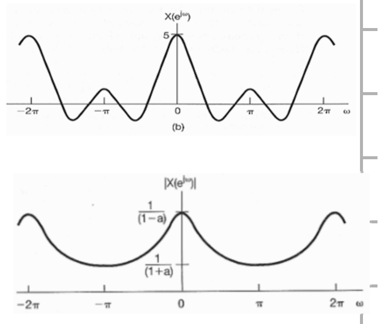
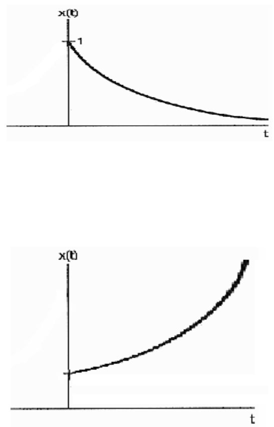
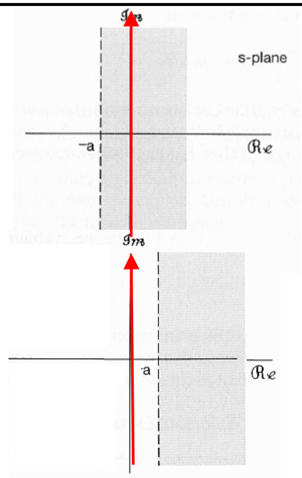
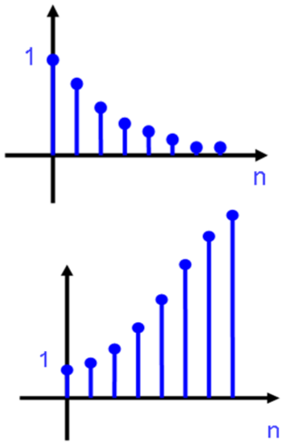
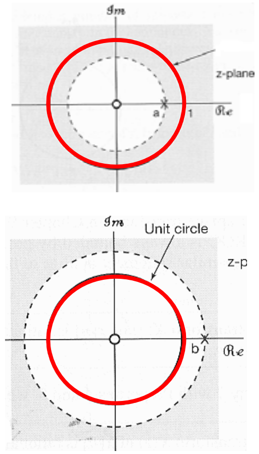
CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)

Fourier Series, Fourier Transform, Laplace Transform, z-Transform

	CT		DT	
	time	frequency	time	frequency
FS				
FT				
LT/zT				

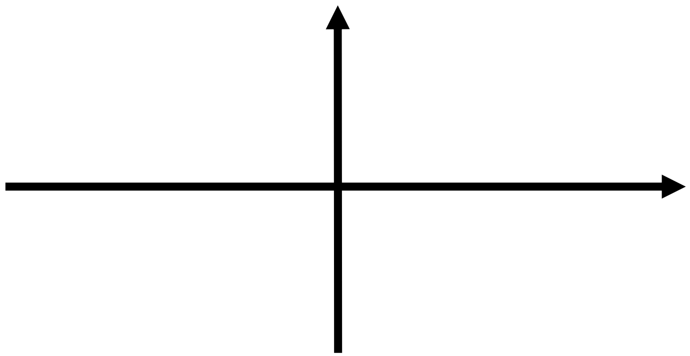
Outline

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

- The Laplace transform of a general signal x(t):

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \qquad\qquad X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$s = \sigma + jw$$



$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw) \qquad\qquad x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(jw) = \mathcal{F}\left\{ x(t) \right\} \qquad\qquad X(s) = \mathcal{L}\left\{ x(t) \right\}$$

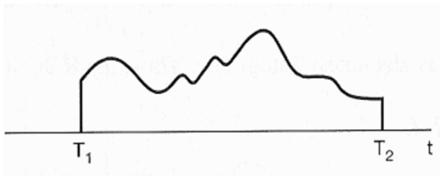
$$x(t) = \mathcal{F}^{-1}\left\{ X(jw) \right\} \qquad\qquad x(t) = \mathcal{L}^{-1}\left\{ X(s) \right\}$$

$$X(s)\Big|_{s=jw} = \mathcal{L}\left\{ x(t) \right\}\Big|_{s=jw} = \mathcal{F}\left\{ x(t) \right\} = X(jw)$$

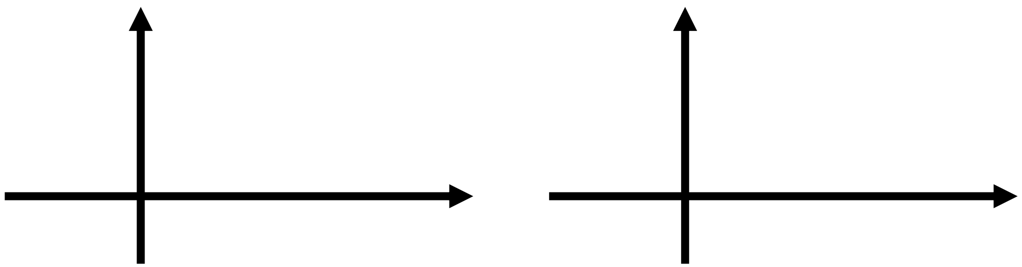
▪ Laplace Transform & Fourier Transform:

$$X(s)\Big|_{s=jw} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = \mathcal{F}\left\{x(t)\right\} = X(jw)$$

$$\mathcal{L}\left\{x(t)\right\} = X(s) \qquad s = \sigma + jw$$



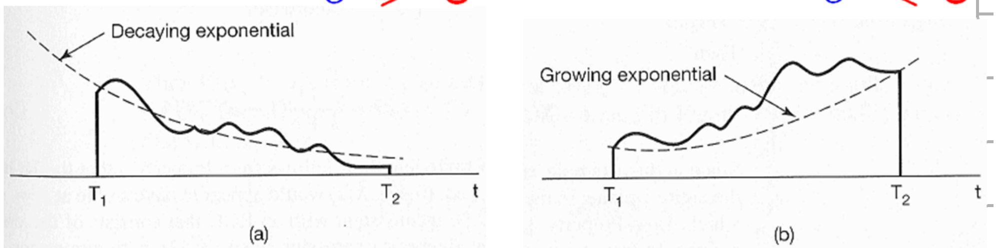
$$= X(\sigma + jw)$$



$$= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + jw)t} dt$$

$\sigma > 0$ $\qquad$ $\sigma < 0$

$$= \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right] e^{-jwt} dt$$



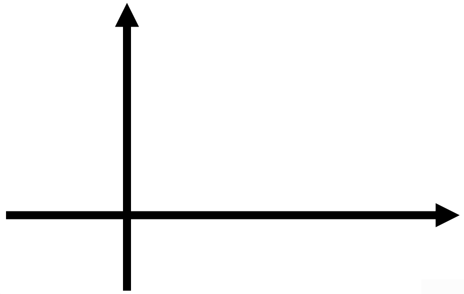
(a) $\qquad$ (b)

$$= \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$

- Example 9.1:

$$X(jw) \triangleq \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

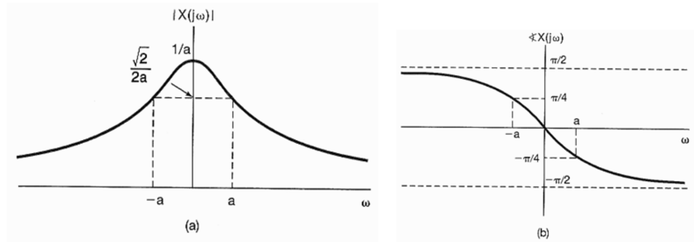


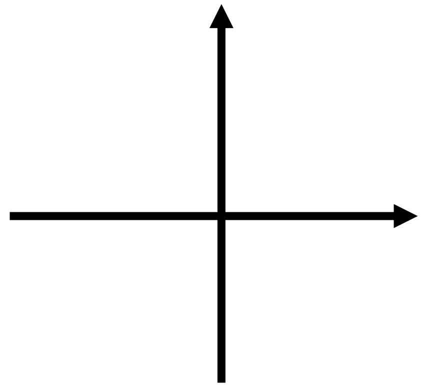
$$x(t) = e^{-at}u(t)$$

$$X(jw) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-jwt}dt \quad = \left. \frac{e^{-(a+jw)t}}{-(a+jw)} \right|_0^{\infty}$$

$$= \frac{1}{jw+a}, \quad a > 0$$





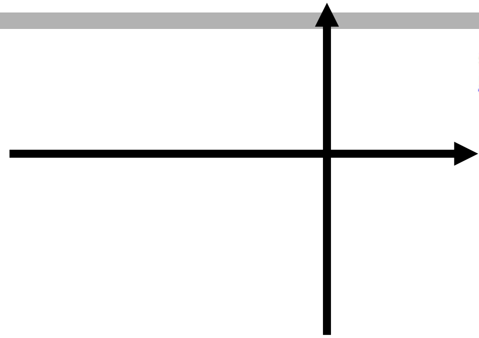
$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt \quad = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$X(\sigma + jw) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-jwt}dt \quad = \frac{1}{(\sigma+a)+jw}, \quad \sigma + a > 0$$

$$= \frac{1}{(\sigma+jw)+a} \quad = \frac{1}{s+a}, \quad \mathcal{Re}\{s\} > -a$$

▪ Example 9.2:

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

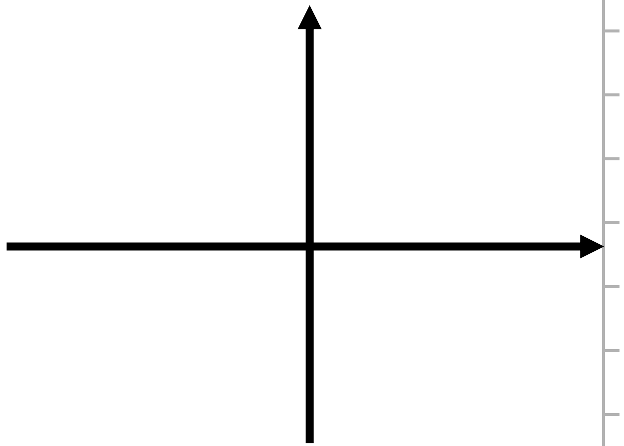


$$x(t) = -e^{-at} u(-t)$$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st} dt$$

$$= -\int_{-\infty}^{0} e^{-at} e^{-st} dt$$

$$= \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$



$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

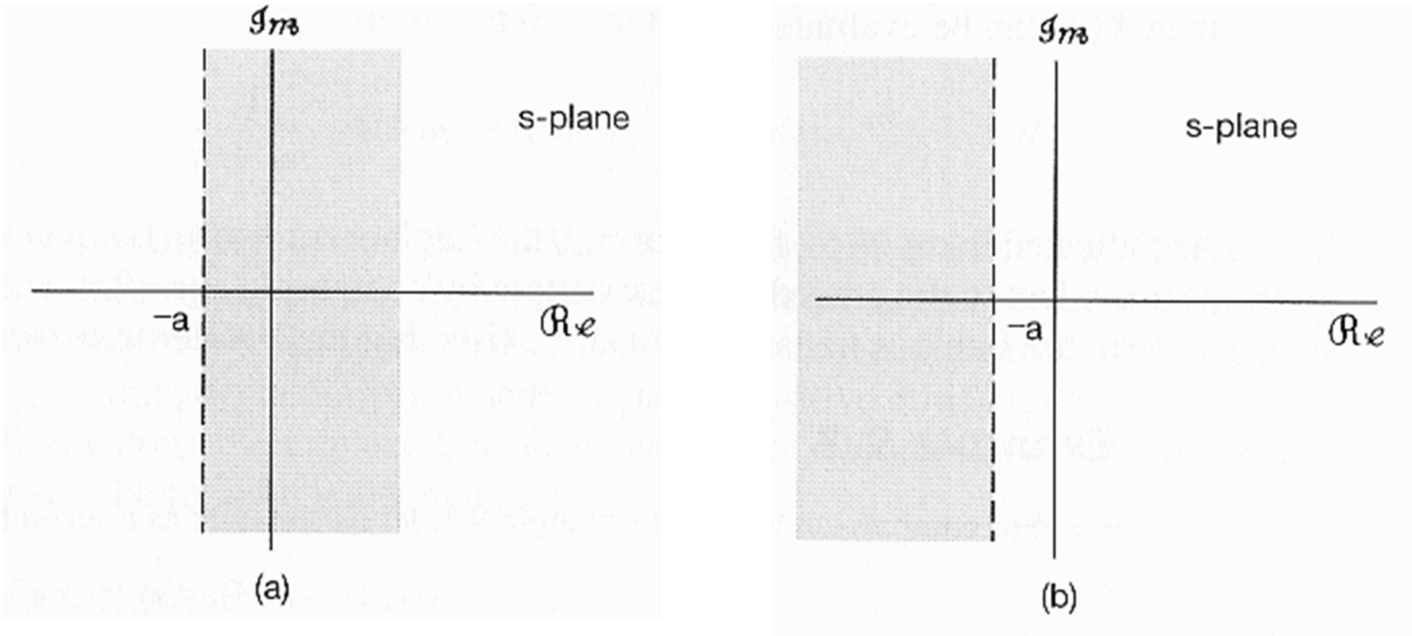
Region of Convergence (ROC)

- Region of Convergence (ROC):

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

where Fourier transform of $x(t)e^{-\sigma t}$ converges



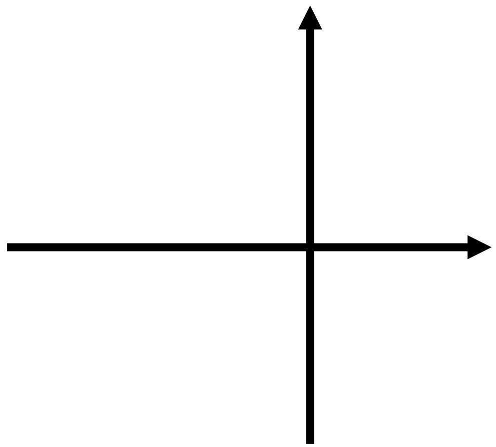
- Example 9.3:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st} dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt$$

$$= 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+1}\right)$$



$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2$$

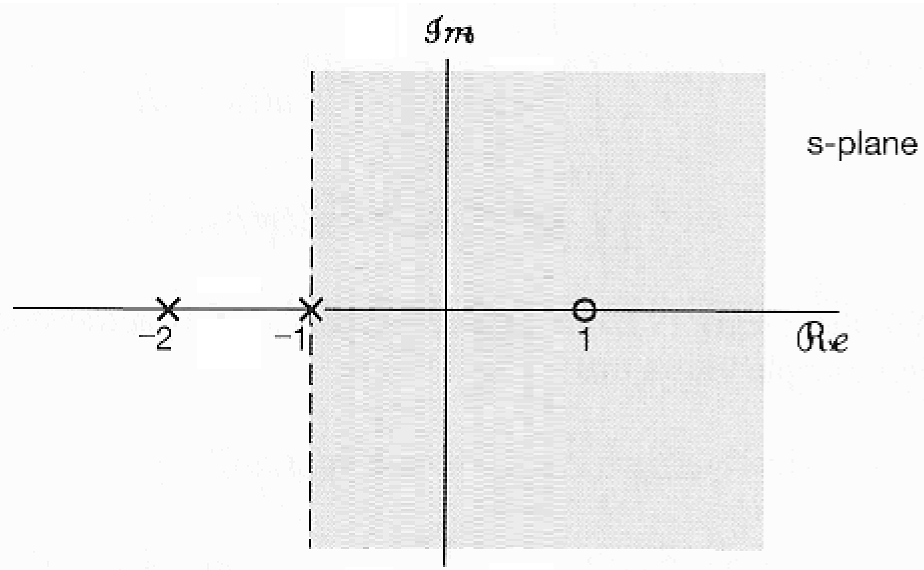
$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

- Example 9.3:

$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{3}{s+2} - \frac{2}{s+1}, \qquad \mathcal{R}e\{s\} > -1$$

$$\overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{(s+2)(s+1)}, \qquad \mathcal{R}e\{s\} > -1$$



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw

- Example 9.4:

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -\mathcal{R}e\{a\}$$

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \qquad \mathcal{R}e\{s\} > -2 \qquad \mathcal{R}e\{s\} > -1$$

$$= \left[e^{-2t} + \frac{1}{2}e^{-t}\left(e^{j3t} + e^{-j3t}\right)\right]u(t)$$

$$e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \qquad \mathcal{R}e\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \qquad \mathcal{R}e\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \qquad \mathcal{R}e\{s\} > -1$$

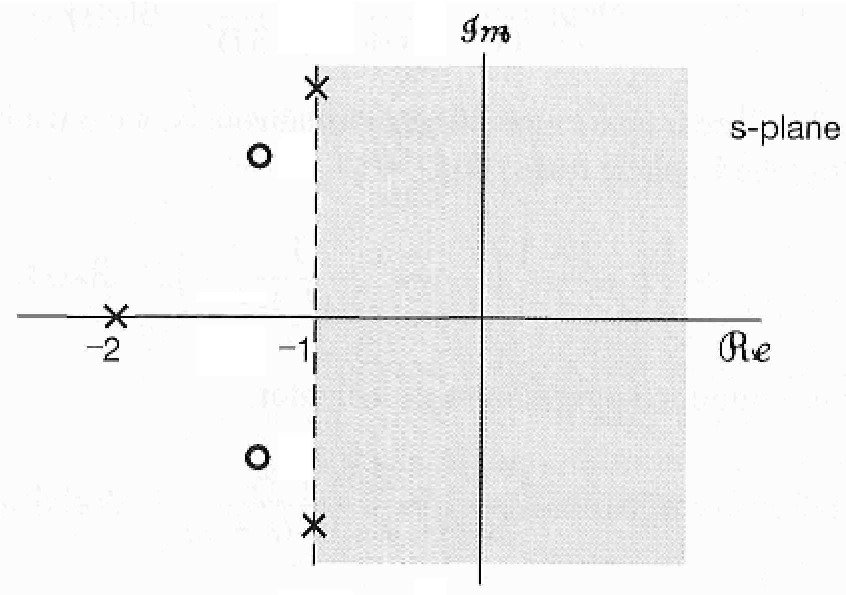
$$X(s) = \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}\right] = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$$

- Example 9.4:

$\mathcal{R}e\{s\} > -2 \quad \mathcal{R}e\{s\} > -1$

$$e^{-2t}u(t) + e^{-t}(\cos(3t))u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$



- The jw-axis is included in the ROC!
- Fourier transform!
  - s = jw

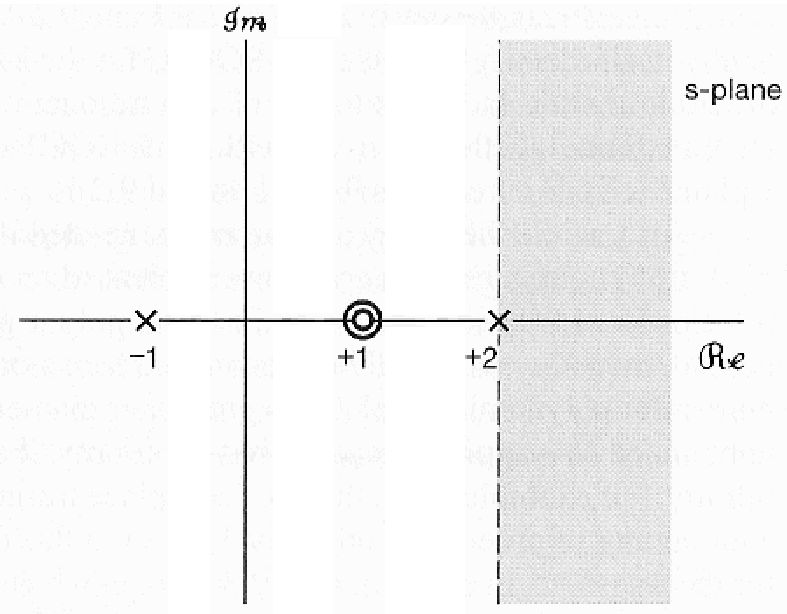
- Example 9.5:

$$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

$$x(t) = \delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

$$X(s) = 1 \quad - \frac{4}{3} \frac{1}{s+1} \quad + \frac{1}{3} \frac{1}{s-2}, \quad \mathcal{R}e\{s\} > 2$$

$$\delta(t) - \frac{4}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{R}e\{s\} > 2$$



- The jw-axis is not included in the ROC!
- Fourier transform?
- Why?

$$e^{-at}u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{jw+a}$$

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

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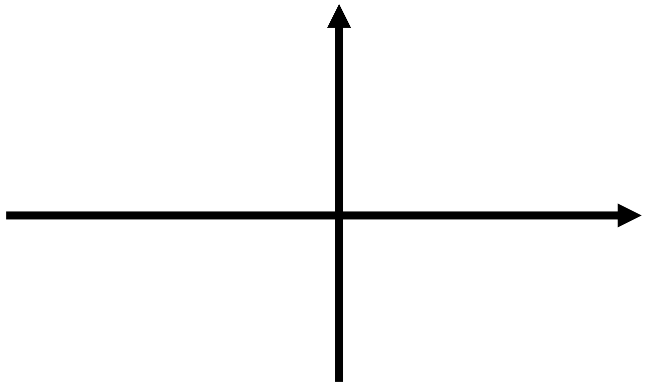
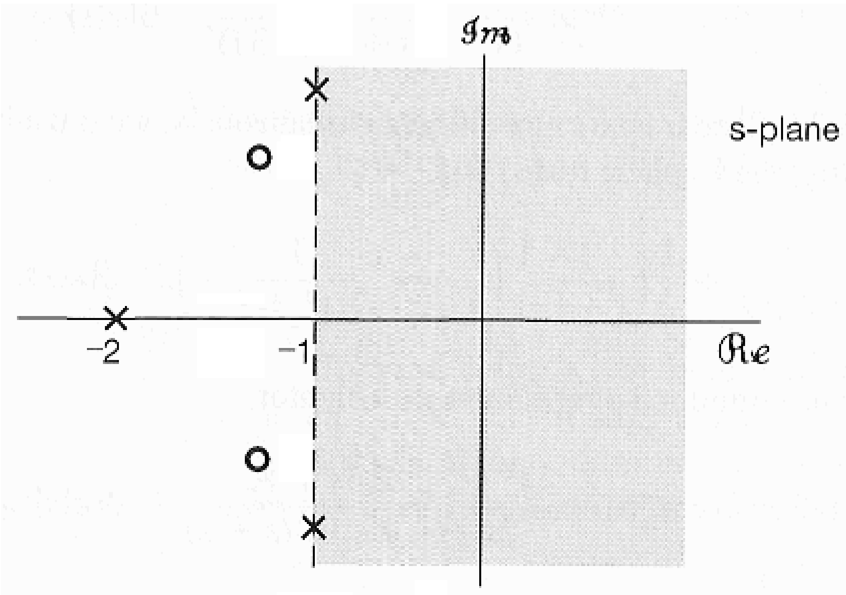
Properties of ROC:

1. The ROC of X(s) consists of strips parallel to the jw-axis in the s-plane



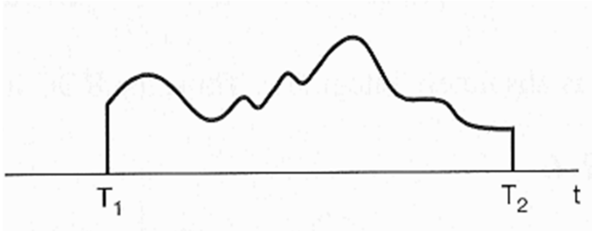
2. For rational Laplace transforms, the ROC does not contain any poles



$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$

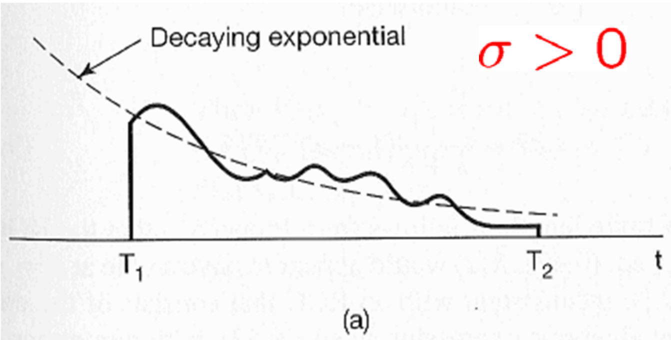
Properties of ROC:

3. If $x(t)$ is of finite duration & is absolutely integrable, then the ROC is the entire s-plane



$$\int_{T_1}^{T_2} |x(t)|dt < \infty$$

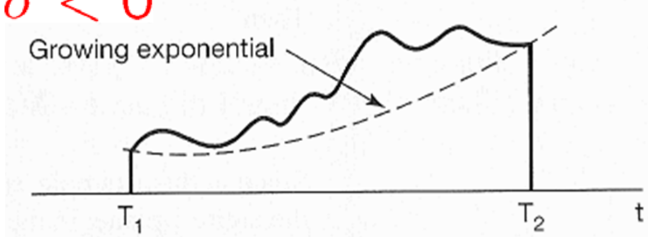
$e^{-\sigma t}$



(a)

$\sigma > 0$

$\sigma < 0$



(b)

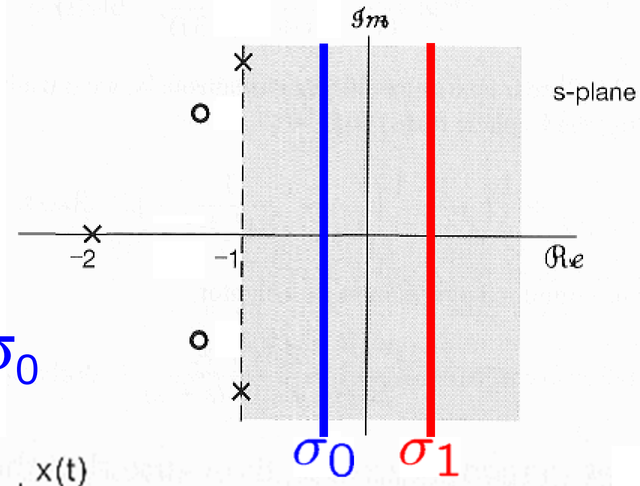
$e^{-\sigma t}$

$$s = \sigma + jw$$

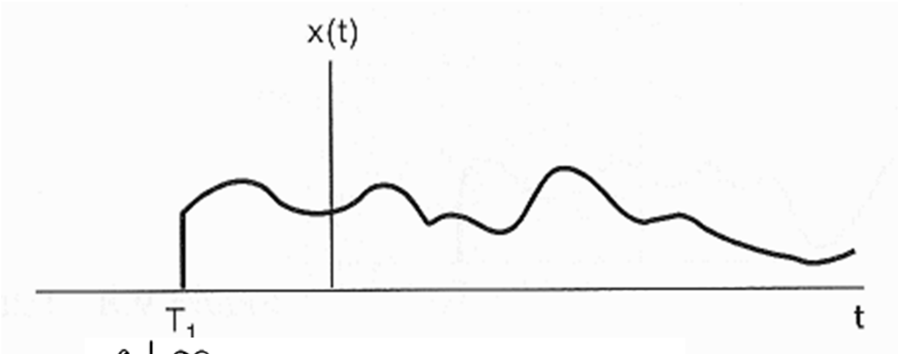
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{T_1}^{T_2} x(t)e^{-st}dt \ < \ e^{-\sigma(T_1 \text{ or } T_2)} \int_{T_1}^{T_2} |x(t)|dt$$

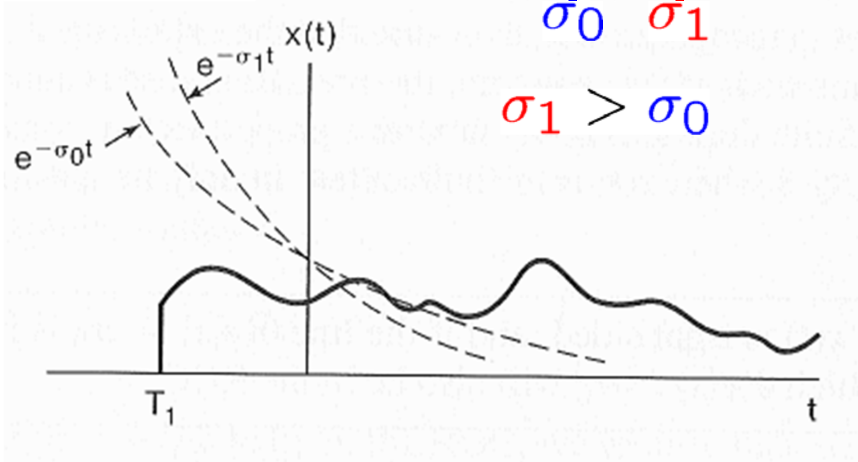
- Properties of ROC:

4. If $x(t)$ is right-sided, and
if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC,
then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$
will also be in the ROC



$\sigma_1 > \sigma_0$





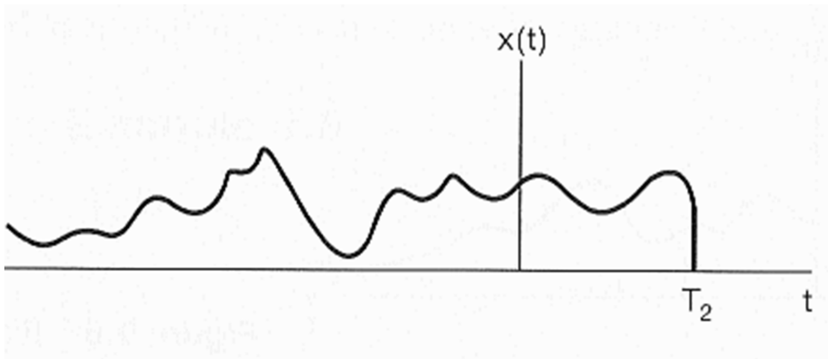
$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$$

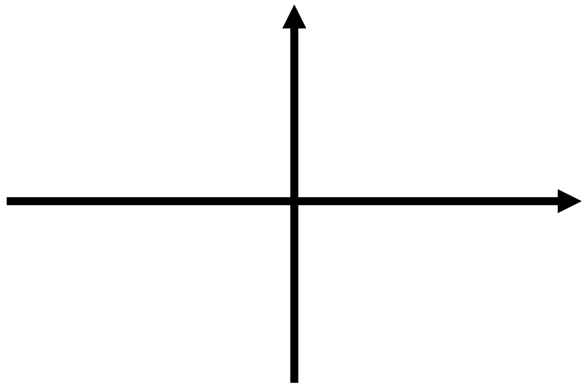
$$\Rightarrow \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{+\infty} |x(t)| e^{-(\sigma_1 - \sigma_0 + \sigma_0)t} dt$$

$$t > T_1 \Rightarrow e^{-(\sigma_1 - \sigma_0)t} \le e^{-(\sigma_1 - \sigma_0)T_1} \qquad \le e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

Properties of ROC:

5. If $x(t)$ is left-sided, and
   if the line $Re\{s\} = \sigma_0$ is in the ROC,
   then all values of s for which $Re\{s\} < \sigma_0$
   will also be in the ROC

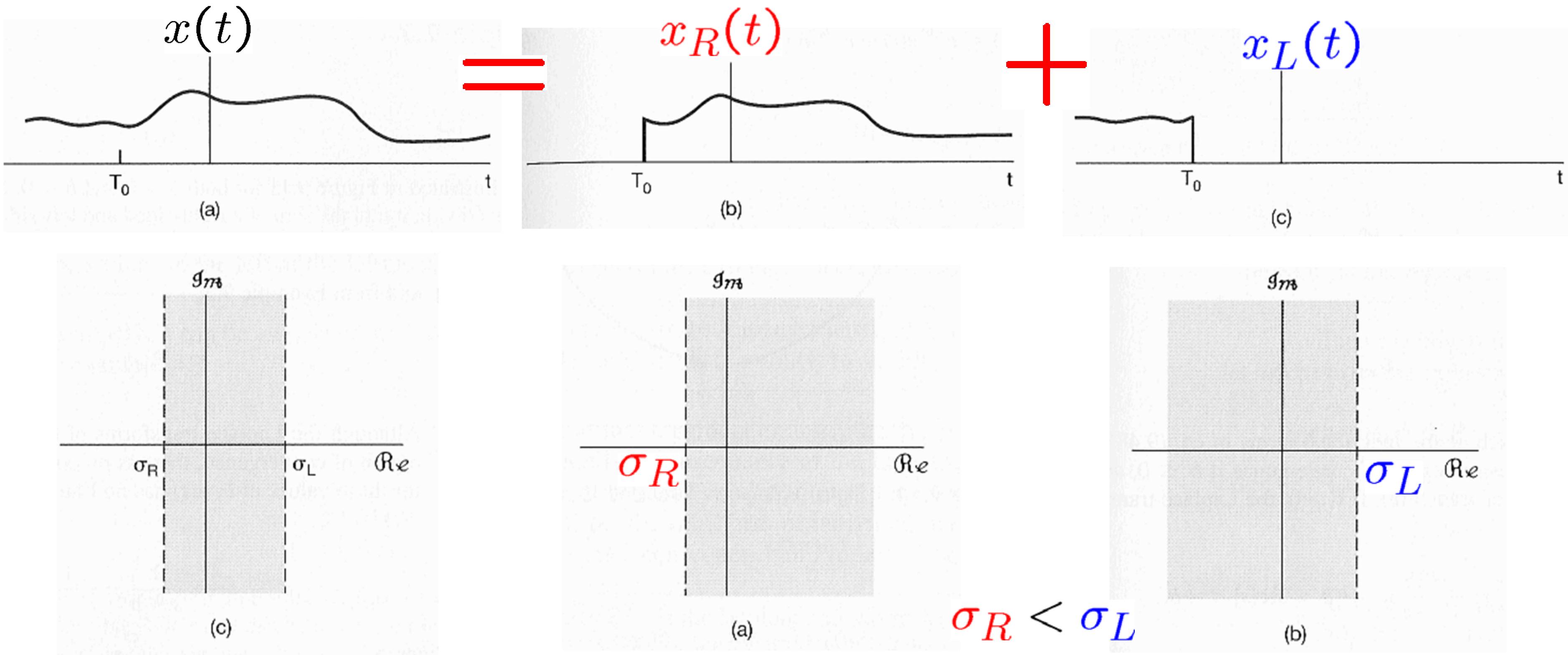




The argument is the similar to that for Property 4.

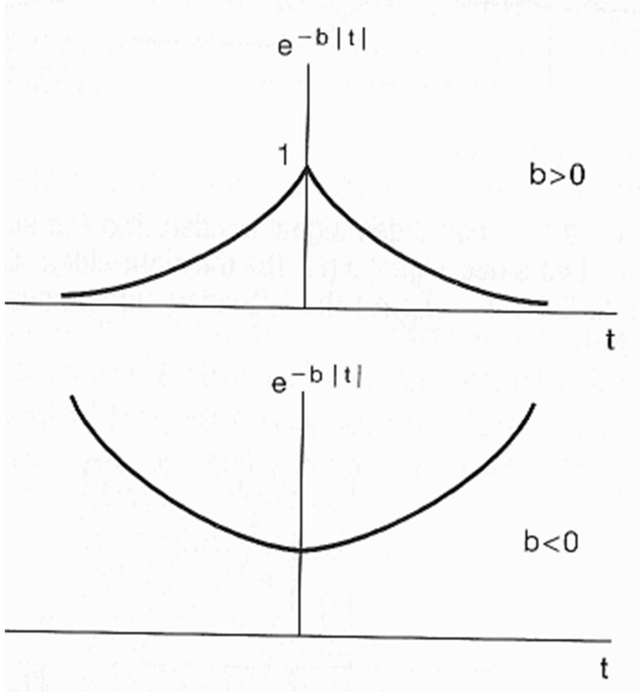
Properties of ROC:

6. If x(t) is two-sided, and
   if the line $Re\{s\} = \sigma_0$ is in the ROC,
   then the ROC will consist of a strip in the s-plane
   that includes the line $Re\{s\} = \sigma_0$



- Example 9.7:

$$x(t) = e^{-b|t|} \qquad =$$



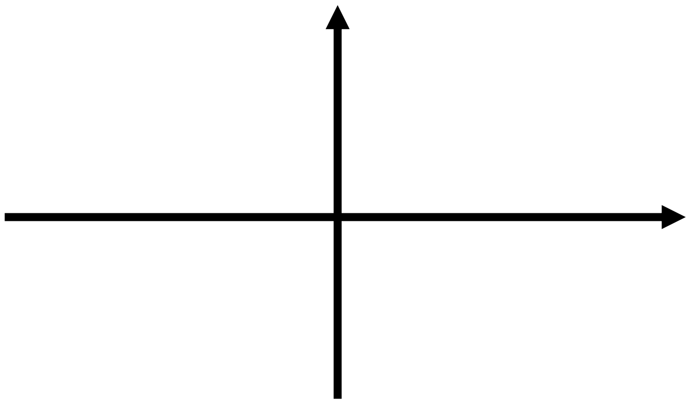
$$e^{-bt}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \underline{\qquad\qquad}, \quad \mathcal{R}e\{s\}$$

$$e^{+bt}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \underline{\qquad\qquad}, \quad \mathcal{R}e\{s\}$$

- $b > 0$:

$$e^{-b|t|} \stackrel{\mathcal{L}}{\longleftrightarrow} \underline{\qquad\qquad} + \underline{\qquad\qquad}, \qquad <\mathcal{R}e\{s\}<$$

$$= \underline{\qquad\qquad\qquad\qquad}$$



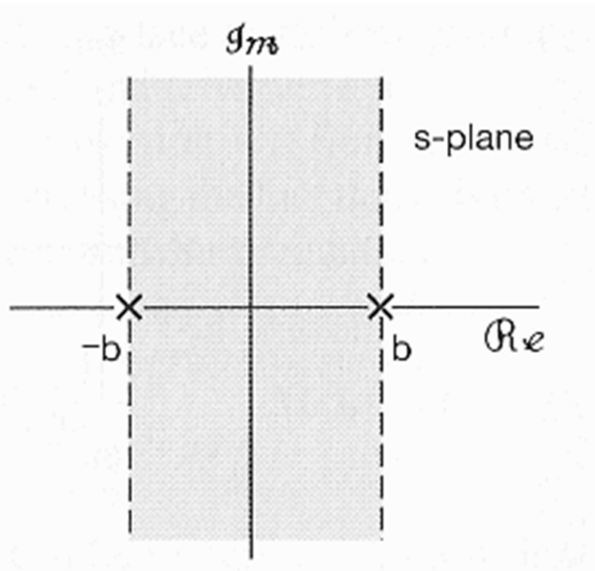
- $b \leq 0$:

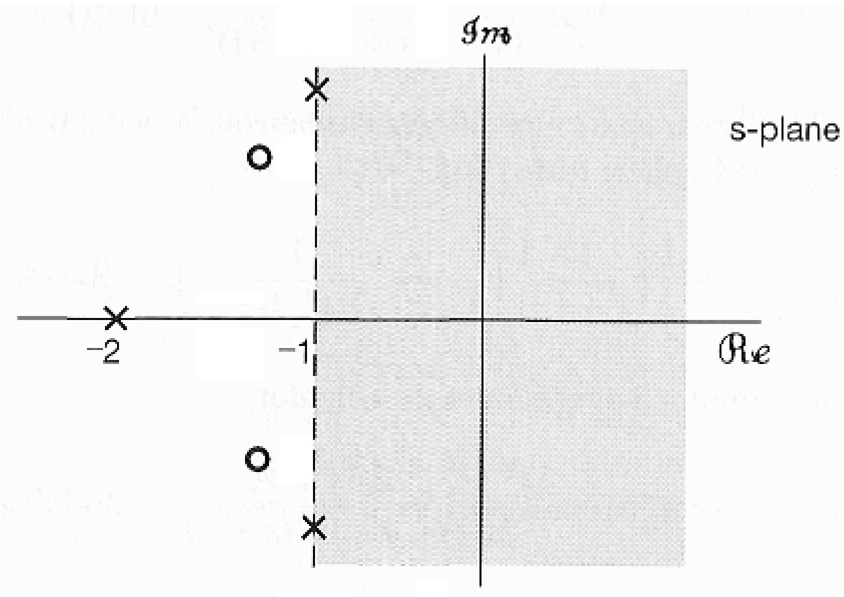
Properties of ROC:

7. If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to ∞. In addition, no poles of X(s) are contained in ROC

$$\frac{-2b}{(s+b)(s-b)}$$

$$\frac{2(s+1.25-2.11j)(s+1.25+2.11j)}{(s+1-3j)(s+1+3j)(s+2)}$$



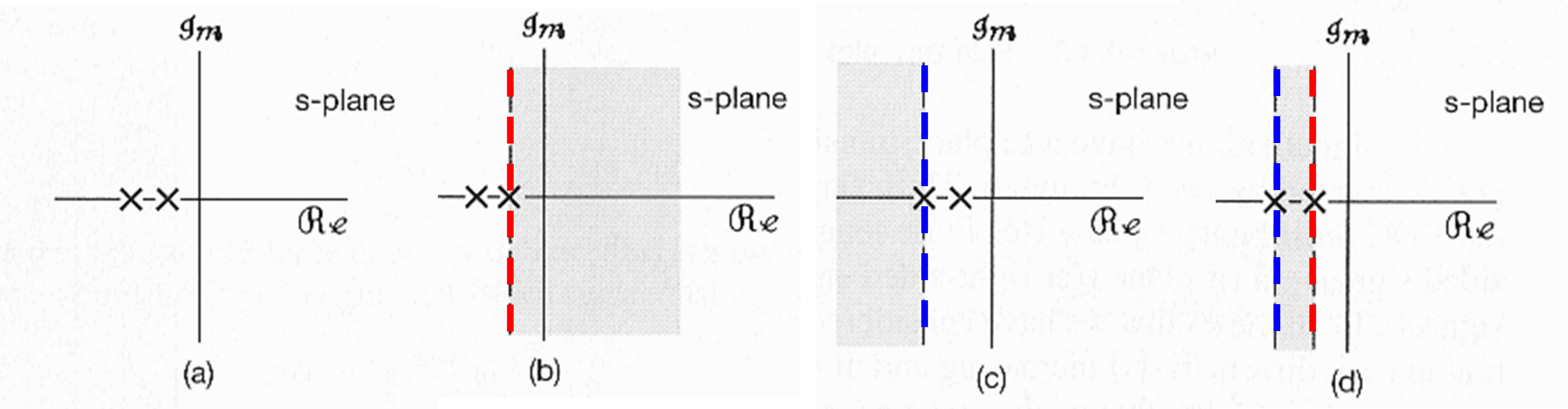


Properties of ROC:

8. If the Laplace transform X(s) of x(t) is rational
   - If x(t) is right-sided, the ROC is the region in the s-plane to the right of the rightmost pole
   - If x(t) is left-sided, the ROC is the region in the s-plane to the left of the leftmost pole

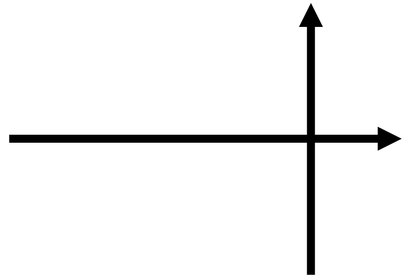
$$X(s) = \frac{1}{(s+2)(s+1)}$$

- Which one has Fourier transform?

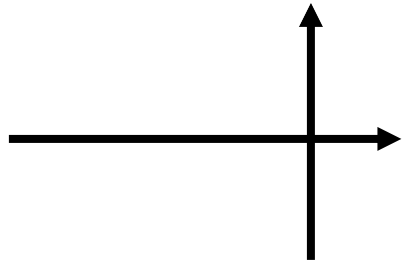


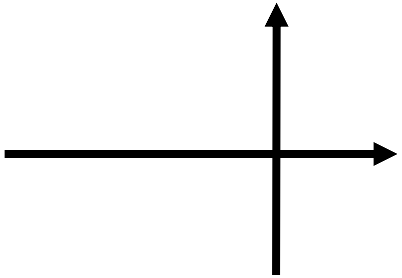
- Examples 9.9, 9.10, 9.11:

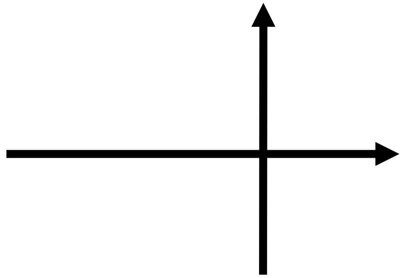
	$\mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)}$	$e^{\quad t}u(\quad t)$	$e^{\quad t}u(\quad t)$

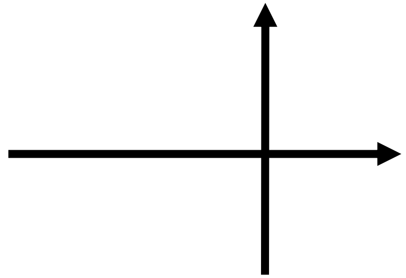


	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\}$
$\frac{1}{(s+2)}$	$e^{\quad t}u(\quad t)$	$e^{\quad t}u(\quad t)$









	$\mathcal{R}e\{s\} < -2$	$-2 < \mathcal{R}e\{s\} < -1$	$-1 < \mathcal{R}e\{s\}$
$\frac{1}{(s+1)} + \frac{1}{(s+2)}$	$e^{\quad t}u(\quad t)$ + $e^{\quad t}u(\quad t)$	$e^{\quad t}u(\quad t)$ + $e^{\quad t}u(\quad t)$	$e^{\quad t}u(\quad t)$ + $e^{\quad t}u(\quad t)$

- The Laplace Transform
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  for Laplace Transforms
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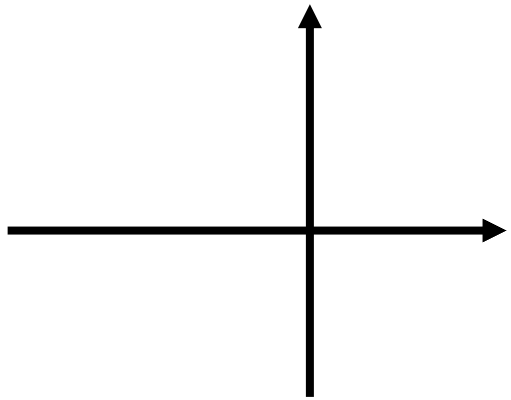
The Inverse Laplace Transform:

- By the use of contour integration



$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(\sigma + jw) = \mathcal{F}\left\{ x(t) e^{-\sigma t} \right\} = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-jwt} dt$$

$\forall s = \sigma + jw$ in the ROC

$$x(t) e^{-\sigma t} = \mathcal{F}^{-1}\left\{ X(\sigma + jw) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{jwt} dw$$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + jw) e^{(\sigma + jw)t} dw$$

$$s = \sigma + jw$$

$$ds = j dw$$

$$\Rightarrow x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

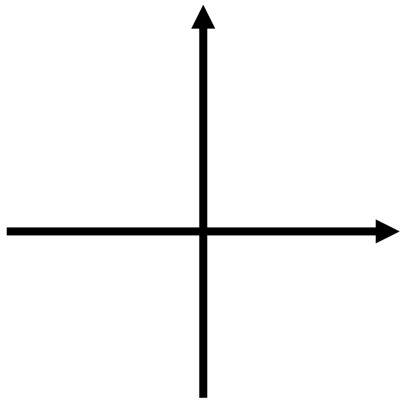
The Inverse Laplace Transform:

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$-e^{-at}u(-t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} < -a$$

- By the technique of partial fraction expansion

$$X(s) = \frac{A}{s+a} + \frac{B}{s+b} + \cdots + \frac{M}{s+m}$$



$$x(t) = A\,e^{-at}\,u(t) \;-\; B\,e^{-bt}\,u(-t) + \cdots + x_m(t)$$

(if R.S.) (if L.S.)

- Example 9.9:

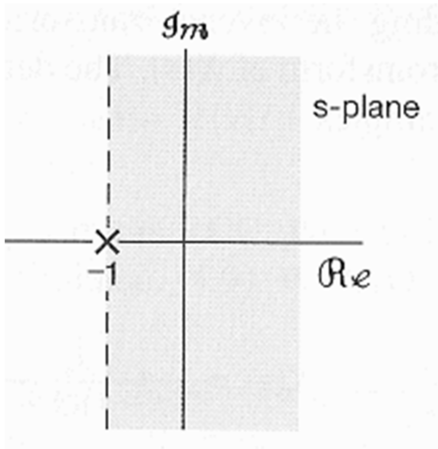
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$= \frac{1}{(s+1)} + \frac{-1}{(s+2)}$$

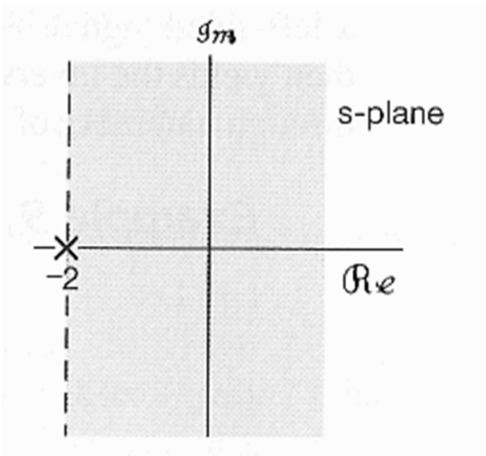
$$e^{-t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{R}e\{s\} > -1$$

$$e^{-2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{R}e\{s\} > -2$$

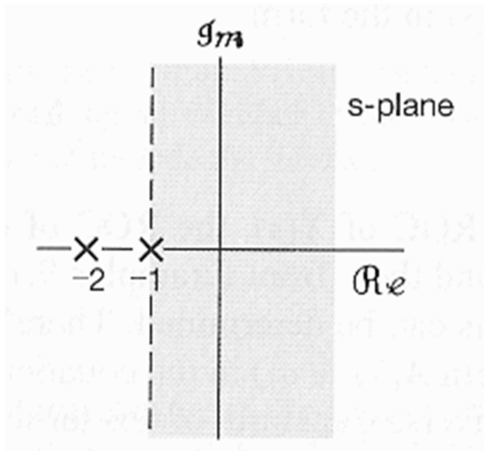
$$\left[e^{-t} + (-1)e^{-2t}\right]u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$



(b)



(c)



(a)

- The Laplace Transform
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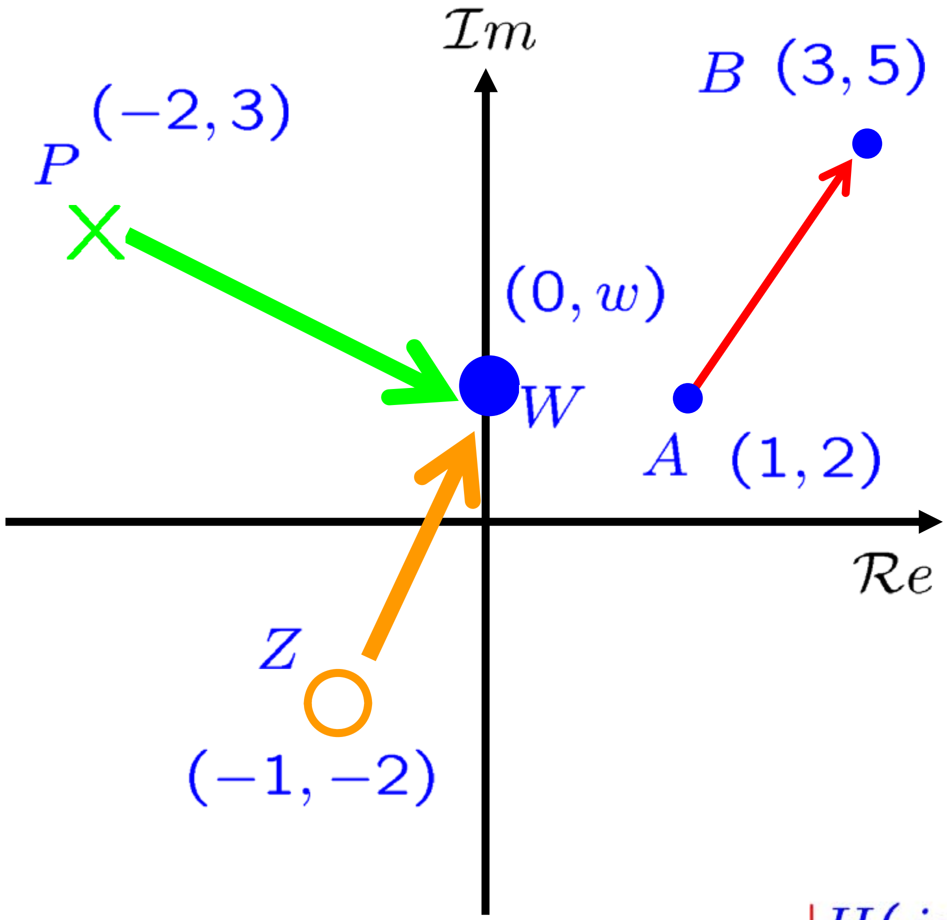
Magnitude-Phase Representation in Complex Plane (s-plane & z-plane)

- In s-plane or z-plane:



$$\overrightarrow{AB} = (3+5j) - (1+2j) = 2+3j$$

$$\overrightarrow{AB} = (3,5) - (1,2) = (2,3)$$

$$|\overrightarrow{AB}| = \sqrt{2^2+3^2}$$

$$\measuredangle \overrightarrow{AB} = \tan^{-1}\frac{(5-2)}{(3-1)}$$

$$H(s) = \frac{s-(-1-2j)}{s-(-2+3j)}$$

$$|H(jw)| = \frac{|jw-(-1-2j)|}{|jw-(-2+3j)|} = \frac{|\overrightarrow{ZW}|}{|\overrightarrow{PW}|}$$

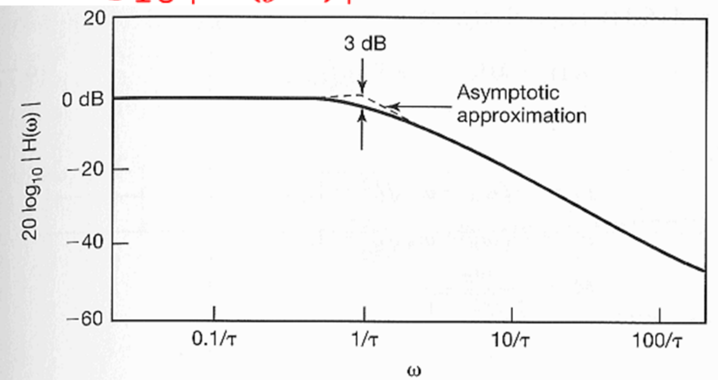
$$\measuredangle H(jw) = \measuredangle \overrightarrow{ZW} - \measuredangle \overrightarrow{PW}$$

- First-Order Systems:

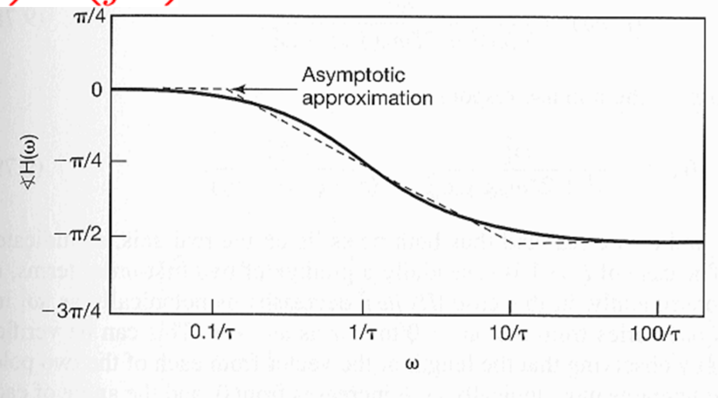
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(jw) = \frac{\frac{1}{\tau}}{jw + \frac{1}{\tau}}$$

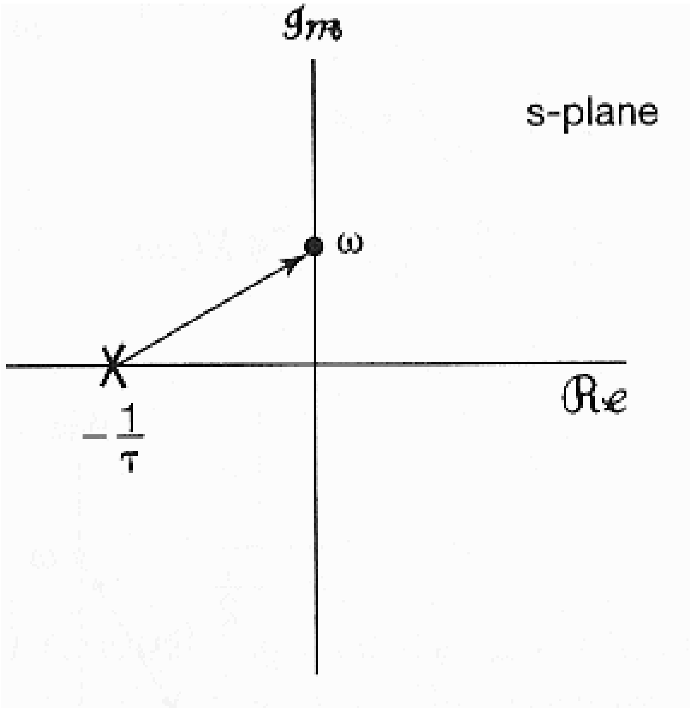
$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}, \quad \mathcal{R}e\{s\} > -\frac{1}{\tau}$$

$20 \log_{10} |H(jw)|$



$\measuredangle H(jw)$





Geometric Evaluation of the Fourier Transform

$$|H(j\quad)| = \left|\frac{\frac{1}{\tau}}{j\quad + \frac{1}{\tau}}\right| = \frac{\quad}{\quad}$$

$$\measuredangle H(j\quad) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\quad + \frac{1}{\tau}\right) = \quad -$$

$$|H(j\quad)| = \left|\frac{\frac{1}{\tau}}{j\quad + \frac{1}{\tau}}\right| = \frac{\quad}{\quad}$$

$$\measuredangle H(j\quad) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\quad + \frac{1}{\tau}\right) = \quad -$$

$$|H(j\quad)| = \left|\frac{\frac{1}{\tau}}{j\quad + \frac{1}{\tau}}\right| = \frac{\quad}{\quad}$$

$$\measuredangle H(j\quad) = \measuredangle\left(\frac{1}{\tau}\right) - \measuredangle\left(j\quad + \frac{1}{\tau}\right) = \quad -$$

Second-Order Systems:

$$H(jw) = \frac{w_n^2}{(jw)^2 + 2\zeta w_n (jw) + w_n^2}$$

$$\Rightarrow h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right] u(t)$$

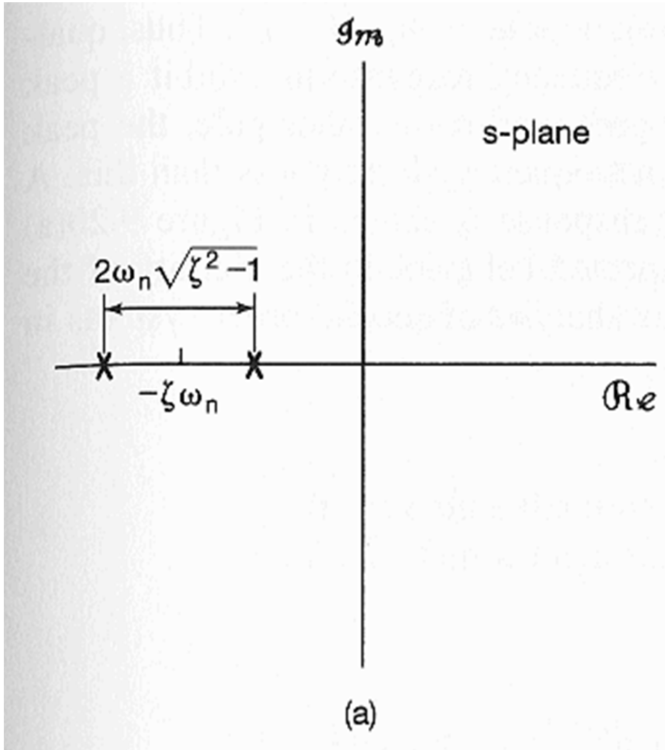
$$\begin{aligned} c_1 &= -\zeta w_n + w_n\sqrt{\zeta^2 - 1} \\ c_2 &= -\zeta w_n - w_n\sqrt{\zeta^2 - 1} \\ M &= \frac{w_n}{2\sqrt{\zeta^2 - 1}} \end{aligned}$$

$$\Rightarrow H(s) = \frac{w_n^2}{(s)^2 + 2\zeta w_n (s) + w_n^2} = \frac{w_n^2}{(s - c_1)(s - c_2)}$$

- $\zeta > 1$: c_1 & c_2 are real
- $0 < \zeta < 1$: c_1 & c_2 are complex

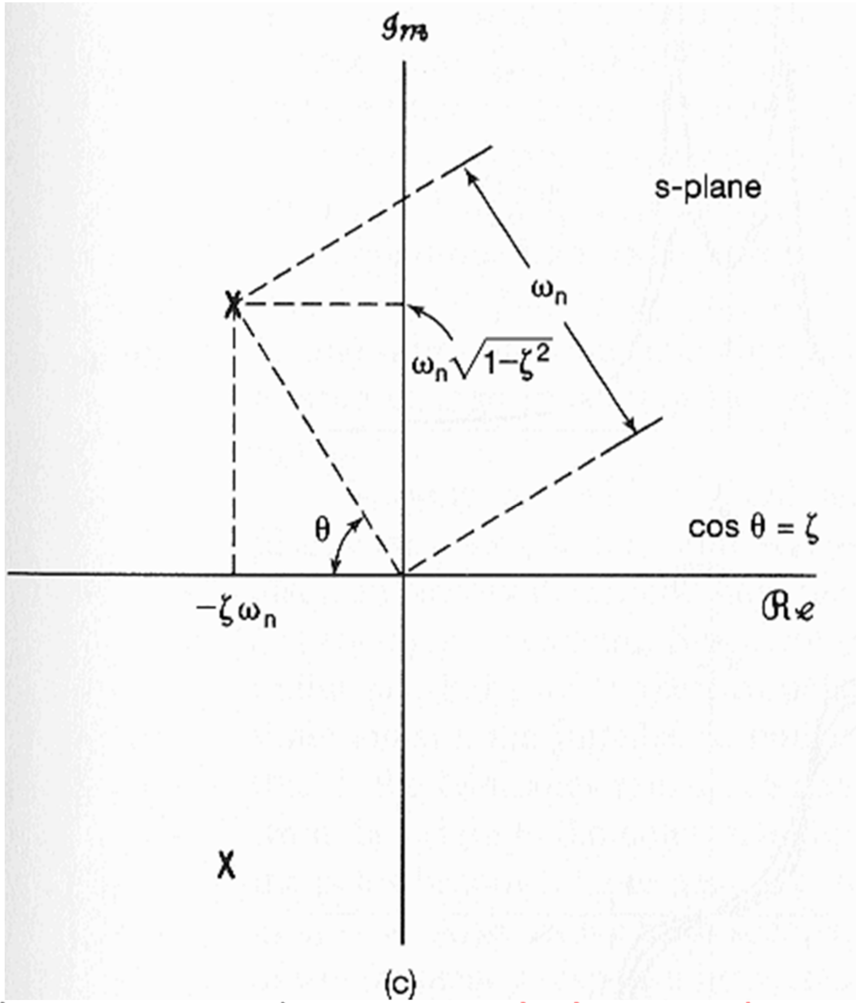
- Pole Locations:

• $\zeta > 1$: c_1 & c_2 are real



(a)

• $0 < \zeta < 1$: c_1 & c_2 are complex



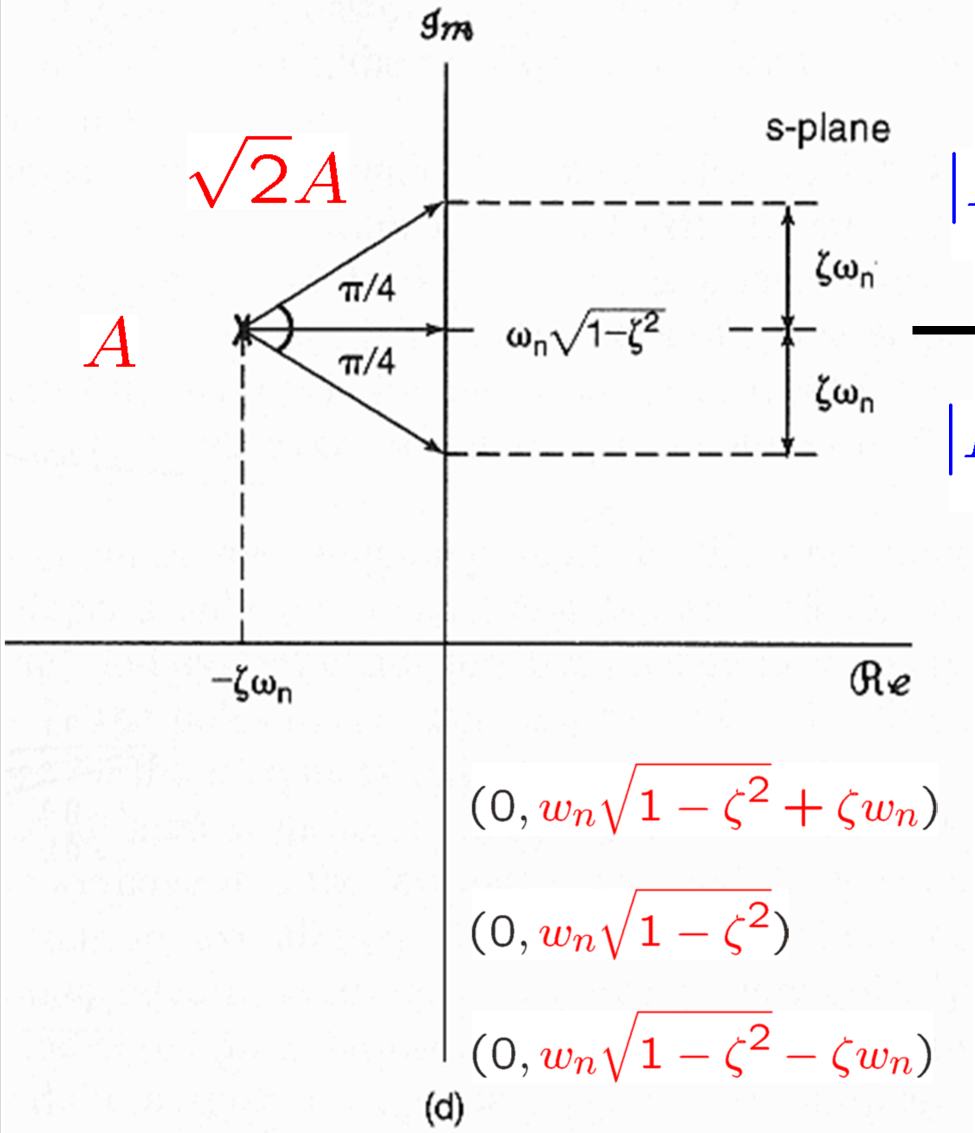
(c)

$|H| =$ ______

$\measuredangle H = \quad - \quad -$

one pole vector has a minimun length at $w = w_n\sqrt{1-\zeta^2}$!!!

- Relative Bandwidth B:



$$\frac{|H(jw)|_{w=w_n\sqrt{1-\zeta^2}}}{|H(jw)|_{w=w_n\sqrt{1-\zeta^2}\pm\zeta w_n}}$$

$$\approx \text{ or } \leq \frac{\sqrt{2}}{1}$$

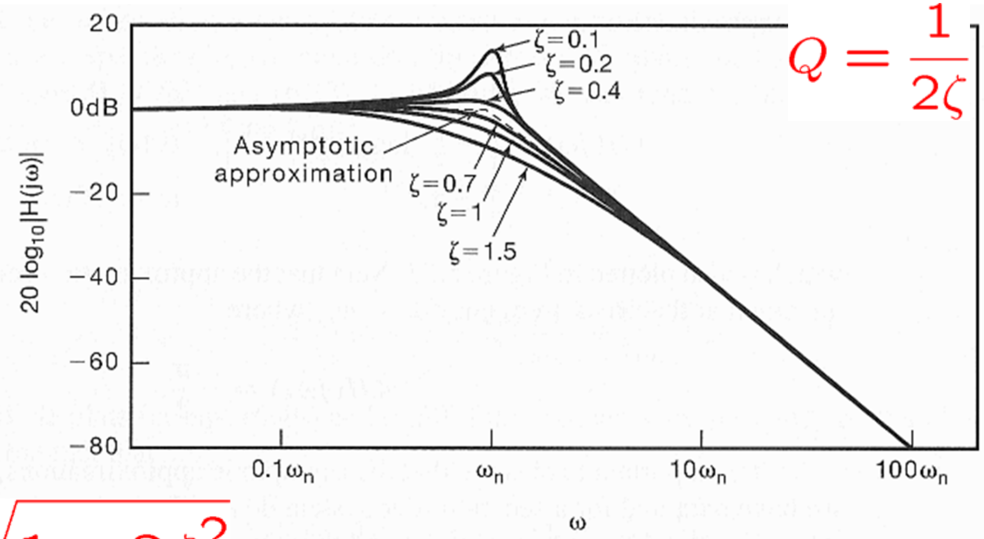
$$\Rightarrow B = 2\zeta$$

$$\Rightarrow \Delta \measuredangle H(jw) = \frac{\pi}{2}$$

- Second-Order CT Systems: $H(jw) = \dfrac{1}{(jw/w_n)^2 + 2\zeta(jw/w_n) + 1}$

$$20\log_{10}|H(jw)| =$$

$$\begin{cases} 0 & w << w_n \\ -20\log_{10}(2\zeta) & w = w_n \\ -40\log_{10}(w) + 40\log_{10}(w_n) & w >> w_n \end{cases}$$

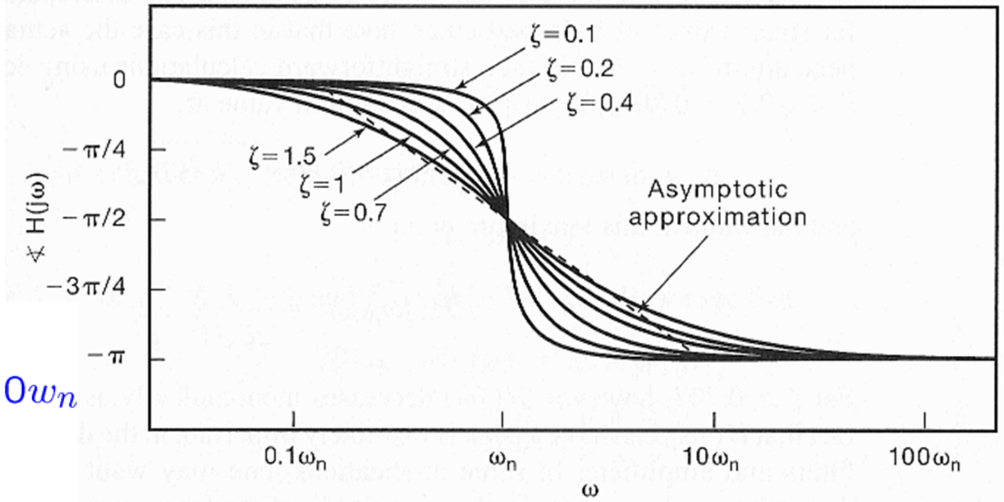


$$Q = \frac{1}{2\zeta}$$

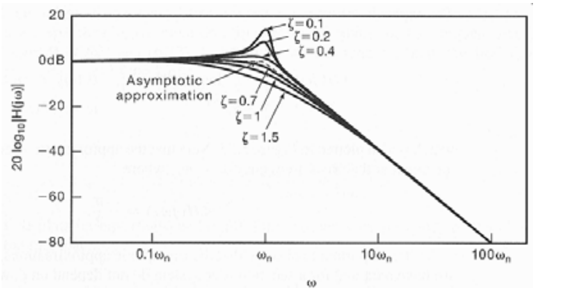
- For $\zeta < \sqrt{2}/2$ $\quad w_{\max} = w_n\sqrt{1-2\zeta^2}$

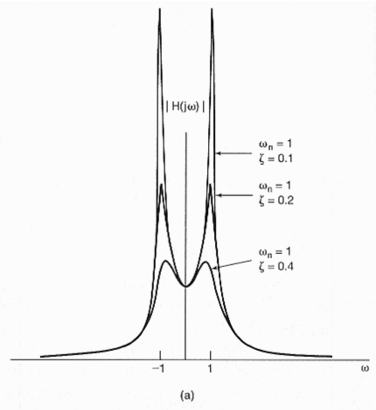
$$\measuredangle H(jw) =$$

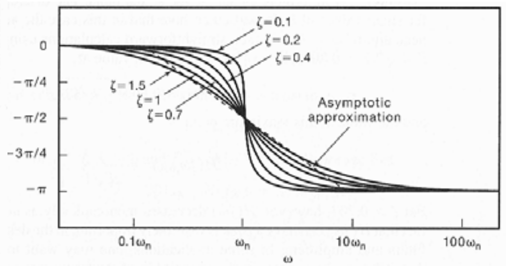
$$\begin{cases} 0 & w \leq 0.1w_n \\ -(\pi/2)[\log_{10}(w/w_n) + 1] & 0.1w_n \leq w \leq 10w_n \\ -\pi/2 & w = w_n \\ -\pi & w \geq 10w_n \end{cases}$$

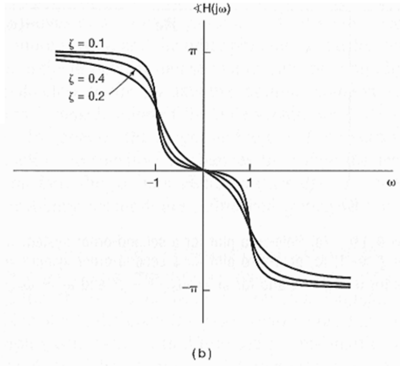


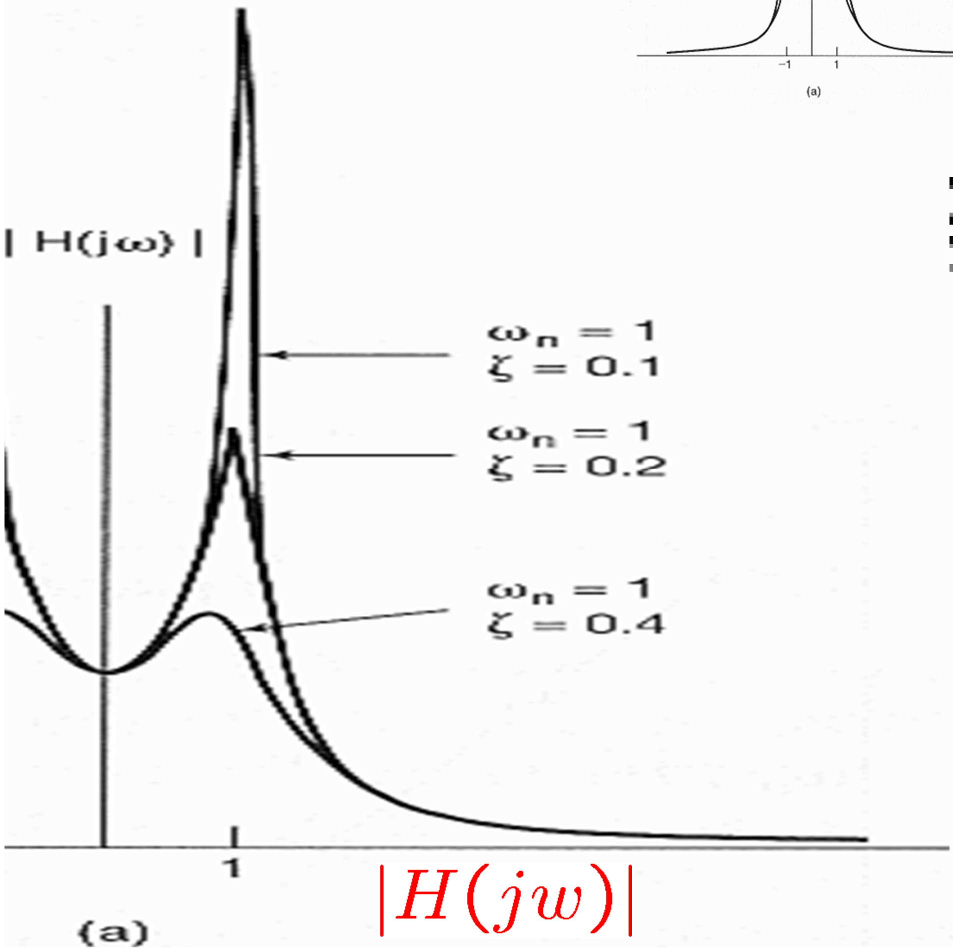
- Frequency Response:

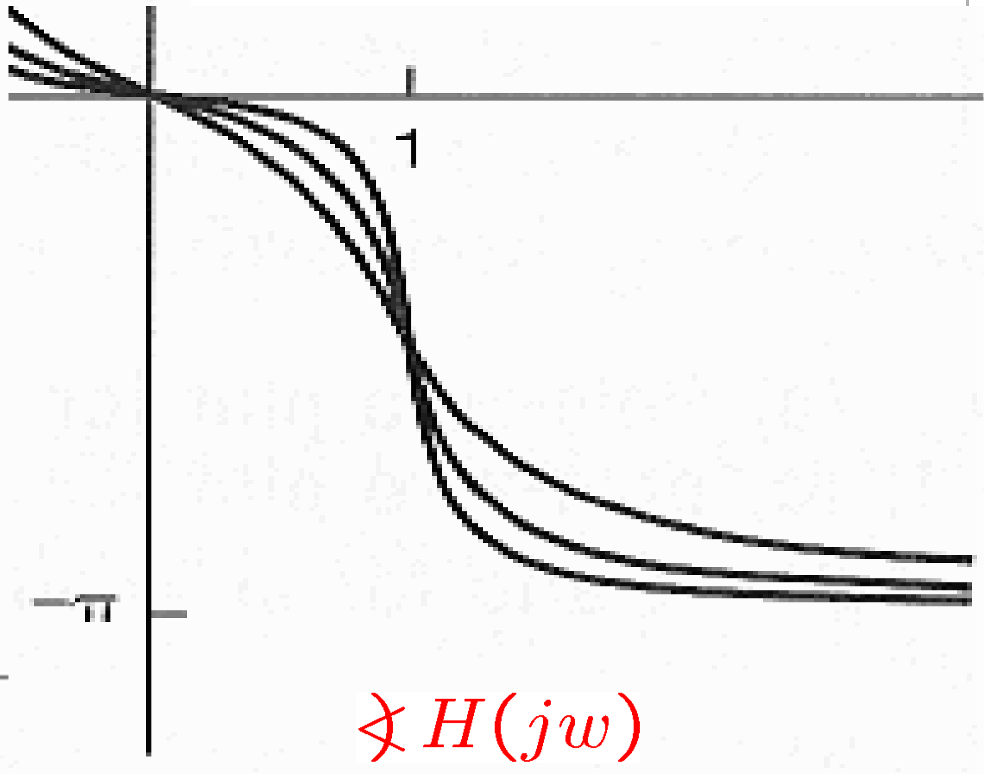








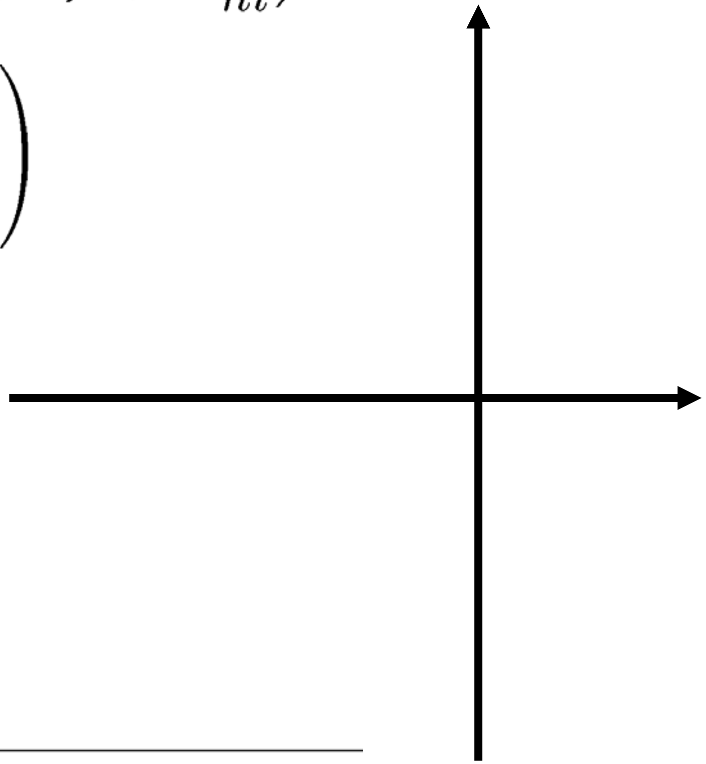




- The Nth-Order Systems:

$$H(jw) = \left(\frac{\frac{1}{\tau}}{jw + \frac{1}{\tau}}\right) \prod_i \left(\frac{w_{ni}^2}{(jw)^2 + 2\zeta_i w_{ni}(jw) + w_{ni}^2}\right)$$

$$H(s) = \left(\frac{b}{s-a}\right) \left(\prod_i \frac{w_{ni}^2}{s^2 + 2\zeta_i w_{ni} s + w_{ni}^2}\right)$$



$$|H(jw)| = \prod_i |H_i(jw)| \quad = \frac{\qquad\qquad}{}$$

$$\measuredangle H(jw) = \sum_i \measuredangle H_i(jw) \quad = (\qquad\qquad) - (\qquad\qquad)$$

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Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

Linearity of the Laplace Transform:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$a\, x_1(t) + b\, x_2(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad a\, X_1(s) + b\, X_2(s),$$

with ROC containing $R_1 \cap R_2$

$$\int_{-\infty}^{\infty} \qquad e^{-st} dt \qquad \int_{-\infty}^{\infty} \qquad e^{-st} dt$$

$$\int_{-\infty}^{\infty} \qquad e^{-st} dt \qquad \int_{-\infty}^{\infty} \qquad e^{-st} dt$$

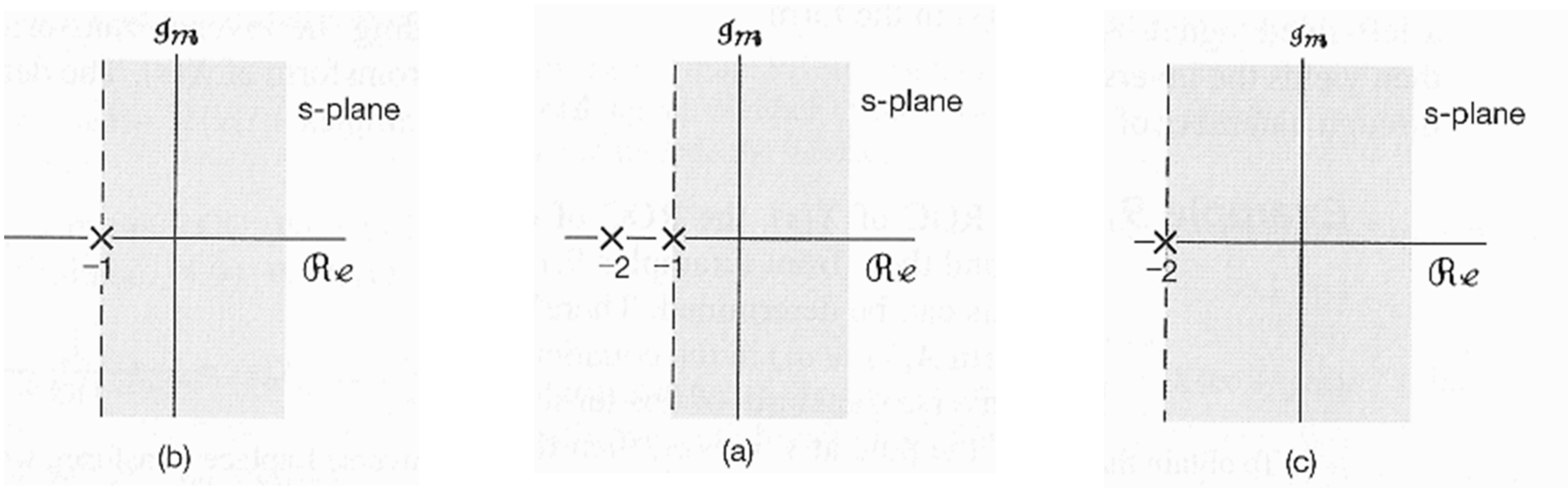
- Example 9.13:

$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathcal{R}e\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{R}e\{s\} > -1$$

$$X(s) = \frac{1}{(s+1)} - \frac{1}{(s+1)(s+2)} \quad = \frac{1}{(s+2)} \quad \mathcal{R}e\{s\}$$



- **Time Shifting:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t - t_0) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s), \quad ROC = R$$

$$X_0(s) = \int_{-\infty}^{\infty} x(t - t_0) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(\qquad) e^{-s(\qquad)} d$$

$$= \int_{-\infty}^{\infty} x(\qquad) e^{-s(\quad)} e^{-s(\quad)} d$$

$$= e^{-s(\quad)} \int_{-\infty}^{\infty} x(\qquad) e^{-s(\quad)} d$$

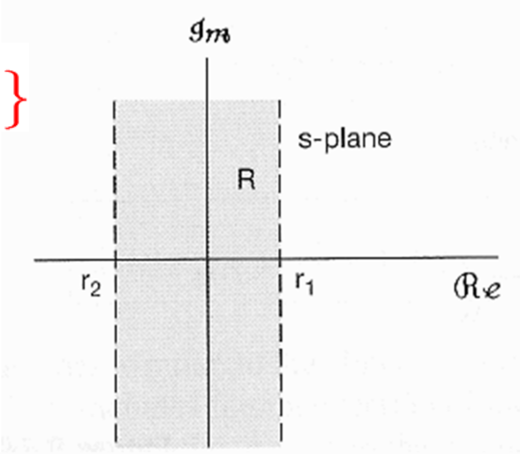
- **Shifting in the s-Domain:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad ROC = R + \mathcal{R}e\{s_0\}$$



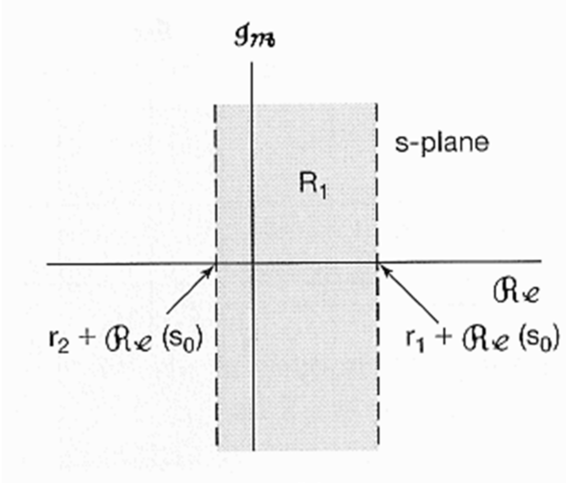
$$X(s = \quad)$$

$$\Rightarrow X(s - s_0 = \quad) = X(s = \quad + s_0)$$

$$X(s - s_0) = \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-(s \quad)}\, e^{+(s_0 \quad)}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{+(s_0 \quad)}\, e^{-(s \quad)}\, dt$$



(b)

- Time Scaling:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X(\frac{s}{a}), \quad ROC = aR$$

$$X(s = \quad)$$

$$X_a(s) = \int_{-\infty}^{\infty} x(a\,t) e^{-st} dt$$

$$\Rightarrow X(\frac{s}{a} = \quad)$$

$$= X(s = \quad)$$

$a \quad 0$

$$= \int_{-\infty}^{\infty} x(\quad) e^{-s(\quad)} \quad d$$

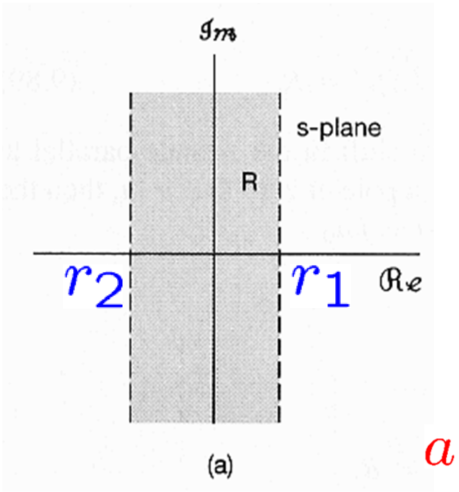
$$= \int_{-\infty}^{\infty} x(\quad) e^{-(\quad)} \quad d \quad = \frac{}{} X(\frac{}{})$$

$a \quad 0$

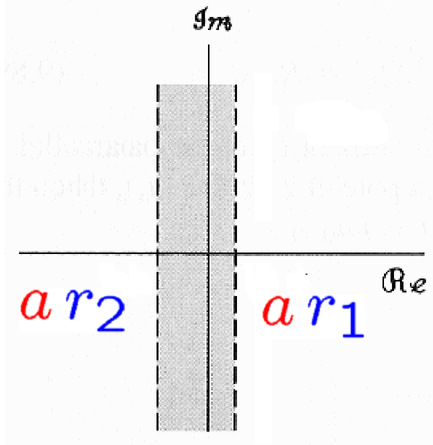
$$= \int_{\infty}^{\infty} x(\quad) e^{-s(\quad)} \quad d \quad = \frac{}{} X(\frac{}{})$$

$$= \int_{-\infty}^{\infty} x(\quad) e^{-(\quad)} \quad d \quad = \frac{}{} X(\frac{}{})$$

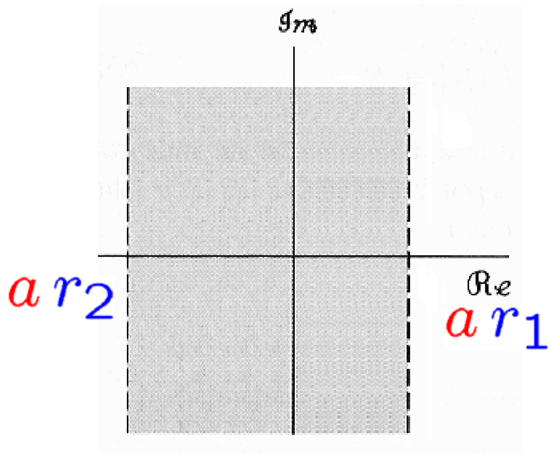
Properties of the Laplace Transform



a 0



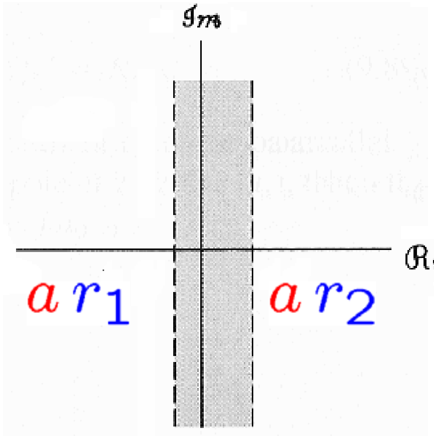
$\boxed{0 < a < 1}$



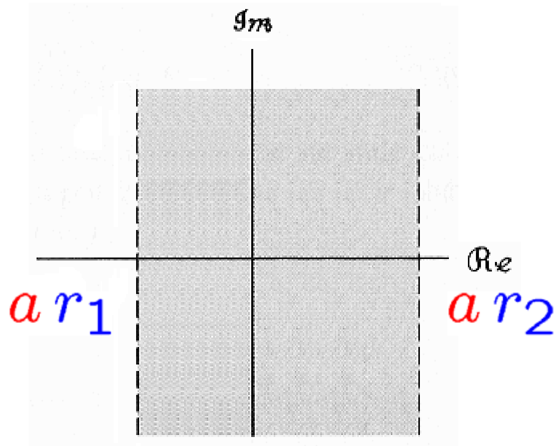
$\boxed{1 < a}$

$$s \longrightarrow \frac{s}{a}$$

a 0



$\boxed{-1 < a < 0}$



$\boxed{a < -1}$

$$x(-t) \xleftrightarrow{\mathcal{L}} X(-s), \quad ROC = -R$$

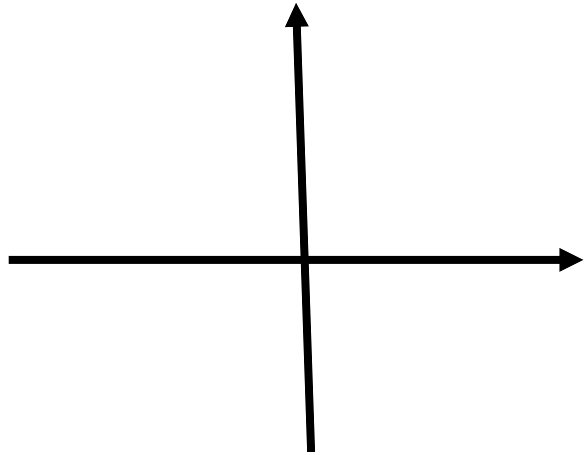
- Conjugation:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad ROC = R$$

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*), \quad ROC = R$$



$$X(s) = \int_{-\infty}^{\infty} x(t) \; e^{- \; s \; t} \; dt$$

$$= \int_{-\infty}^{\infty} x(t) \; e^{- \; s \; t} \; dt$$

$$= \int_{-\infty}^{\infty} x(t) \; e^{- \; s \; t} \; dt$$

$$X(\quad) = \int_{-\infty}^{\infty} x(t) \; e^{- \quad t} \; dt$$

- Convolution Property:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x_1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s), \quad ROC = R_1$$

$$x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_2(s), \quad ROC = R_2$$

$$x_1(t) * x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) X_2(s), \qquad ROC \text{ containing } R_1 \cap R_2$$

$$\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) d\tau$$

$$\int_{-\infty}^{\infty} x_1(\tau) \qquad x_2(t-\tau) \qquad d\tau$$

$$\int_{-\infty}^{\infty} x_1(\tau) \qquad x_2(\quad) \qquad d\tau$$

$$\int_{-\infty}^{\infty} x_1(\tau) \qquad x_2(\quad) \qquad d\tau$$

$$\int_{-\infty}^{\infty} x_1(\tau) \qquad d\tau$$

Differentiation in the Time & s-Domain:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\frac{d}{dt} x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sX(s), \quad ROC \text{ containing } R$$

$$-tx(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds} X(s), \quad ROC = R$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \quad e^{st} ds$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \quad e^{st} ds$$

$$X(s) = \int_{-\infty}^{\infty} x(t) \quad e^{-st} dt$$

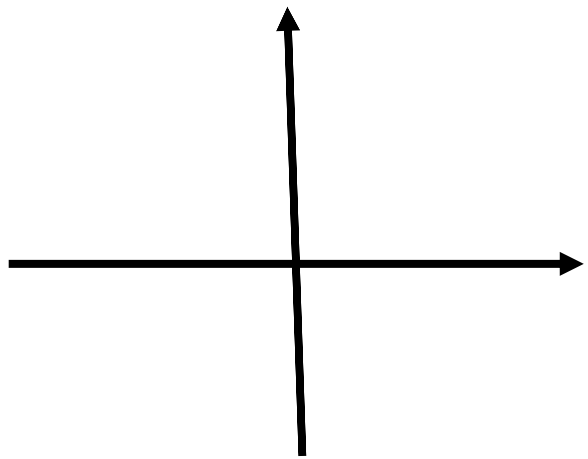
$$= \int_{-\infty}^{\infty} x(t) \quad e^{-st} dt$$

Integration in the Time Domain:

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad ROC = R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s),$$

ROC containing $R \cap \{\mathcal{R}e\{s\} > 0\}$



$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\int_{-\infty}^{t} x(\tau)d\tau$$

$$\boxed{\int_0^{\infty} u'v dt = uv\Big|_0^{\infty} - \int_0^{\infty} uv' dt}$$

- The Initial-Value Theorem:

If $x(t) = 0$ for $t < 0$ $\quad \Rightarrow \quad x(0^+) = \lim_{s \to \infty} sX(s)$

- The Final-Value Theorem:

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finit limit as $t \to \infty$,

$$\Rightarrow \quad \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

$$\boxed{\int_0^\infty u'v dt = uv\Big|_0^\infty - \int_0^\infty uv' dt}$$

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \frac{d}{dt}x(t) \; e^{-st}dt \quad = x(t) \; e^{-st}\Big|_0^\infty - \int_0^\infty x(t)(-s)e^{-st}dt$$

$$= x(\infty)\,\frac{1}{e^{s\infty}} - x(0^+)\,\frac{1}{e^{s0}} + (s)\int_0^\infty x(t) \; e^{-st}dt \qquad = s\;X(s) - x(0^+)$$

$$\lim_{s \to \infty} \int_0^\infty \frac{d}{dt}x(t) \; e^{-st}dt \; = \qquad 0 \qquad = \lim_{s \to \infty}\left\{s\;X(s) - x(0^+)\right\}$$

$$\lim_{s \to 0} \int_0^\infty \frac{d}{dt}x(t) \; e^{-st}dt = \lim_{t \to \infty} x(t) \; - \; x(0) = \lim_{s \to 0}\left\{s\;X(s) - x(0^+)\right\}$$

TABLE 9.1 PROPERTIES OF THE LAPLACE TRANSFORM

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$	$X(s)$	R
		$x_1(t)$	$X_1(s)$	R_1
		$x_2(t)$	$X_2(s)$	R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
9.5.3	Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathcal{R}e\{s\} > 0\}$

Initial- and Final-Value Theorems

9.5.10 If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \longrightarrow \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to \infty} sX(s)$$

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the Laplace Transform
- Some Laplace Transform Pairs
- Analysis & Characterization of LTI Systems Using the Laplace Transform
- System Function Algebra and Block Diagram Representations
- The Unilateral Laplace Transform

Some Laplace Transform Pairs

TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13	$[e^{-\alpha t}\cos \omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
- The Inverse Laplace Transform
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- The Unilateral Laplace Transform

▪ Analysis & Characterization of LTI Systems:

$h(t)$

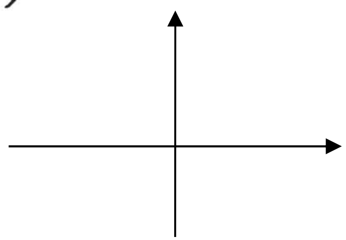
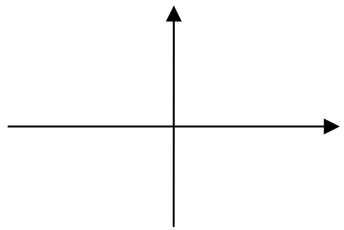
$x(t)$ → LTI System → $y(t) = h(t) * x(t)$

$X(s)$ → LTI System → $Y(s) = H(s)X(s)$

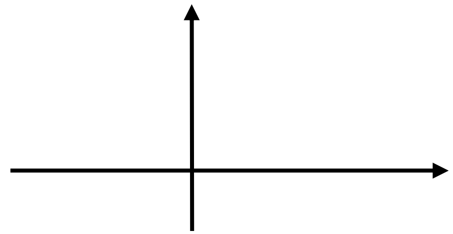
$H(s) = \mathcal{L}\{h(t)\}$

$H(s)$: system function

or transfer function

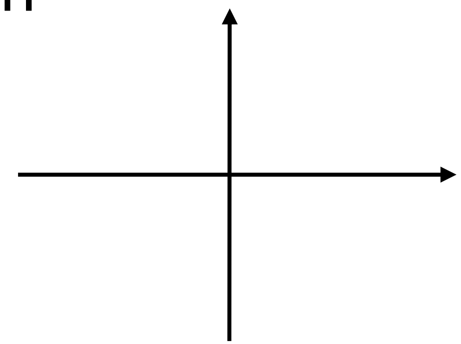
		$h(t)$	$H(s)$
▪ Causality	$x(t)$ $y(t)$	$h(t)$	
▪ Stability	$x(t)$ $y(t)$	$h(t)$	

- **Causality:**



  - For a causal LTI system,
    $h(t) = 0$ for $t < 0$, and thus is right sided

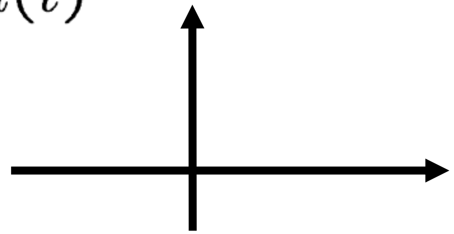
  - The ROC associated with the system function
    for a causal system is a right-half plane

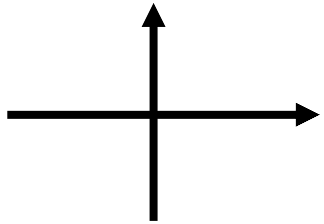


  - For a system with a rational system function,
    causality of the system is equivalent to
    the ROC being
    the right-half plane to the right of the rightmost pole

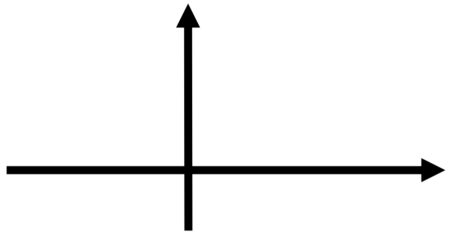
▪ Examples 9.17, 9.18, 9.19:

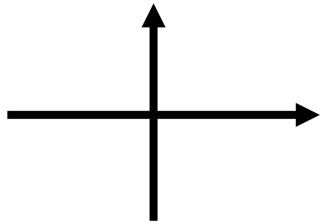
$$h(t) = e^{-t}u(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{Re}\{s\}$$



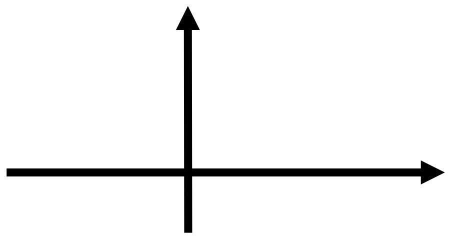


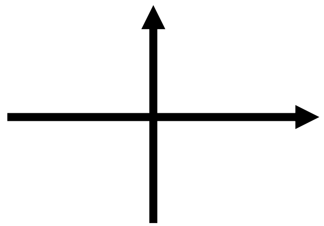
$$h(t) = e^{-|t|} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{-2}{s^2-1}, \quad -1 < \mathcal{Re}\{s\} < +1$$





$$h(t) = e^{-(t+1)}u(t+1) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{e^s}{s+1}, \quad -1 < \mathcal{Re}\{s\}$$



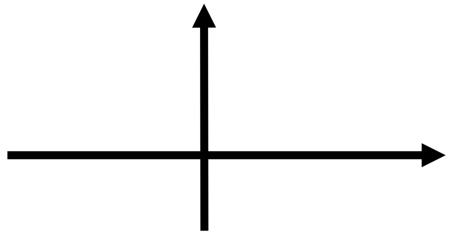


$\begin{cases} h(t): \\ H(s): \\ ROC: \end{cases}$ causal ? rational ? right-sided ?

$\begin{cases} h(t): \\ H(s): \\ ROC: \end{cases}$ causal ? rational ? right-sided ?

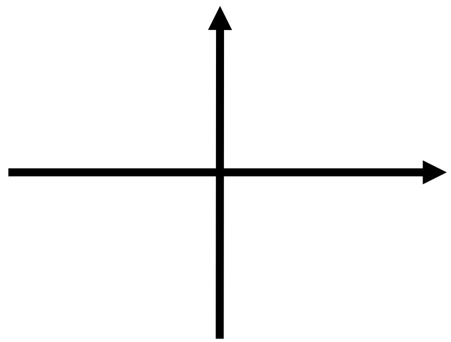
$\begin{cases} h(t): \\ H(s): \\ ROC: \end{cases}$ causal ? rational ? right-sided ?

- Anti-causality:



  - For a anti-causal LTI system,
    $h(t) = 0$ for $t > 0$, and thus is left sided

  - The ROC associated with the system function
    for a anti-causal system is a left-half plane



  - For a system with a rational system function,
    anti-causality of the system is equivalent to
    the ROC being
    the left-half plane to the left of the leftmost pole

Stability:

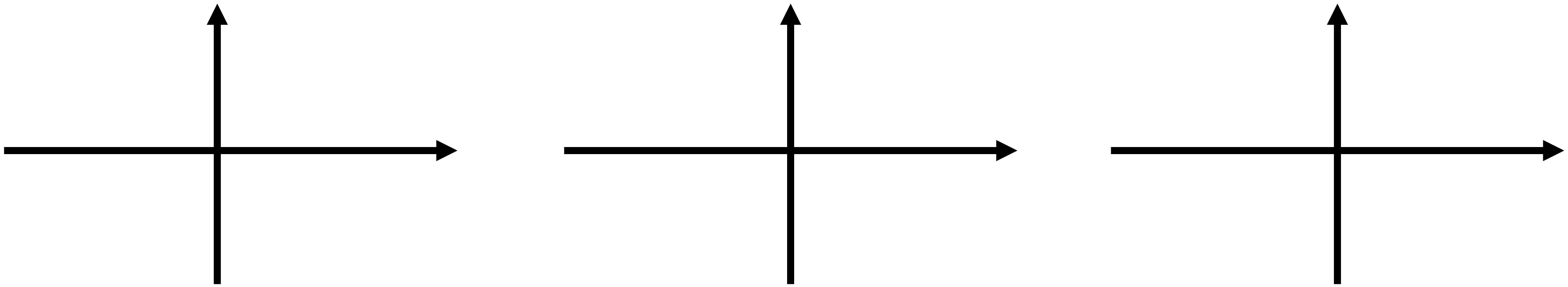
- An LTI system is stable
  if and only if
  the ROC of its system function H(s)
  includes the entire jw-axis [i.e., Re{s} = 0]

1.

2.

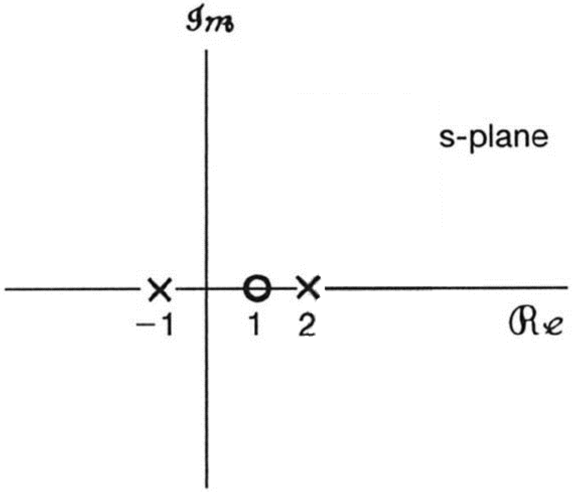
3.

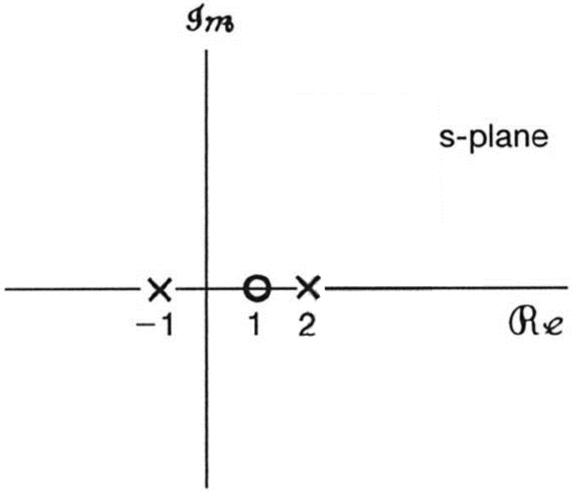
4.

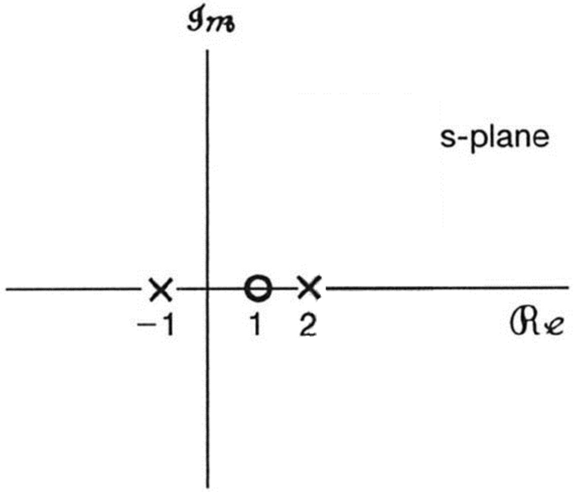


Example 9.20:

$$H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s-2}$$







$$h(t) = \frac{2}{3}e^{\ t}u(\ t) \quad \frac{1}{3}e^{\ t}u(\ t)$$

$$h(t) = \frac{2}{3}e^{\ t}u(\ t) \quad \frac{1}{3}e^{\ t}u(\ t)$$

$$h(t) = \frac{2}{3}e^{\ t}u(\ t) \quad \frac{1}{3}e^{\ t}u(\ t)$$

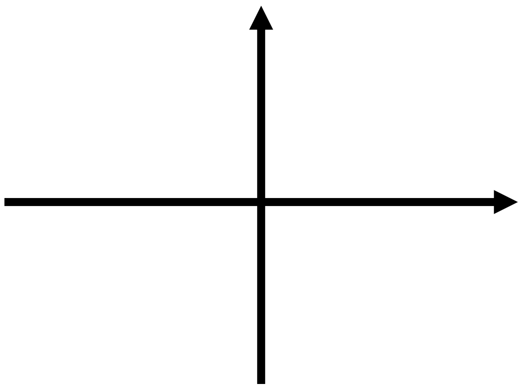
causal ? causal ? causal ?

stable ? stable ? stable ?

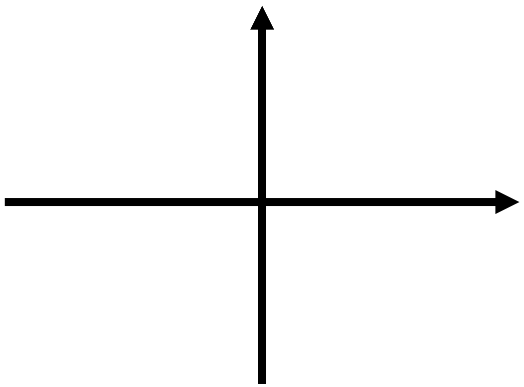
- **Stability:**

  - A causal system with rational system function H(s)
    is stable
    if and only if
    all of the poles of H(s) lie in the left-half of s-plane,
    i.e., all of the poles have negative real parts

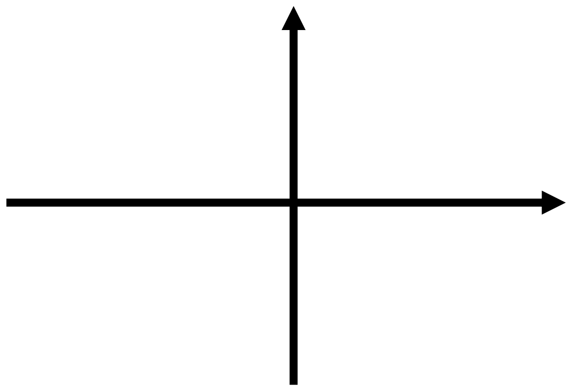
causal



stable



causal & stable

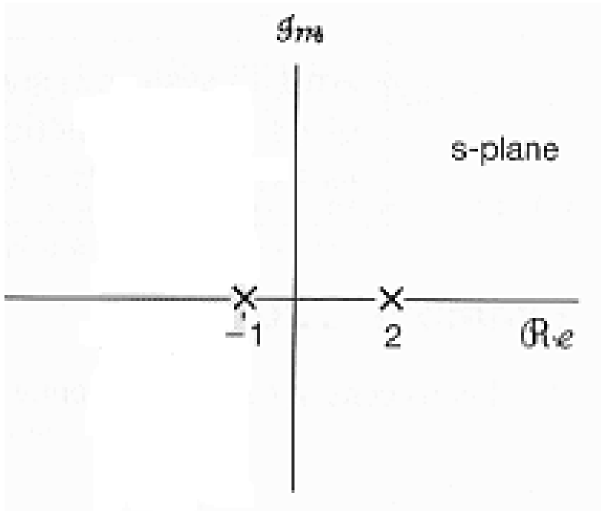


- Examples 9.17, 9.21:

$$h(t) = e^{-t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s+1}, \quad -1 < \mathcal{R}e\{s\}$$

$$h(t) = e^{2t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} H(s) = \frac{1}{s-2}, \quad 2 < \mathcal{R}e\{s\}$$

$$\begin{cases} h(t): \\ H(s): \end{cases} \qquad \begin{cases} h(t): \\ H(s): \end{cases}$$



- LTI Systems by Linear Constant-Coef Differential Equations:

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$

$$= b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$x(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t)$$

$$Y(s) = X(s) H(s) \qquad\qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\mathcal{L}\left\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{L}\left\{ \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^{N} a_k \, \mathcal{L}\left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^{M} b_k \, \mathcal{L}\left\{ \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^{N} a_k \, s^k \, Y(s) = \sum_{k=0}^{M} b_k \, s^k \, X(s)$$

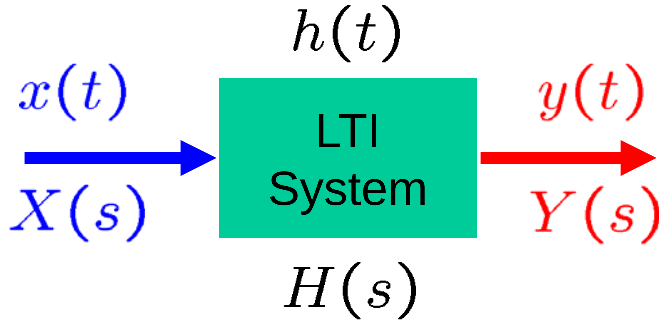
$$Y(s) \left[\sum_{k=0}^{N} a_k \, s^k \right] = X(s) \left[\sum_{k=0}^{M} b_k \, s^k \right]$$

zeros

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + \cdots + b_1 s + b_0}{a_N s^N + \cdots + a_1 s + a_0}$$

poles

- Example 9.23:



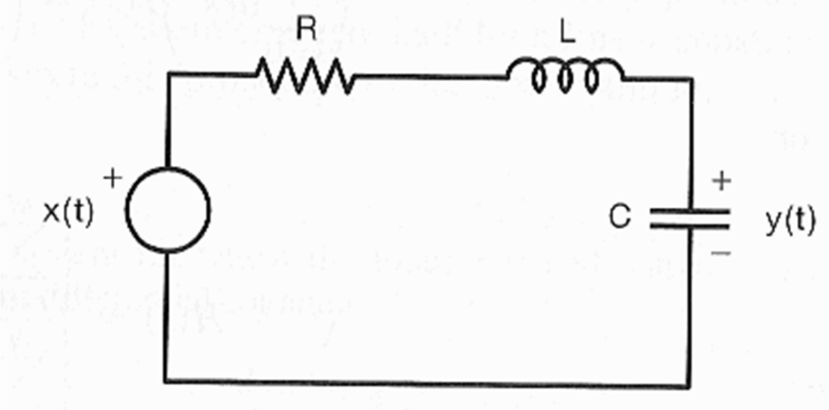
$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$\Rightarrow \qquad H(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow (\qquad) Y(s) = X(s) \qquad \Rightarrow H(s) = \frac{\qquad}{\qquad}$$

- If causal, $\Rightarrow \mathcal{R}\{s\}$ $\qquad \Rightarrow h(t) = e^{\ t} u(\ t)$
- If anti-causal, $\Rightarrow \mathcal{R}\{s\}$ $\qquad \Rightarrow h(t) = e^{\ t} u(\ t)$

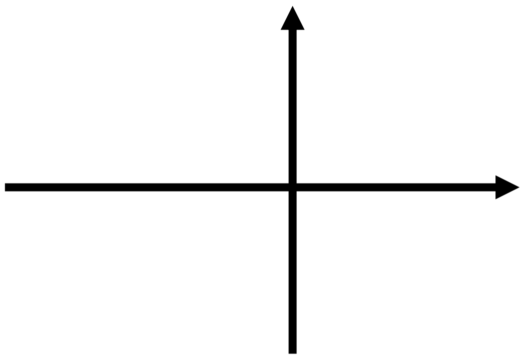
- Example 9.24:



$$LC \frac{d^2y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

$$\Rightarrow H(s) = \frac{(\qquad)}{\left(s^2 + \qquad\right)} = \frac{(\qquad)}{(s-a)(s-b)}$$

- If $R, L, C > 0$, $\Rightarrow \mathcal{R}e\{a\}, \mathcal{R}e\{b\} < 0$



i.e., poles with negative real parts

causal?

stable?

- Example 9.25:

$$x(t) = e^{-3t}u(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow y(t) = \left[e^{-t} - e^{-2t}\right]u(t)$$

?

$X(s) = \rule{3cm}{0.4pt}, \quad \mathcal{R}e\{s\}$

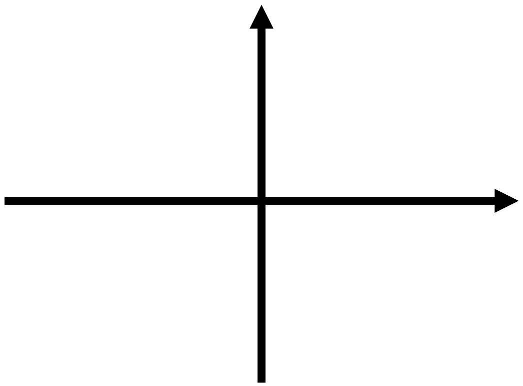
$Y(s) = \rule{5cm}{0.4pt}, \quad \mathcal{R}e\{s\}$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{X(s)} = \rule{5cm}{0.4pt} = \rule{5cm}{0.4pt}$$

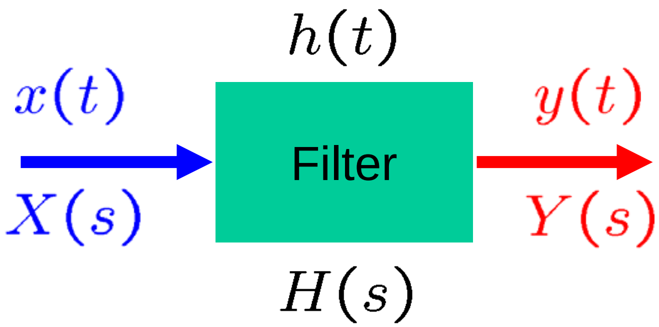
ROC : $\mathcal{R}e\{s\}$

causal?

stable?

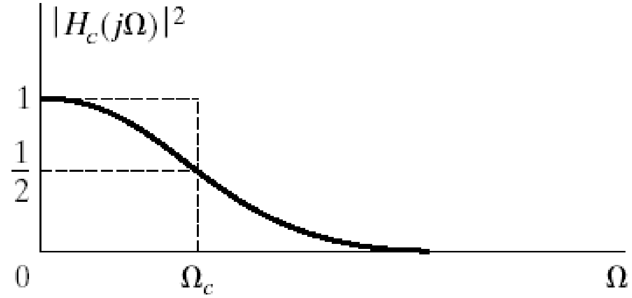


$$\Rightarrow \frac{d^2y(t)}{dt^2} + \quad \frac{dy(t)}{dt} + \quad y(t) = \frac{dx(t)}{dt} + \quad x(t)$$

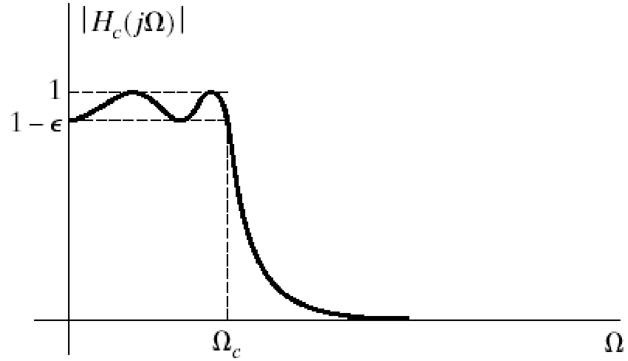


$H(jw)$

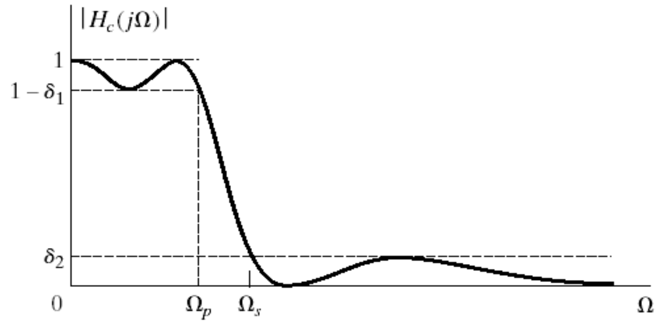
- Butterworth Lowpass Filters:



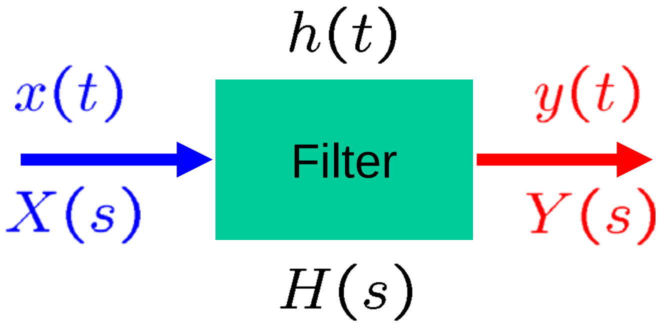
- Chebyshev Filters:



- Elliptic Filters:



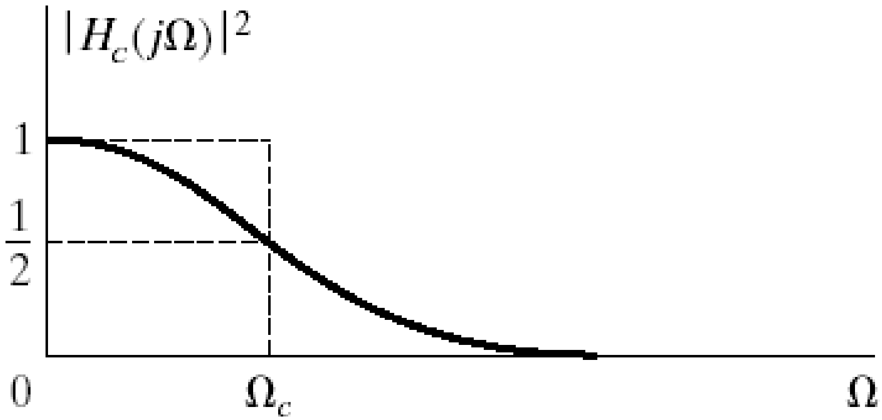
Butterworth Lowpass Filters:

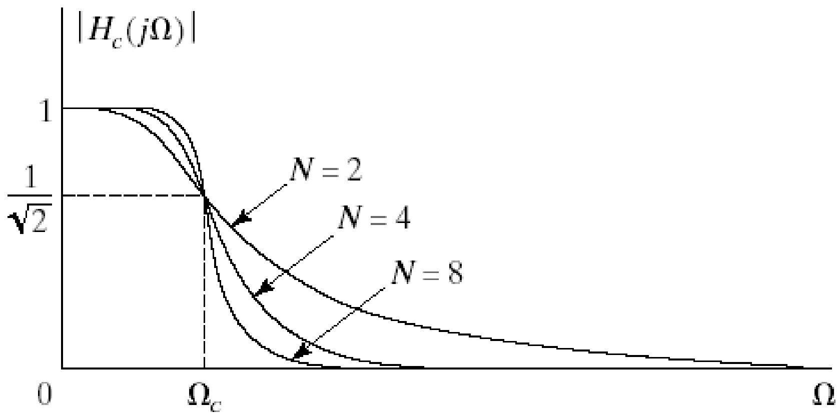


N-th order

$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1 + (\frac{j\Omega}{j\Omega_c})^{2N}}$$

$$1 + (\frac{s}{j\Omega_c})^{2N}$$





An Nth-Order Lowpass Butterworth Filters:

$$|B(jw)|^2 = \frac{1}{1+(jw/jw_c)^{2N}}$$

- If impulse response is real, $\Rightarrow \; B^*(jw) = B(-jw)$

$$|B(jw)|^2 \;=\; B(jw)\;B^*(jw) \;=\; B(jw)\;B(-jw)$$

$$\Rightarrow \; B(jw)\;B(-jw) = \frac{1}{1+(jw/jw_c)^{2N}}$$

$$\Rightarrow \; B(s)\;B(-s) = \frac{1}{1+(s/jw_c)^{2N}}$$

at $s_p = (-1)^{1/2N}(jw_c)$

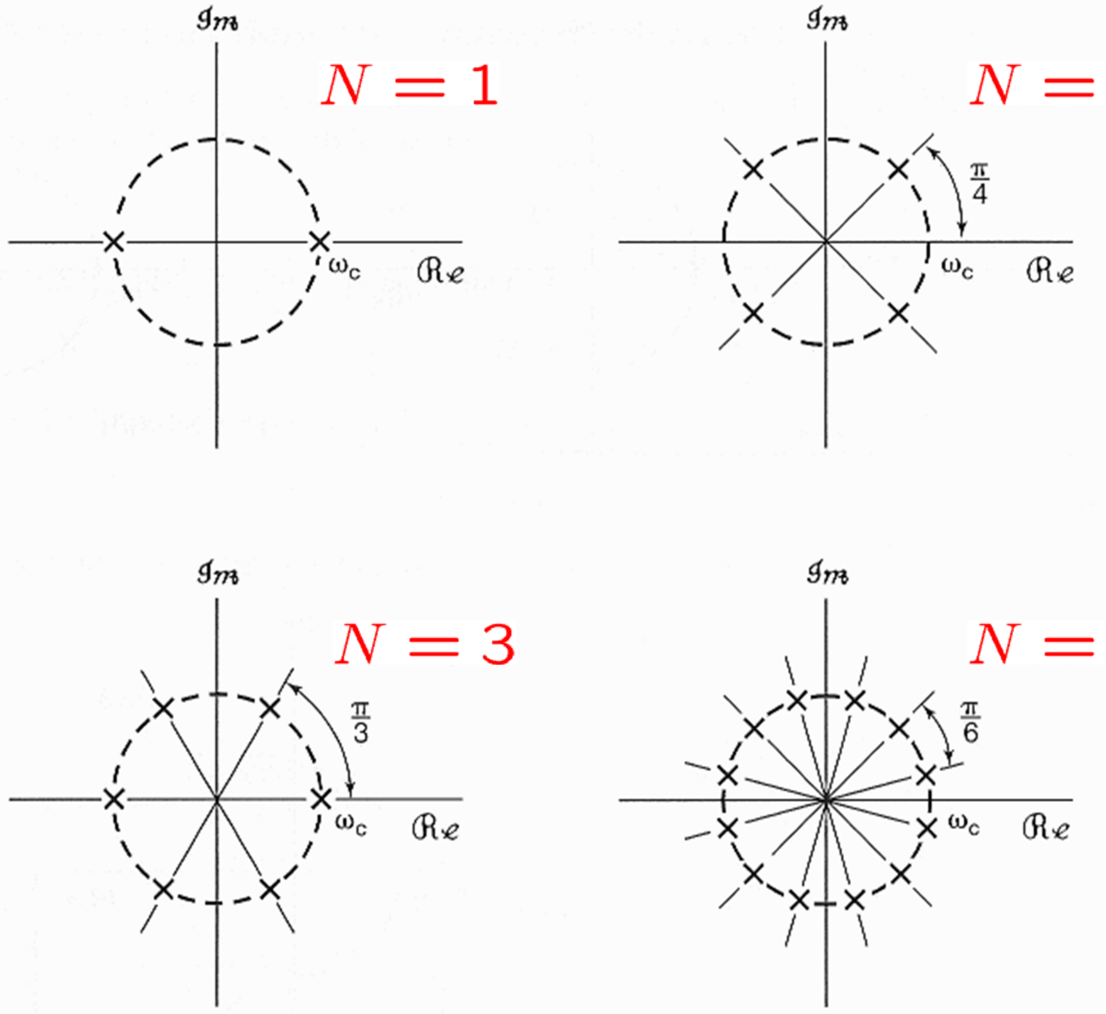
$$\Rightarrow \; s_p = w_c \exp\left(j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]\right)$$

$$\Rightarrow \begin{cases} |s_p| \;=\; w_c \\ \measuredangle\, s_p \;=\; \frac{\pi(2k+1)}{2N}+\frac{\pi}{2} \end{cases} \qquad k = 0, 1, 2, ..., 2N-1$$

▪ An Nth-Order Lowpass Butterworth Filters:

$$B(s)\,B(-s) = \frac{1}{1+(s/jw_c)^{2N}}$$

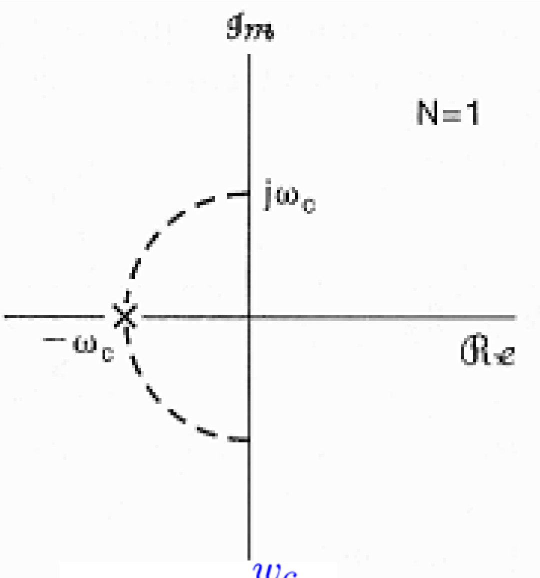
$$s_p = w_c \exp\left(j\left[\frac{\pi(2k+1)}{2N}+\frac{\pi}{2}\right]\right)$$



An Nth-Order Lowpass Butterworth Filters:

- Both $s = s_p$ and $s = -s_p$ are poles of $B(s)\ B(-s)$
- If the system is stable & causal

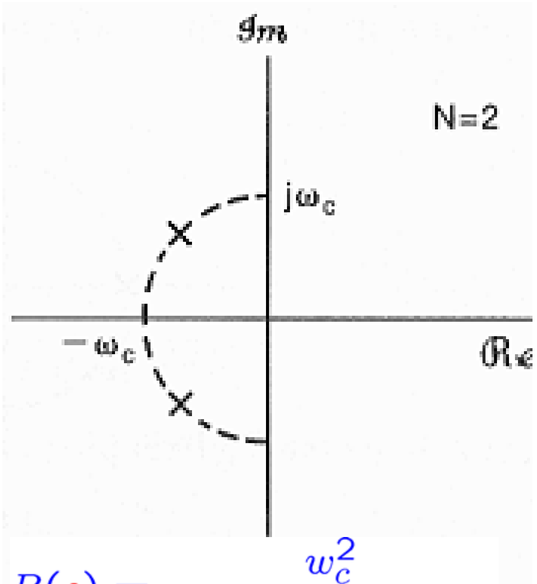
$$\Rightarrow \text{ poles of } B(s) \text{ are in the left-hand plane}$$



$$B(s) = \frac{w_c}{s + w_c}$$

$$B(-s) = \frac{w_c}{-s + w_c}$$

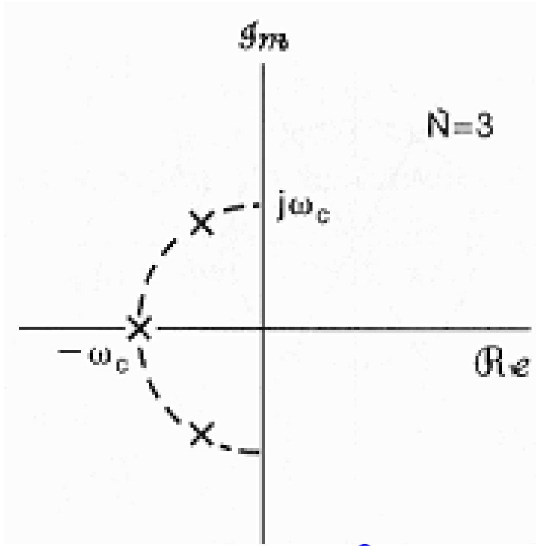
$$\frac{dy(t)}{dt} + w_c\, y(t) = w_c\, x(t)$$



$$B(s) = \frac{w_c^2}{s^2 + \sqrt{2} w_c s + w_c^2}$$

$$B(-s) = \frac{w_c^2}{s^2 - \sqrt{2} w_c s + w_c^2}$$

$$\frac{d^2y(t)}{dt^2} + \sqrt{2} w_c \frac{dy(t)}{dt} + w_c^2\, y(t) = w_c^2\, x(t)$$



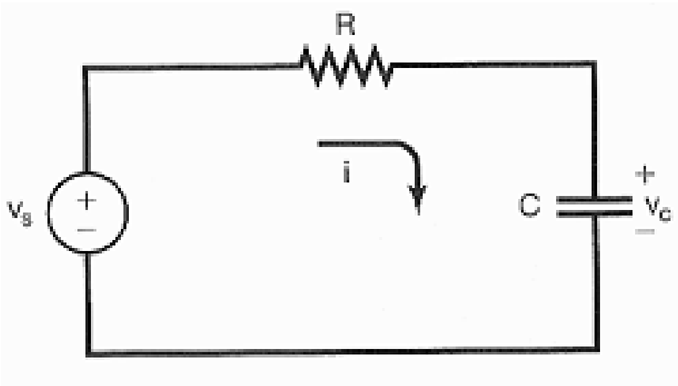
$$B(s) = \frac{w_c^3}{s^3 + 2w_c s^2 + 2w_c^2 s + w_c^3}$$

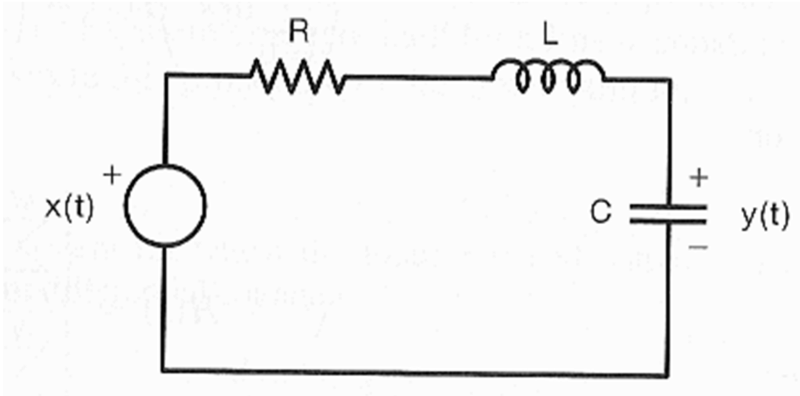
$$B(-s) = \frac{w_c^3}{-s^3 + 2w_c s^2 - 2w_c^2 s + w_c^3}$$

▪ An Nth-Order Lowpass Butterworth Filters:

$$B(s) = \frac{w_c}{s + w_c} \qquad B(s) = \frac{w_c^2}{s^2 + \sqrt{2} w_c s + w_c^2} \qquad B(s) = \frac{w_c^3}{s^3 + 2 w_c s^2 + 2 w_c^2 s + w_c^3}$$

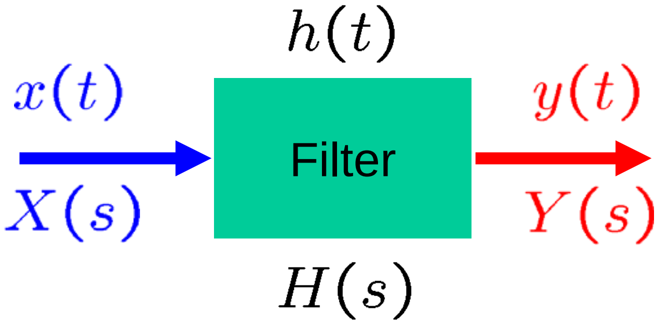
$$\frac{dy(t)}{dt} + w_c\, y(t) = w_c\, x(t) \qquad \frac{d^2 y(t)}{dt^2} + \sqrt{2} w_c \frac{dy(t)}{dt} + w_c^2\, y(t) = w_c^2\, x(t)$$





$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t) \qquad LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t)$$

Chebyshev Filters:

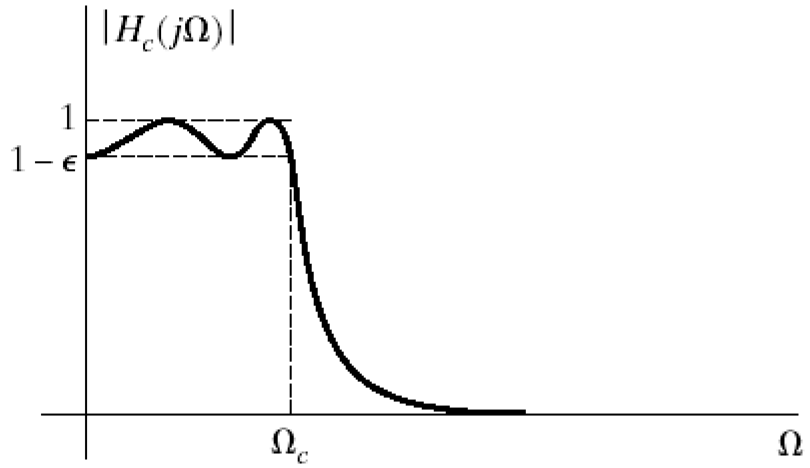


$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\epsilon^2 V_N^2(\frac{\Omega}{\Omega_c})}$$

$$V_N(x) = \cos\left(N \cos^{-1} x\right)$$

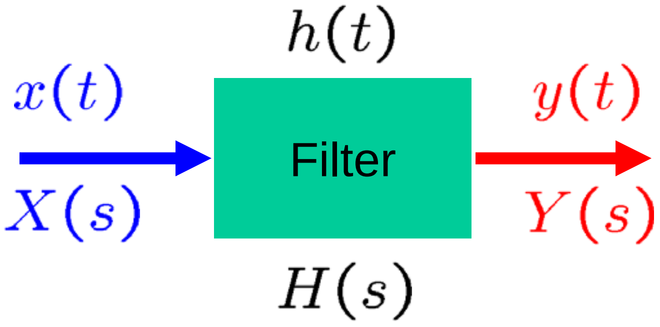
Nth-order Chebyshev polynomial

$$\begin{aligned} V_0(x) &= 1 \\ V_1(x) &= x \\ V_2(x) &= 2x^2 - 1 \\ V_3(x) &= 4x^3 - 3x \\ \cdots & \quad \cdots \end{aligned}$$



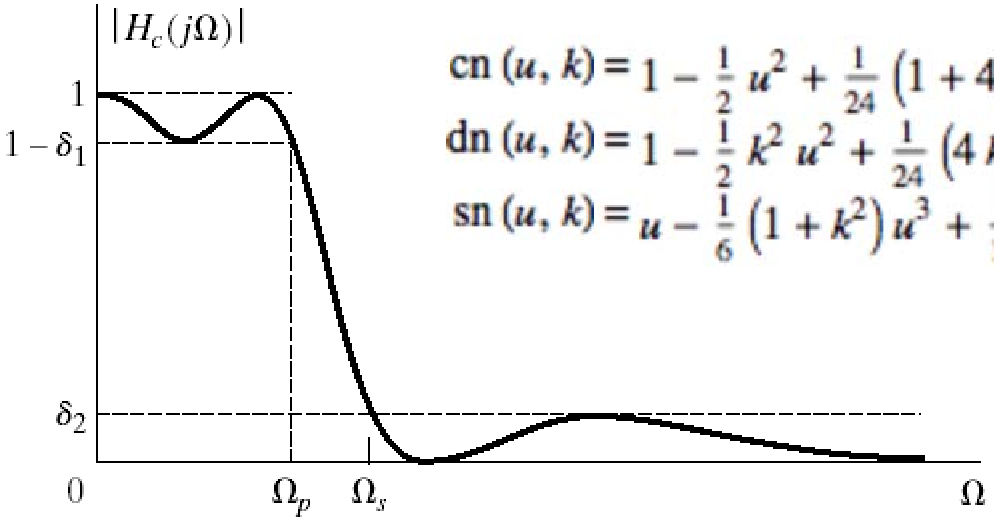
$$\begin{aligned} \cos(0\theta) &= 1 \\ \cos(1\theta) &= \cos(\theta) \\ \cos(2\theta) &= 2\cos(\theta)\cos(\theta) - \cos(0\theta) \\ &= 2\cos^2(\theta) - 1 \\ \cos(3\theta) &= 2\cos(\theta)\cos(2\theta) - \cos(\theta) \\ &= 4\cos^3(\theta) - 3\cos(\theta) \end{aligned}$$

Elliptic Filters:



$$\left|H_c(j\Omega)\right|^2 = \frac{1}{1+\epsilon^2 U_N^2(\Omega)}$$

$U_N(x)$: Jacabian elliptic function

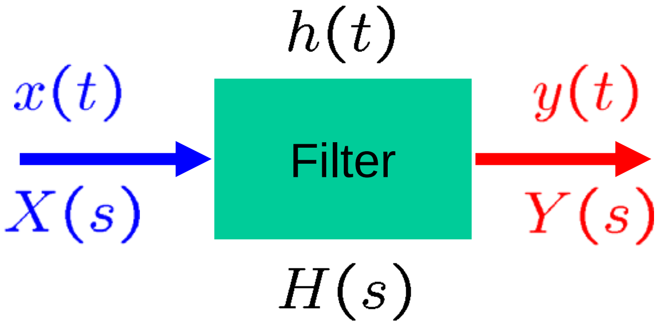


$$\mathrm{cn}\,(u,k) = 1 - \frac{1}{2}u^2 + \frac{1}{24}\left(1+4k^2\right)u^4 - \frac{1}{720}\left(1+44k^2+16k^4\right)u^6 + \ldots$$

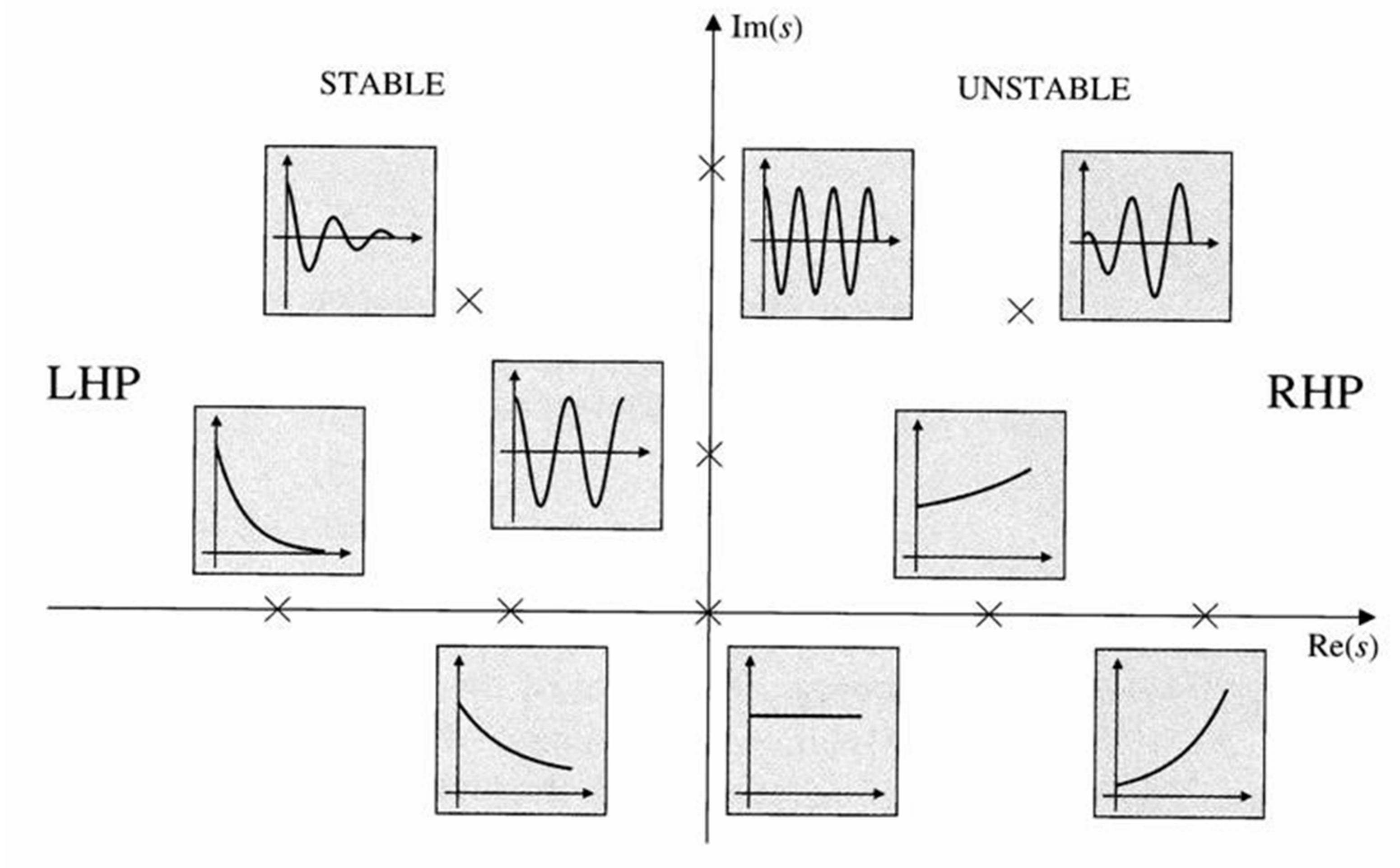
$$\mathrm{dn}\,(u,k) = 1 - \frac{1}{2}k^2u^2 + \frac{1}{24}\left(4k^2+k^4\right)u^4 - \frac{1}{720}\left(16k^2+44k^4+k^6\right)u^6 + \ldots$$

$$\mathrm{sn}\,(u,k) = u - \frac{1}{6}\left(1+k^2\right)u^3 + \frac{1}{120}\left(1+14k^2+k^4\right)u^5 + \ldots$$

System Characteristics and Pole Location



$$H(s) = \frac{s+c}{(s+a)(s+b)}$$



Outline

- The Laplace Transform
- The Region of Convergence (ROC) for Laplace Transforms
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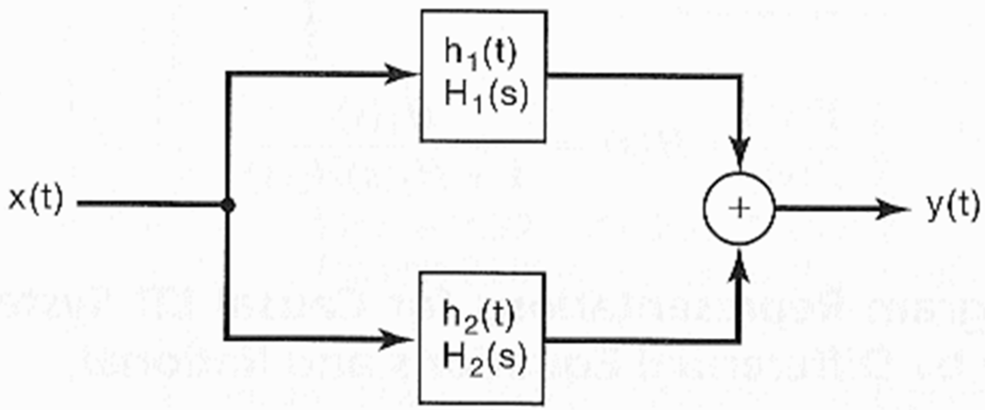
System Function Blocks:

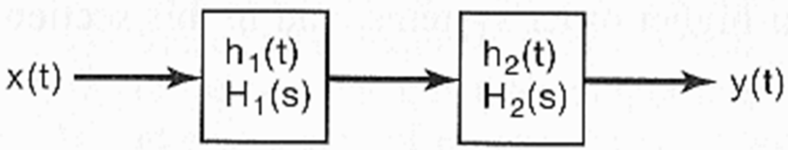
- parallel interconnection

$$h(t) = h_1(t) + h_2(t)$$
$$H(s) = H_1(s) + H_2(s)$$



- series interconnection

$$h(t) = h_1(t) * h_2(t)$$
$$H(s) = H_1(s)\ H_2(s)$$

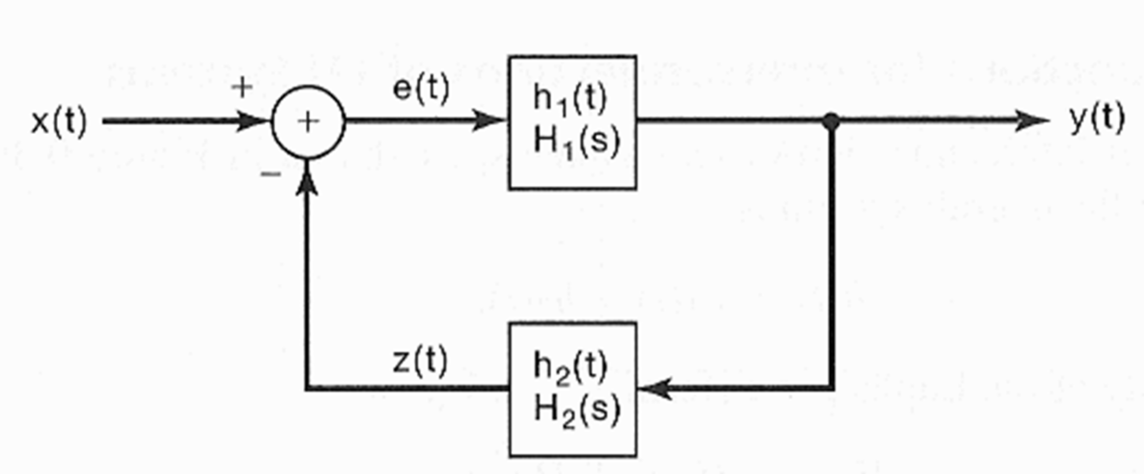


(b)

System Function Blocks:

- feedback interconnection

$$H(s) = \frac{H_1(s)}{1+H_1(s)H_2(s)}$$



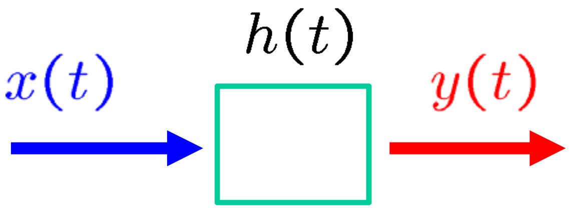
$$Y = H_1 \, E$$

$$Z = H_2 \, Y$$

$$E = X - Z$$

- Example 9.28:
  - Consider a causal LTI system with system function

$$H(s) = \frac{1}{s+3} \quad \Rightarrow Y(s) = \frac{\qquad}{\qquad} X(s)$$



$$\Rightarrow \frac{d}{dt} y(t) + \qquad y(t) = x(t)$$

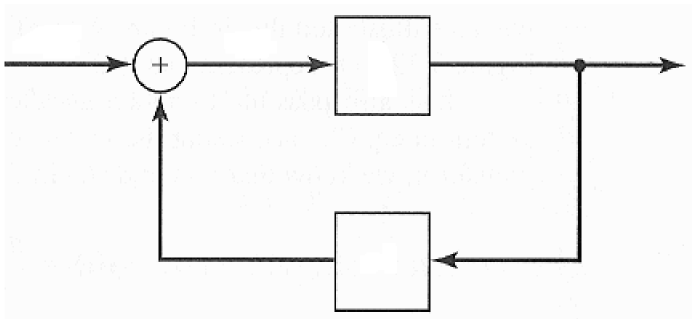
$$\Rightarrow \frac{d}{dt} y(t) = x(t) - \qquad y(t)$$

Example 9.29:

- Consider a causal LTI system with system function

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+\quad}\right)(s+\quad)$$

$$\Rightarrow \quad Z(s) \triangleq \frac{1}{s+3}X(s) \quad \& \quad Y(s) = (s+2)Z(s)$$



- Example 9.30:
  - Consider a causal LTI system with system function

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

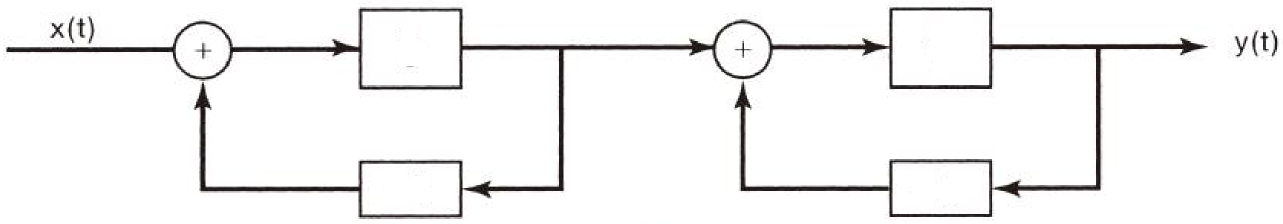
$$\Rightarrow s^2Y + 3sY + 2Y = X$$

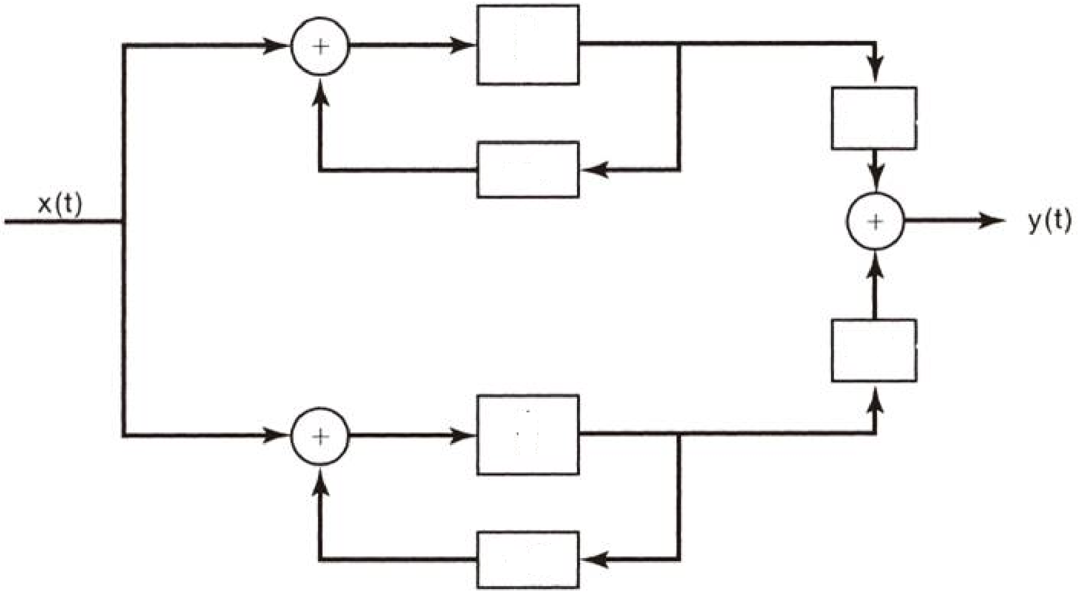
$$\Rightarrow \begin{cases} sY = F \\ s^2Y = E = sF \end{cases} \qquad \Rightarrow E = s^2Y =$$

- Example 9.30:

$$H(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+\quad}\right)\left(\frac{1}{s+\quad}\right)$$

$$= \left(\frac{\quad}{s+\quad}\right) + \left(\frac{\quad}{s+\quad}\right)$$

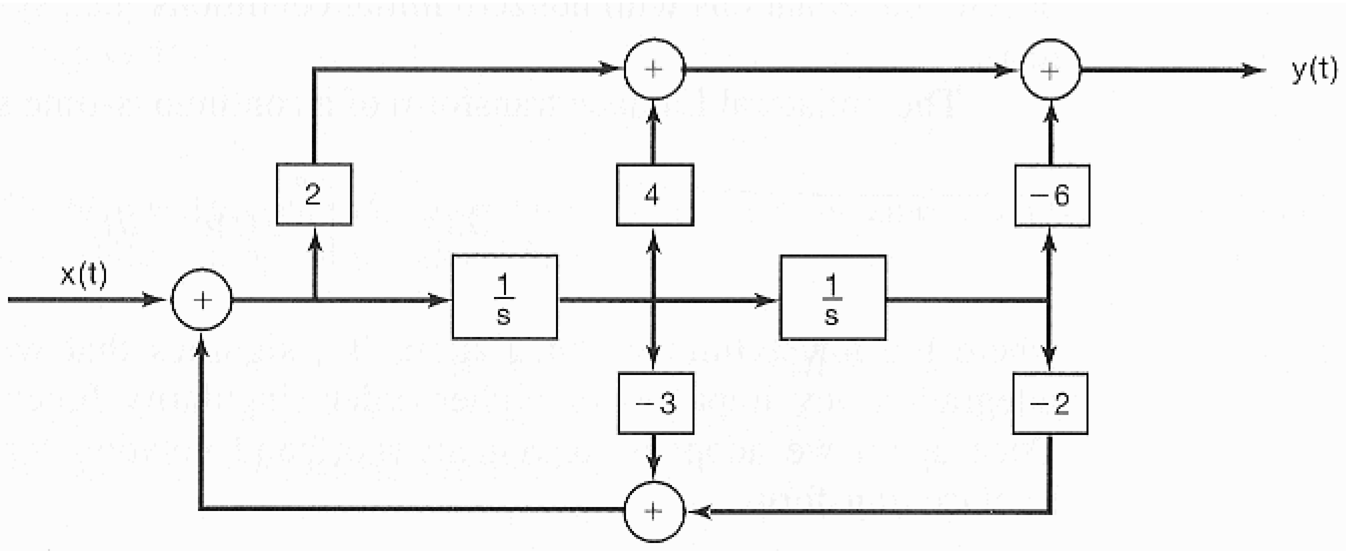




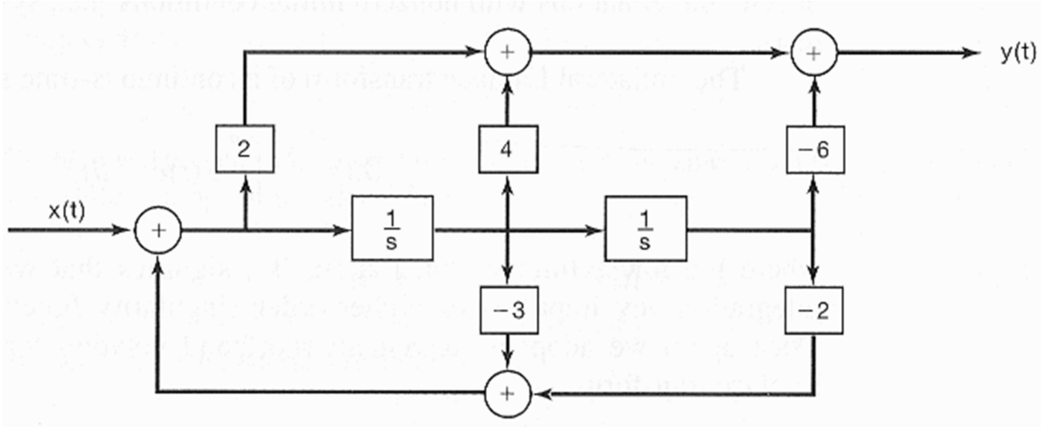
- Example 9.31:

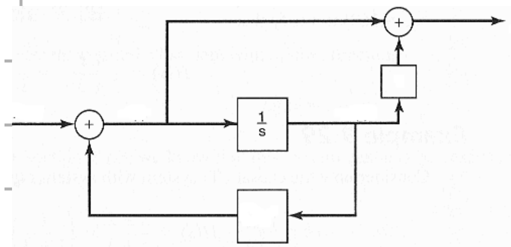
$$H(s) = \frac{2s^2+4s-6}{s^2+3s+2} = \left(\frac{1}{s^2+3s+2}\right)(2s^2+4s-6)$$

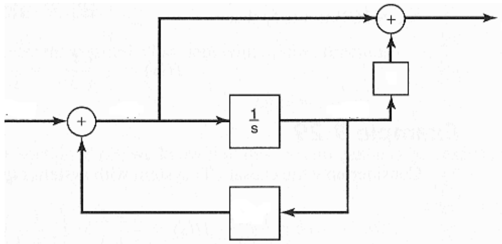
$$\Rightarrow \quad Z(s) \triangleq \frac{1}{s^2+3s+2}X(s) \quad \& \quad Y(s) = (2s^2+4s-6)Z(s)$$

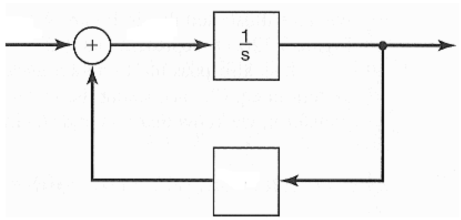


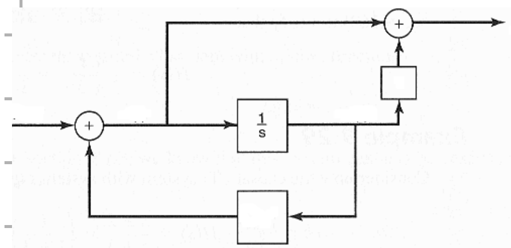
$$H(s) = \begin{cases} \dfrac{2s^2+4s-6}{s^2+3s+2} \\[2ex] \left(\dfrac{2(s-1)}{s+2}\right)\left(\dfrac{s+3}{s+1}\right) \\[2ex] \left(\dfrac{2(s-1)}{s+1}\right)\left(\dfrac{s+3}{s+2}\right) \\[2ex] 2+\dfrac{6}{s+2}+\dfrac{-8}{s+1} \end{cases}$$

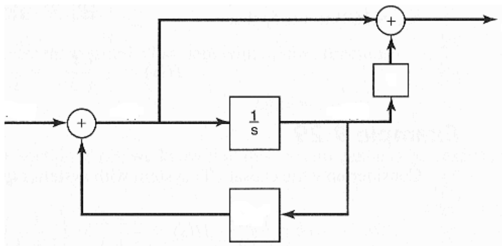


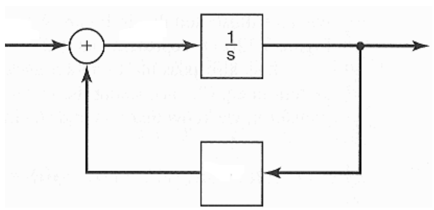






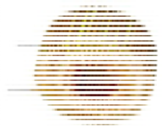




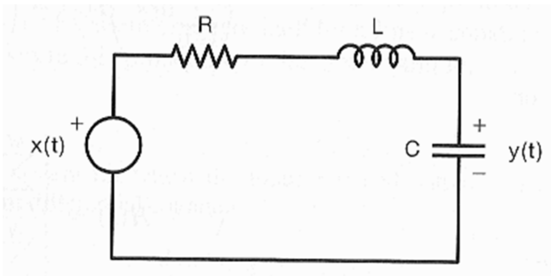


- Technology





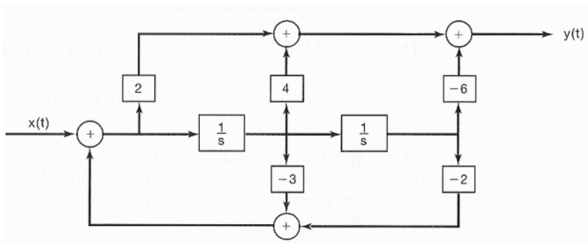

- Engineering

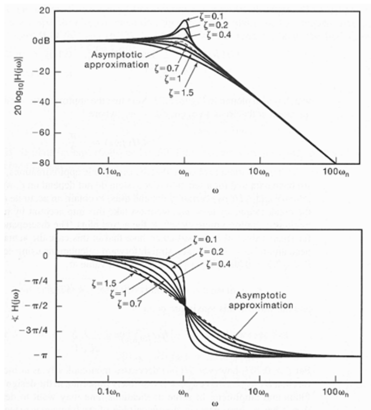


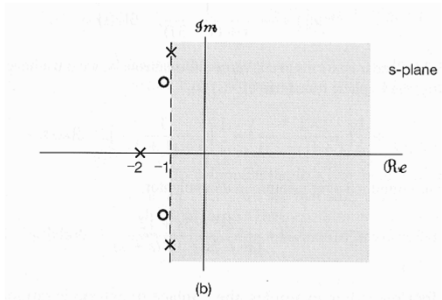
- Mathematics

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

- Graph







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▪ The Unilateral Laplace Transform of x(t):

bilateral LT

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{0} x(t) e^{-st} dt + \int_{0}^{\infty} x(t) e^{-st} dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

$$X(s) = \mathcal{L}\{x(t)\}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

unilateral LT

for causal system &

with nonzero initial condition

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$$x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$$

$$\mathcal{X}(s) = \mathcal{UL}\{x(t)\}$$

$$x(t) = \mathcal{UL}^{-1}\{\mathcal{X}(s)\}$$

ROC : a right-half plane

The Unilateral Laplace Transform

TABLE 9.3 PROPERTIES OF THE UNILATERAL LAPLACE TRANSFORM

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s-domain	$e^{s_0 t}x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a}\mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x*(t)$	$x*(s)$
$\boxed{\text{Convolution (assuming that } x_1(t) \text{ and } x_2(t) \text{ are identically zero for } t < 0)}$	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$
Differentiation in the time domain	$\frac{d}{dt}x(t)$	$\boxed{s\mathfrak{X}(s) - x(0^-)}$
Differentiation in the s-domain	$-tx(t)$	$\frac{d}{ds}\mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} s\mathfrak{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathfrak{X}(s)$$

$x_1(t) = x_2(t) \equiv 0$
$t < 0$

$s\mathcal{X}(s) - x(0^-)$

Differentiation Property:

$$\boxed{\int_0^\infty u'v\,dt = uv\Big|_0^\infty - \int_0^\infty uv'\,dt}$$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^\infty \frac{dx(t)}{dt} e^{-st}dt \quad = x(t)e^{-st}\Big|_{0^-}^\infty + s\int_{0^-}^\infty x(t)e^{-st}dt$$

$$= s\mathcal{X}(s) - x(0^-)$$

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_{0^-}^\infty \frac{d^2x(t)}{dt^2} e^{-st}dt \quad = s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

- Example 9.38:

$$\frac{d^2y(t)}{dt^2} + 3\,\frac{dy(t)}{dt} + 2\,y(t) = x(t) \qquad \begin{cases} y(0^-) = \beta \\ y'(0^-) = \gamma \end{cases}$$

$$x(t) = \alpha\, u(t)$$

$$\Rightarrow \left[s^2\mathcal{Y}(s) - \qquad - \qquad\right] + 3\left[s\mathcal{Y}(s) - \qquad\right] + 2\left[\mathcal{Y}(s)\right] = \alpha$$

$$\Rightarrow \mathcal{Y}(s) = \alpha \,\frac{\qquad}{\qquad}$$

zero- response

only response

$$+\,\beta\,\frac{\qquad}{\qquad}$$

$$+\,\gamma\,\frac{\qquad}{\qquad}$$

zero- response

only response

Example 9.38:

- If $\alpha = 2, \quad \beta = 3, \quad \gamma = -5$

$$\Rightarrow \mathcal{Y}(s) = \frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad}$$

$$\Rightarrow y(t) = \left[\quad + \quad e^{\quad t} + \quad e^{\quad t} \right] u(t), \quad \text{for } t > 0$$

Summary of Fourier Transform and Laplace Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

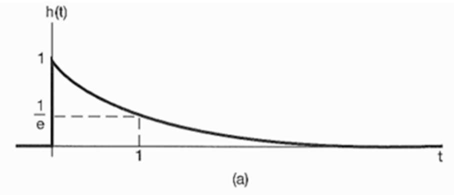
$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

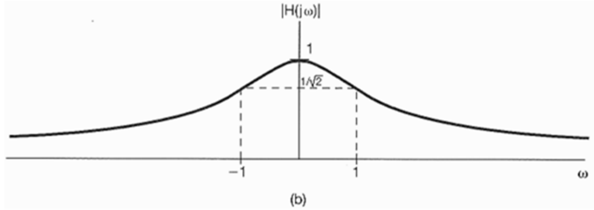
$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$X(s) = \mathcal{L}\left\{x(t)\right\} = \mathcal{F}\left\{x(t) e^{-\sigma t}\right\}$$

$$X(jw) = \mathcal{F}\left\{x(t)\right\} = \mathcal{L}\left\{x(t)\right\}\Big|_{s=jw} = X(s)\Big|_{s=jw}$$





$$h(t) = e^{-at}u(t), \quad a > 0 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(jw) = \frac{1}{jw + a}$$

$$h(t) = e^{-at}u(t), \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad H(s) = \frac{1}{s + a},$$

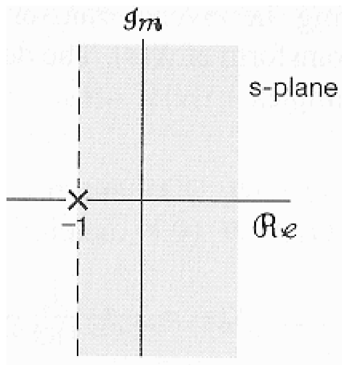
$$\mathcal{R}e\{s\} > -a$$

Definition
Theorem
Property

Causality

Stability

ROC



- Example 9.26: $x(t) = 1 \rightarrow y(t) = 0$

$$H(s) = \frac{4s}{(s+2)(s-4)} = \frac{4/3}{s+2} + \frac{8/3}{s-4}$$

$$h(t) = \frac{4}{3} e^{-2t} u(t) - \frac{8}{3} e^{4t} u(-t)$$

$$Y(s) = H(s)X(s) = H(s) \;\; 2\pi j \; \delta(s) = H(0) = 0$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) \; x(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left\{ \frac{4}{3} e^{-2\tau} \; u(\tau) - \frac{8}{3} e^{4\tau} \; u(-\tau) \right\} \; x(t-\tau) d\tau$$

$$= \int_{0}^{\infty} \frac{4}{3} e^{-2\tau} \; d\tau - \int_{-\infty}^{0} \frac{8}{3} e^{4\tau} d\tau$$

$$= \frac{4}{3(-2)} e^{-2\tau} \Big|_{0}^{\infty} - \frac{8}{3(4)} e^{4\tau} \Big|_{-\infty}^{0} = \frac{-2}{3}(0-1) - \frac{2}{3}(1-0)$$

- The Laplace Transform
- The ROC for LT
- The Inverse LT
- Geometric Evaluation of the FT
- Properties of the LT
  - Linearity
  - Time Shifting
  - Shifting in the s-Domain
  - Time Scaling
  - Conjugation
  - Convolution
  - Differentiation in the Time Domain
  - Differentiation in the s-Domain
  - Integration in the Time Domain
  - Initial- and Final-Value Theorems
- Some LT Pairs
- Analysis & Charac. of LTI Systems Using the LT
- System Function Algebra, Block Diagram Repre.
- The Unilateral LT

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS
CT
DT
(Chap 3)

Aperiodic

FT
CT (Chap 4)
DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)