Spring 2013

信號與系統
Signals and Systems

Chapter SS-5
The Discrete-Time Fourier Transform

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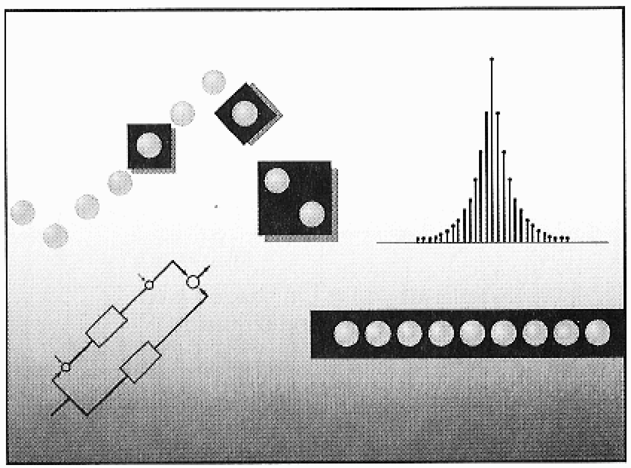
NTU-EE

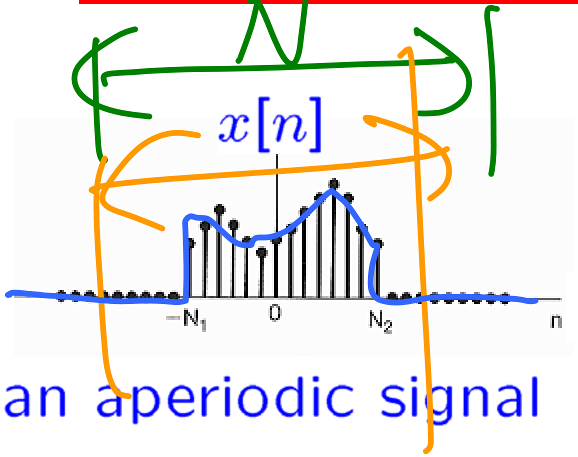
Feb13 – Jun13



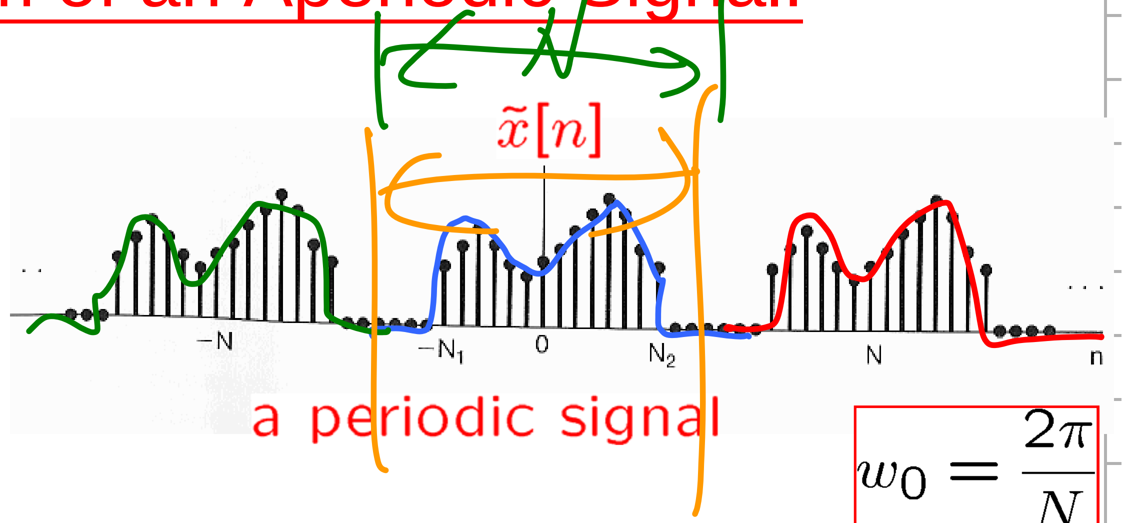
Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of Aperiodic Signals:
  the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by
  Linear Constant-Coefficient Difference Equations

- DT Fourier Transform of an Aperiodic Signal:



an aperiodic signal



a periodic signal

$$w_0 = \frac{2\pi}{N}$$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(w_0)n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$\Rightarrow \quad a_k = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n] e^{-jk(w_0)n} = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

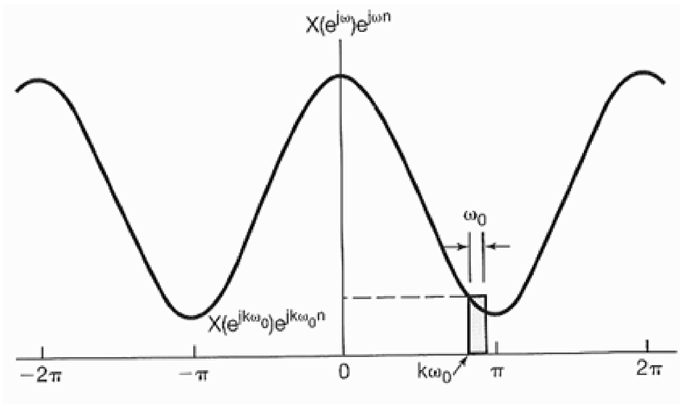
$$\Rightarrow a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n] e^{-jk(w_0)n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n] e^{-jk(w_0)n}$$

ss4-10

DT Fourier Transform of an Aperiodic Signal:

- Define $X(e^{jw})$:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$



- Then,

$$a_k = \frac{1}{N} X(e^{jkw_0})$$

$$w = kw_0$$

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\boxed{\frac{1}{N} = \frac{1}{2\pi} w_0}$$

$$\boxed{w_0 N = 2\pi}$$

- Hence,

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jkw_0}) e^{jkw_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=<N>} X(e^{jkw_0}) e^{jkw_0 n} w_0$$

kw_0

N

$-\infty \to +\infty$

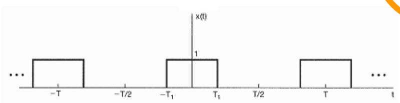
- Example 3.5: $T \; a_k = T \dfrac{\sin(k 2\pi \frac{T_1}{T})}{k\pi}$

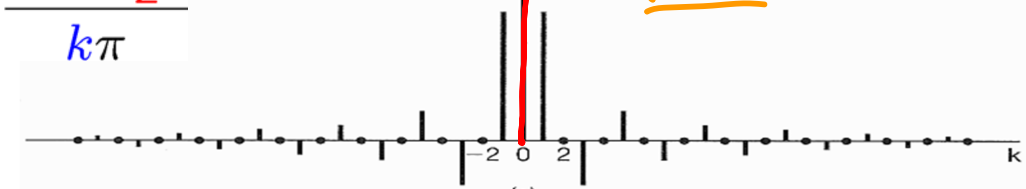
$4 \times \frac{1}{2} = 2$

$T = 4T_1$

$T a_k = T \dfrac{\sin(k\frac{\pi}{2})}{k\pi}$

$T a_k$





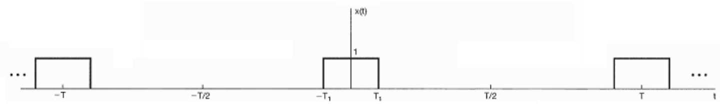
(a)

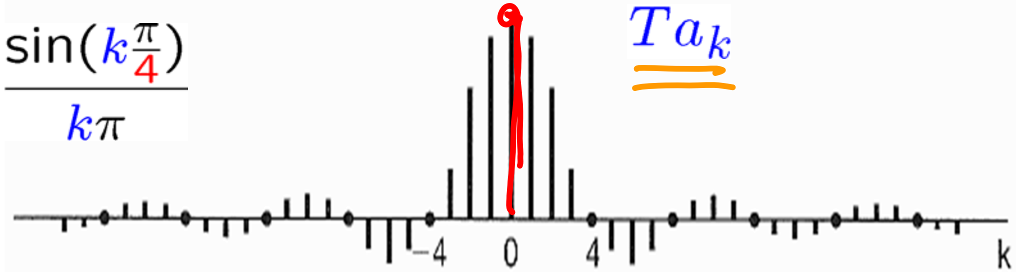
$8 \times \frac{1}{4} = 2$

$T = 8T_1$

$T a_k = T \dfrac{\sin(k\frac{\pi}{4})}{k\pi}$

$T a_k$





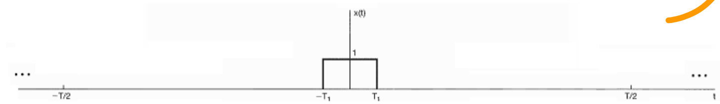
(b)

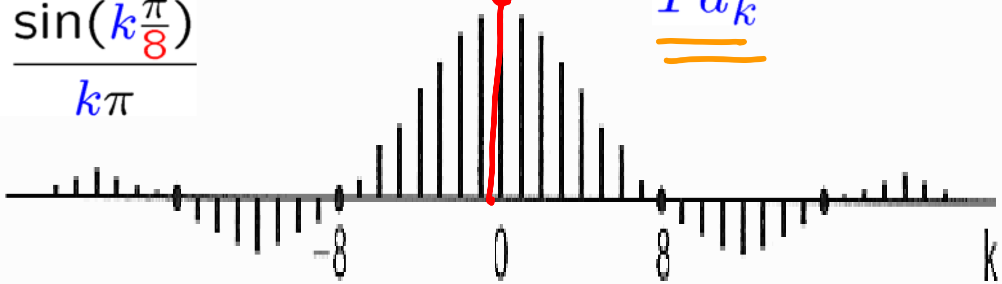
$16 \times \frac{1}{8} = 2$

$T = 16T_1$

$T a_k = T \dfrac{\sin(k\frac{\pi}{8})}{k\pi}$

$T a_k$





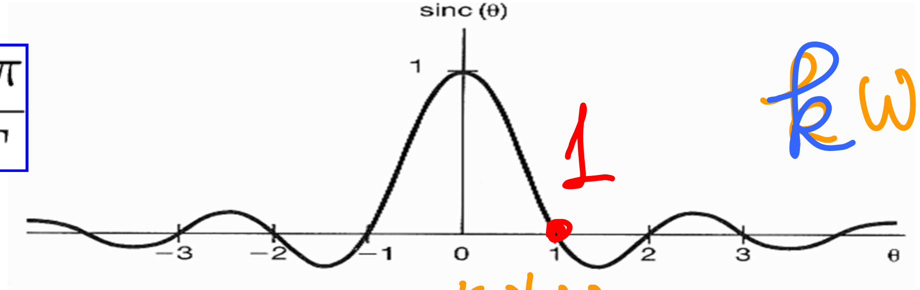
- Example 3.5:

$\omega_0 = \frac{2\pi}{T}$

$$T a_k = T \, \frac{2\sin(k w_0 T_1)}{k w_0 T}$$

$$= T_1 \, \frac{2\sin(k w_0 T_1)}{k w_0 T_1}$$

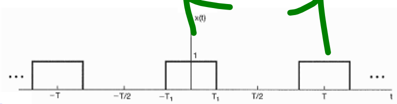
$$\boxed{w_0 = \frac{2\pi}{T}}$$



$k\omega_0$

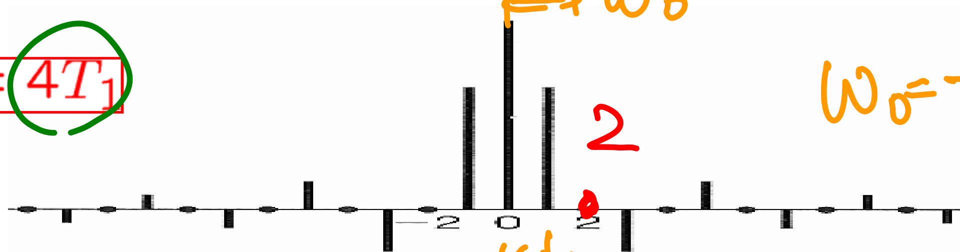
1

$\overset{\omega_0}{\leftrightarrow}$



T

$\boxed{T = 4T_1}$



2

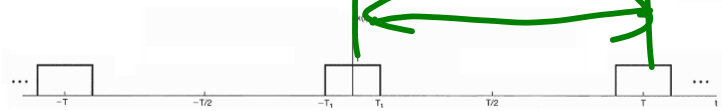
$\omega_0 = \frac{2\pi}{4T_1}$

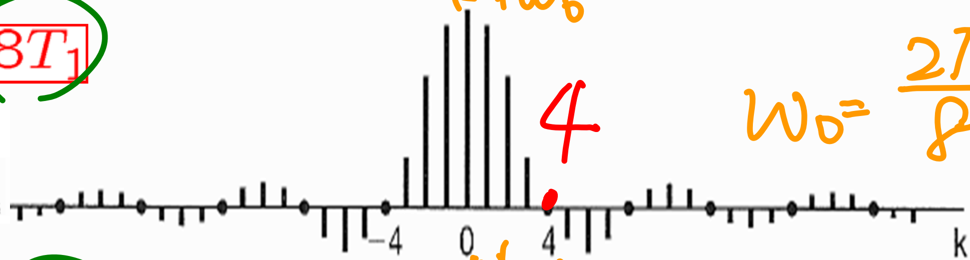
$$= \frac{1}{2} \; T \; \text{sinc}\left(\frac{k}{2}\right)$$

$\overset{\omega_0}{\leftrightarrow}$

T

$\boxed{T = 8T_1}$





4

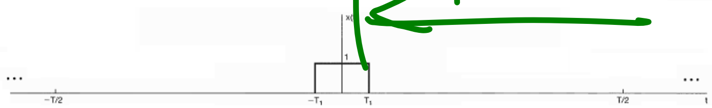
$\omega_0 = \frac{2\pi}{8T_1}$

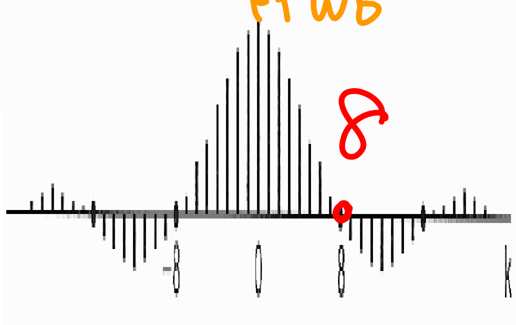
$$= \frac{1}{4} \; T \; \text{sinc}\left(\frac{k}{4}\right)$$

$\overset{\omega_0}{\leftrightarrow}$

T

$\boxed{T = 16T_1}$





8

$\omega_0 = \frac{2\pi}{16T_1}$

$$= \frac{1}{8} \; T \; \text{sinc}\left(\frac{k}{8}\right)$$

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$-\infty$ w_0 $\to$ $+\infty$ w_0

ss4-11

▪ DT Fourier Transform of an Aperiodic Signal:

- As $N \to \infty$, $\tilde{x}[n] \to x[n]$

$\boxed{w_0 N = 2\pi}$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- inverse Fourier transform eqn
- synthesis eqn

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

- $X(e^{jw})$: Fourier transform of $x[n]$
  spectrum
- analysis eqn

$$a_k = \frac{1}{N}X(e^{jw})\Big|_{w=kw_0}$$

$\boxed{w_0 = \frac{2\pi}{N}}$

▪ Periodicity of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$X(e^{j(w+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}e^{-j(2\pi)n}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

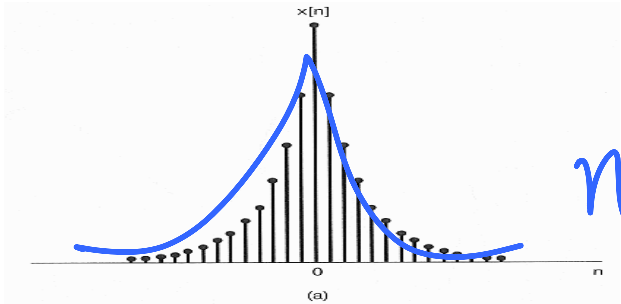
$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(w)n}$$

$$= X(e^{jw})$$

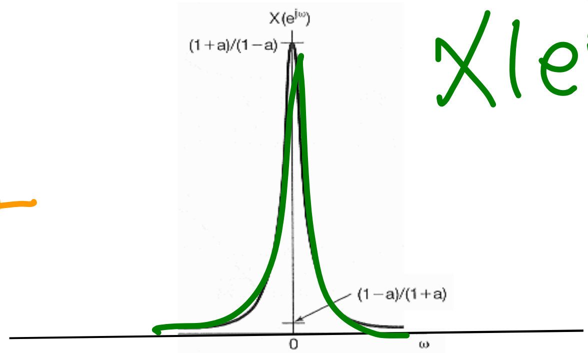
$(-1)^n$

1

1



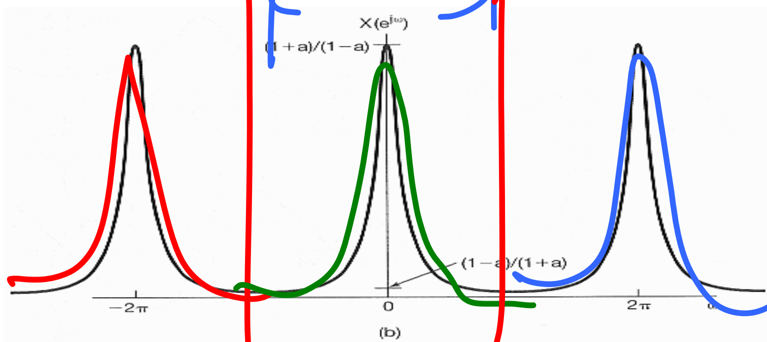
n



$X(e^{jw})$

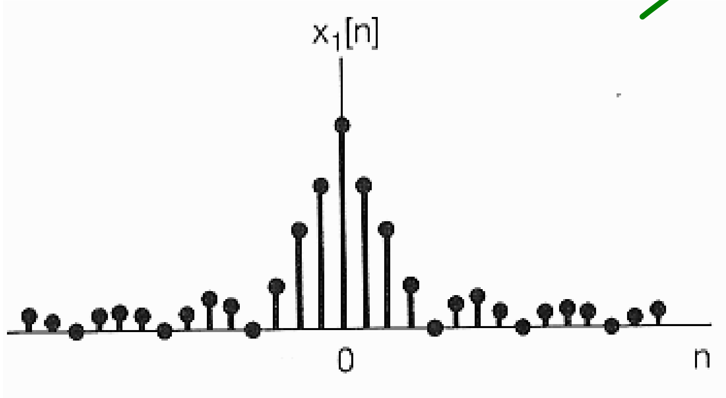
w

2π

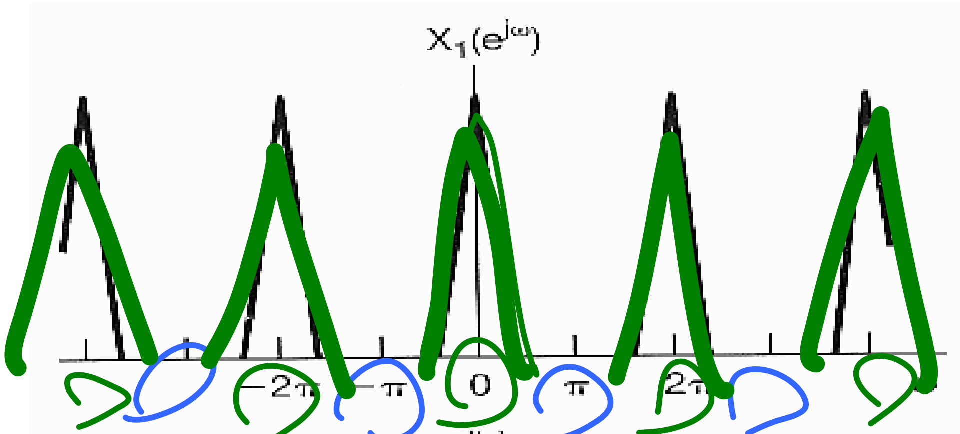


$-\pi$ π

■ High-Frequency & Low-Frequency Signals:

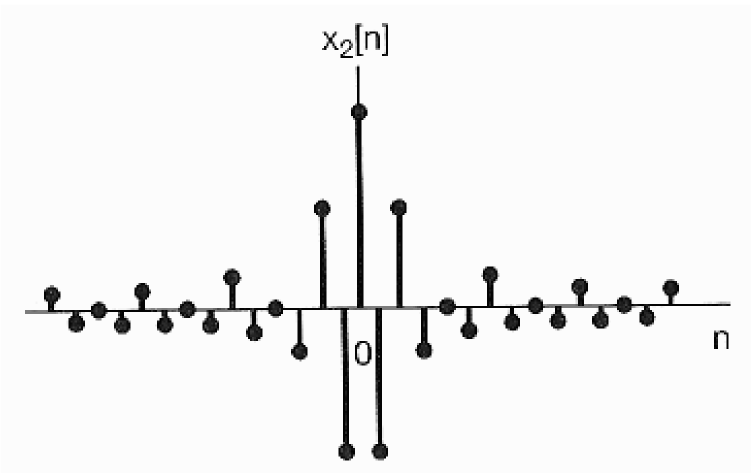


(a)

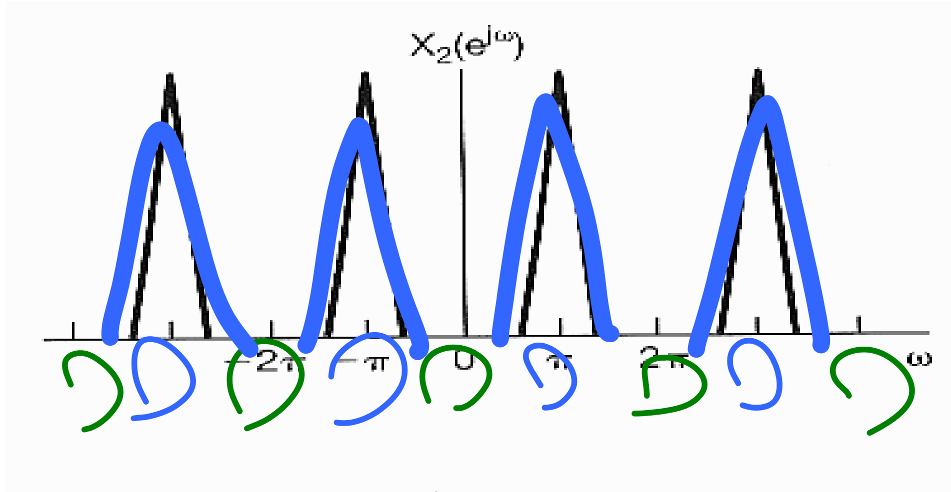


(b)

low-frequency signal



(c)



(d)

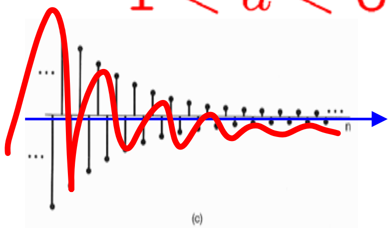
high-frequency signal

- **Example 5.1:**

$$x[n] = a^n u[n], \quad |a| < 1$$

$0 < a < 1$

$-1 < a < 0$

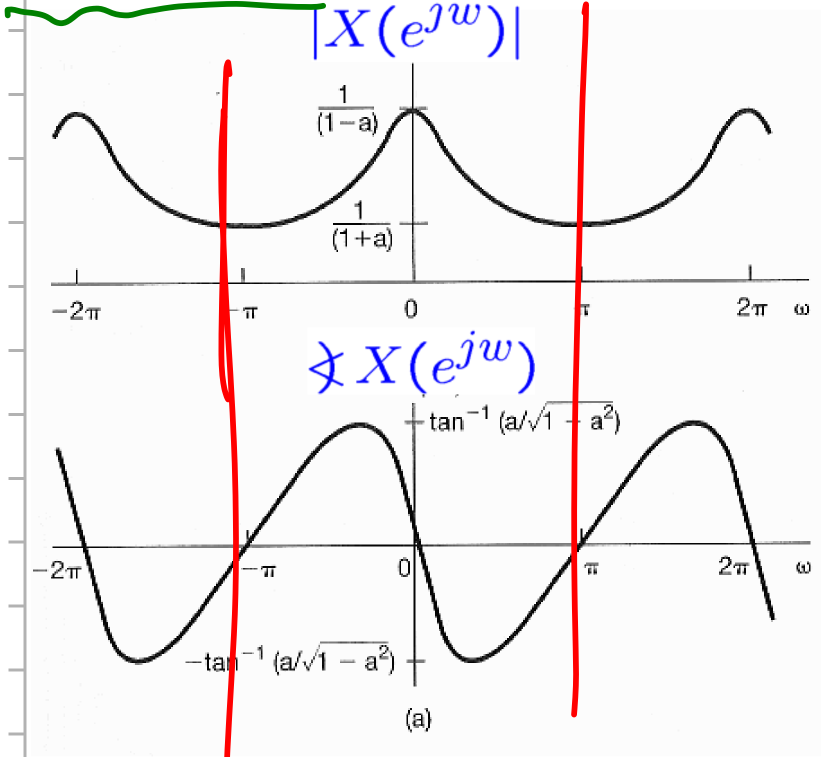


$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-jwn} = \sum_{n=0}^{\infty} \left(ae^{-jw}\right)^n = \frac{1}{1 - ae^{-jw}}$$

$0 < a < 1$

$|X(e^{jw})|$

$\measuredangle X(e^{jw})$

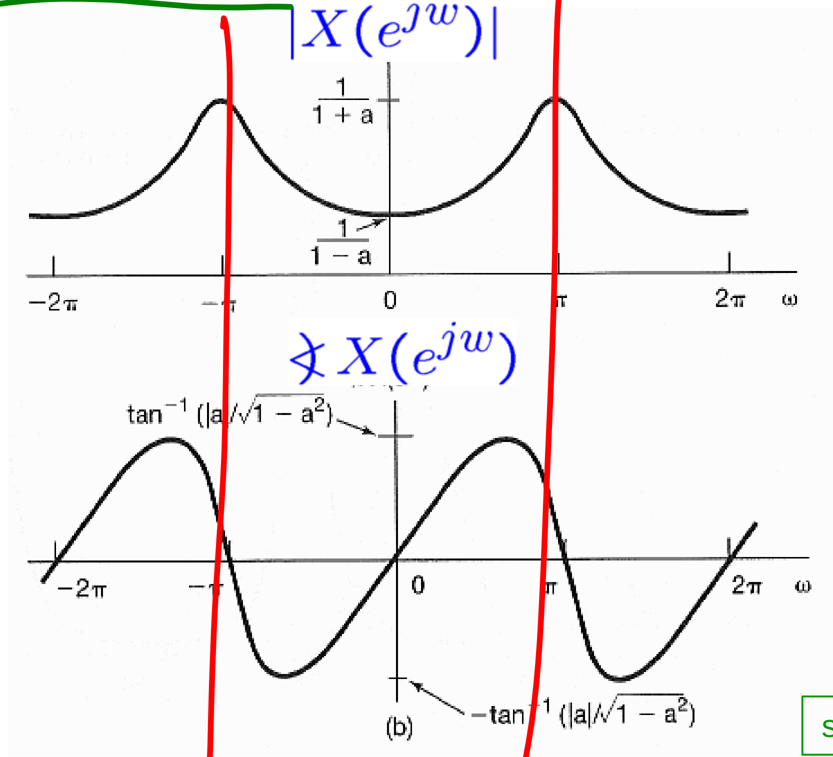


(a)

$-1 < a < 0$

$|X(e^{jw})|$

$\measuredangle X(e^{jw})$



(b)

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

- **Example 5.2:** $x[n] = a^{|n|}, \quad 0 < a < 1$

$-1 < a < 0$

$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} a^{n} e^{-jwn} + \sum_{n=-\infty}^{-1} a^{-n} e^{-jwn}$$

$$= \sum_{n=0}^{+\infty} (ae^{-jw})^{n} + \sum_{m=1}^{\infty} (ae^{jw})^{m}$$

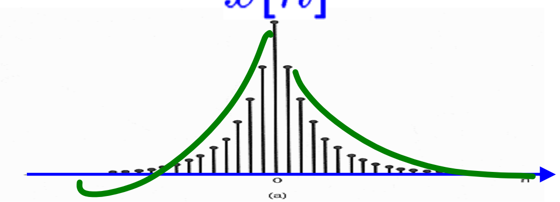
$$= \frac{1}{1 - ae^{-jw}} + \frac{ae^{jw}}{1 - ae^{jw}}$$

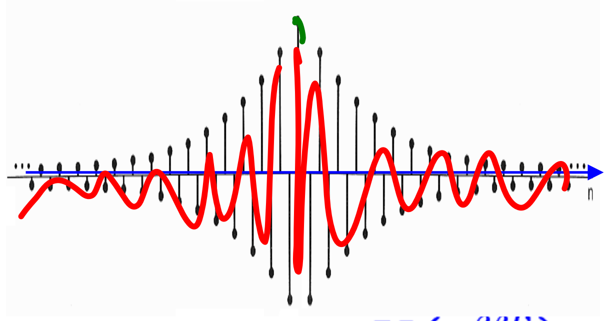
$$= \frac{1 - a^2}{1 - 2a\cos w + a^2}$$

$$= \frac{1-a^2}{(1-a)^2} \quad w=0$$

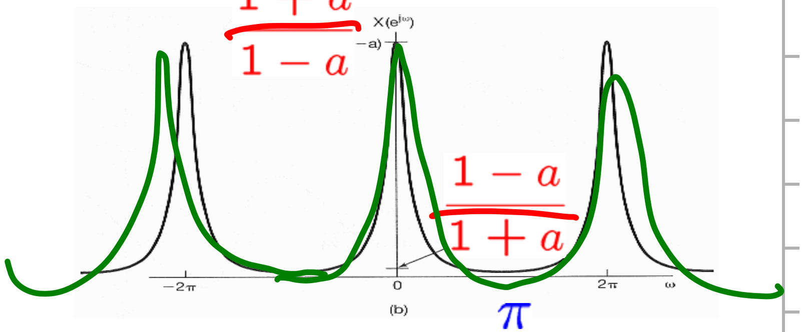
$$= \frac{1-a^2}{(1+a)^2} \quad w=\pi$$

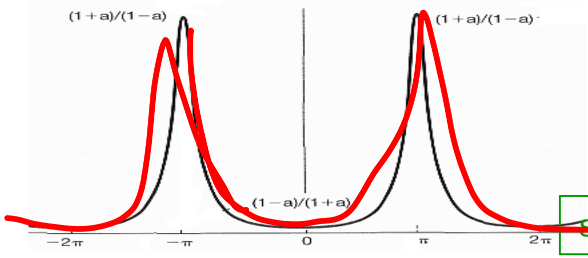
$x[n]$





$X(e^{jw})$





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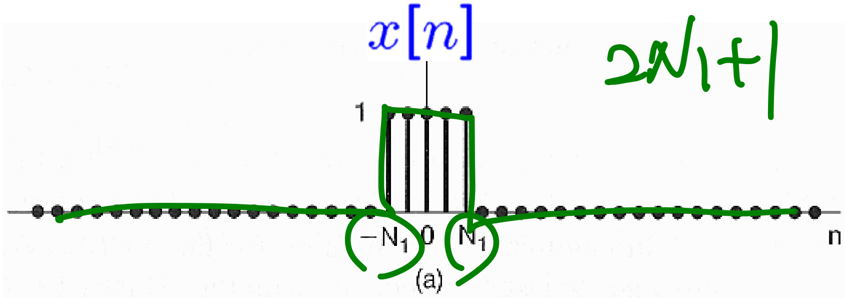
$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

Example 5.3:

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$



$2N_1+1$

$$\Rightarrow X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn} \quad (e^{jw})^n$$

$$= e^{-jw(-N_1)} + \cdots + e^{-jw(N_1)} \quad 2N_1+1$$

$$= e^{-jw(-N_1)} \left(\frac{1-(e^{-jw})^{2N_1+1}}{1-(e^{-jw})} \right)$$

$$1 - e^{-j\theta}$$
$$= e^{-j\theta/2} e^{j\theta/2} - e^{-j\theta/2} e^{-j\theta/2}$$
$$= e^{-j\theta/2} \left(e^{j\theta/2} - e^{-j\theta/2} \right)$$

$$= \cancel{e^{jw(N_1)}} \left(\frac{\cancel{(e^{-jw})^{N_1+1/2}} \left((e^{jw})^{N_1+1/2} - (e^{-jw})^{N_1+1/2} \right)}{\cancel{(e^{-jw/2})} \left((e^{jw/2}) - (e^{-jw/2}) \right)} \right) \quad \frac{2j}{2j}$$

$$= \frac{\sin\left(w(N_1 + \frac{1}{2})\right)}{\sin(w/2)}$$

- Example 5.3:

$$\Rightarrow \quad X(e^{jw}) = \sum_{n=-N_1}^{N_1} e^{-jwn}$$

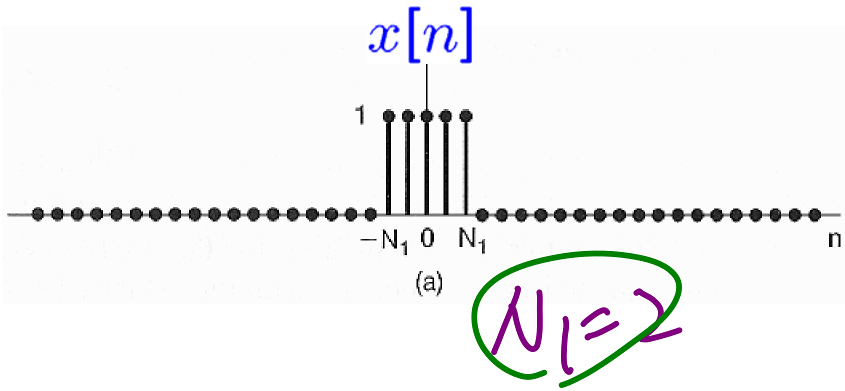
$$= \frac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$$

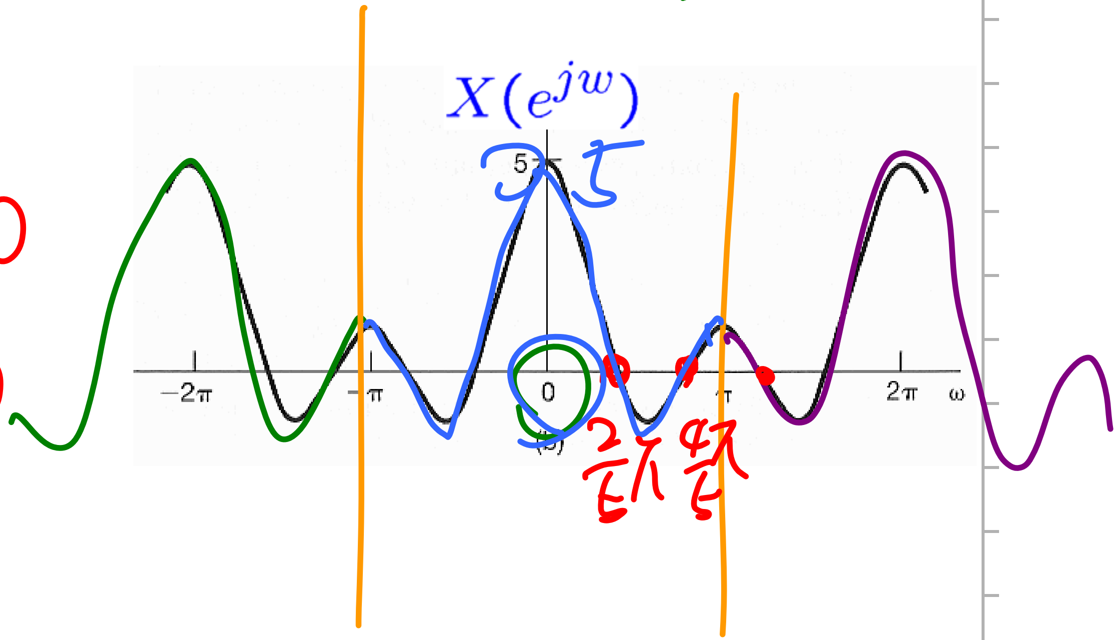
$N_1 = 2$

$$= \frac{\sin\left(w(\frac{5}{2})\right)}{\sin(w/2)} \begin{matrix} = 0 \\ \neq 0 \end{matrix}$$

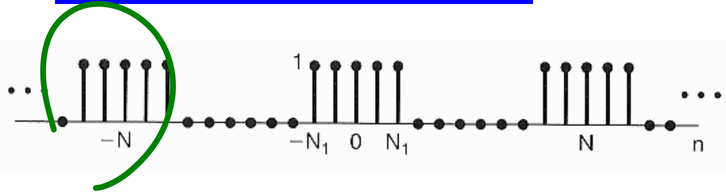
$w = 0 \Rightarrow \frac{0}{0} \to \frac{\frac{5}{2}}{\frac{1}{2}} = 5$

$w\frac{5}{2} = \pi, 2\pi, 3\pi \Rightarrow w = \frac{2\pi}{5}\ \frac{4\pi}{5}\ \frac{6\pi}{5}$



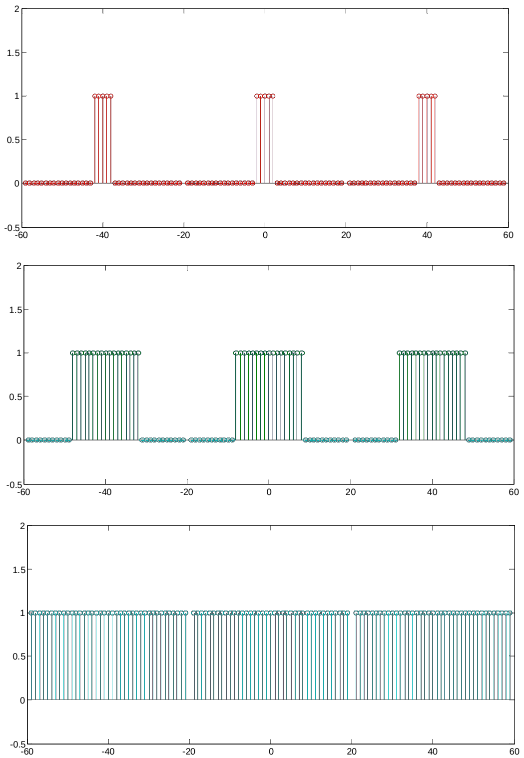


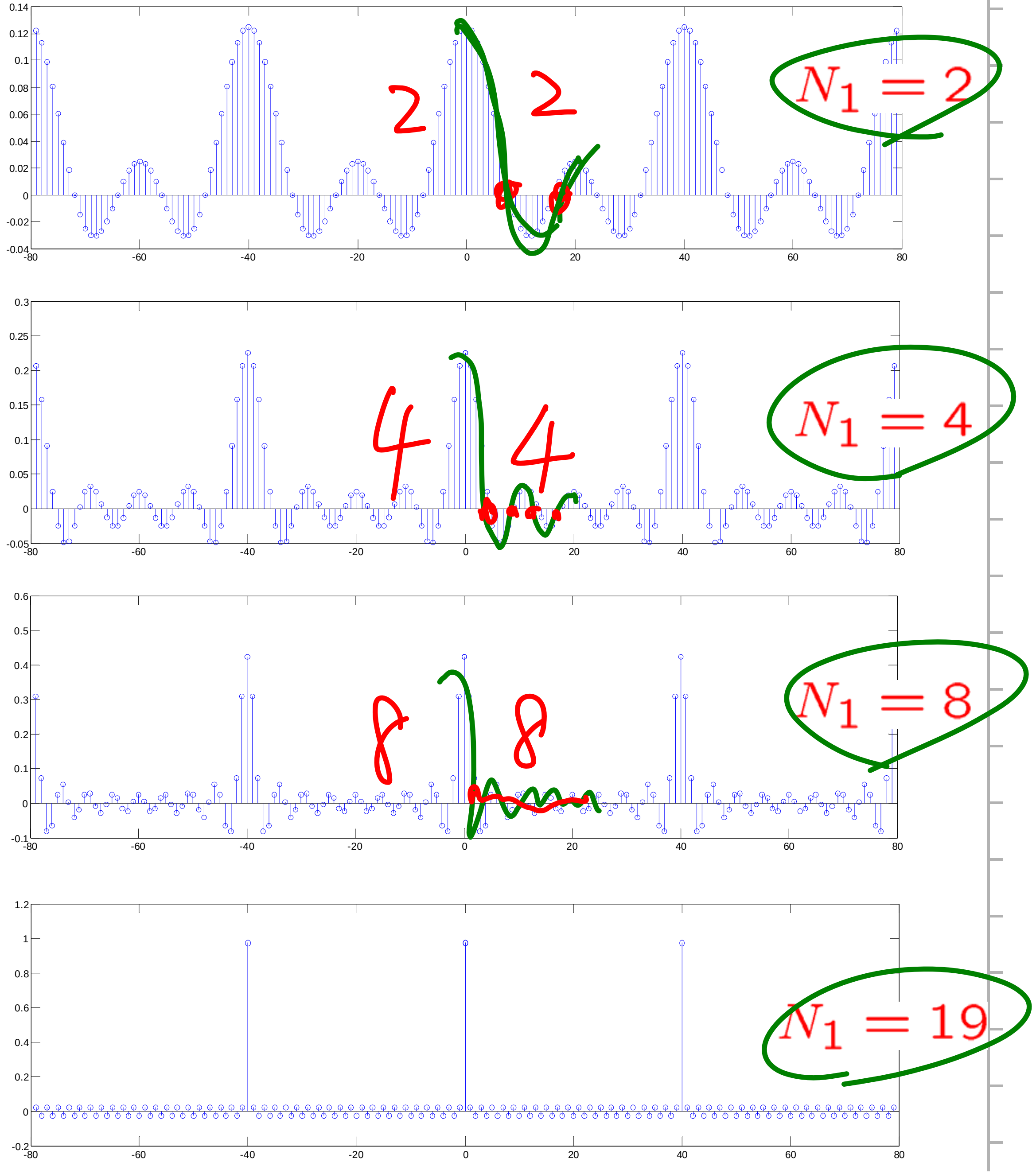
Example 3.12:



- $N = 40$

$$a_k = \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k (N_1 + \frac{1}{2})\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$





- Convergence of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

- The analysis equation will converge:
  - Either if x[n] is absolutely summable, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

  - Or, if x[n] has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- Example 5.4:

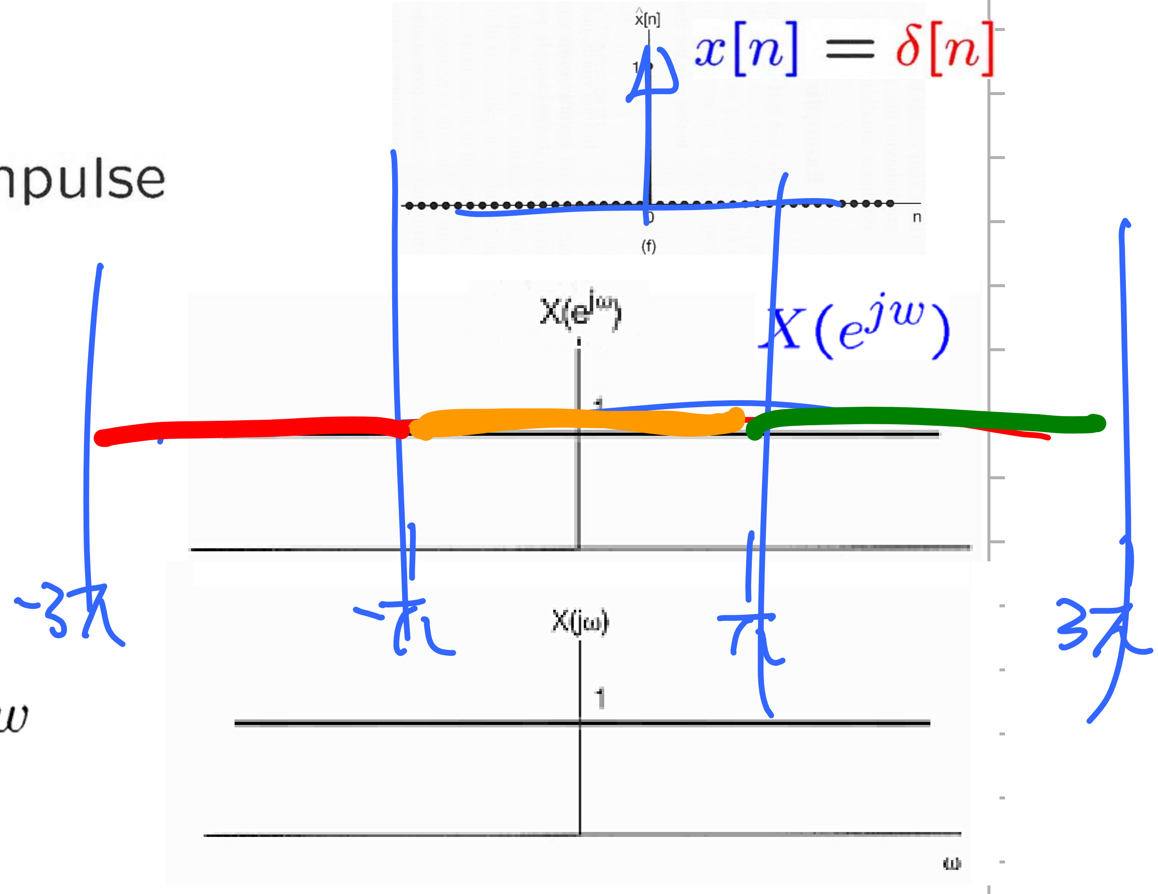
$$x[n] = \delta[n], \quad \text{i.e., unit impulse}$$

$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} \delta[n] e^{-jwn} = 1$$

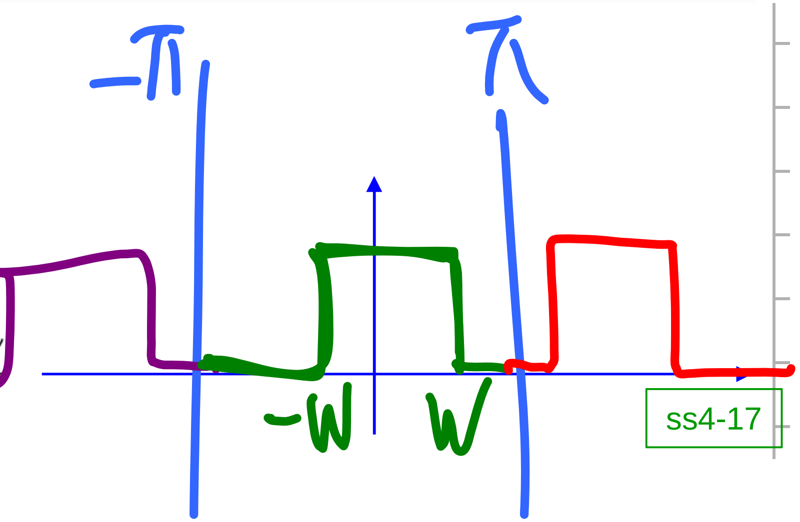
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$= \frac{1}{2\pi} \int_{2\pi} e^{jwn} dw$$



- Approximation

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^{+W} X(e^{jw}) e^{jwn} dw = \frac{\sin Wn}{\pi n}$$



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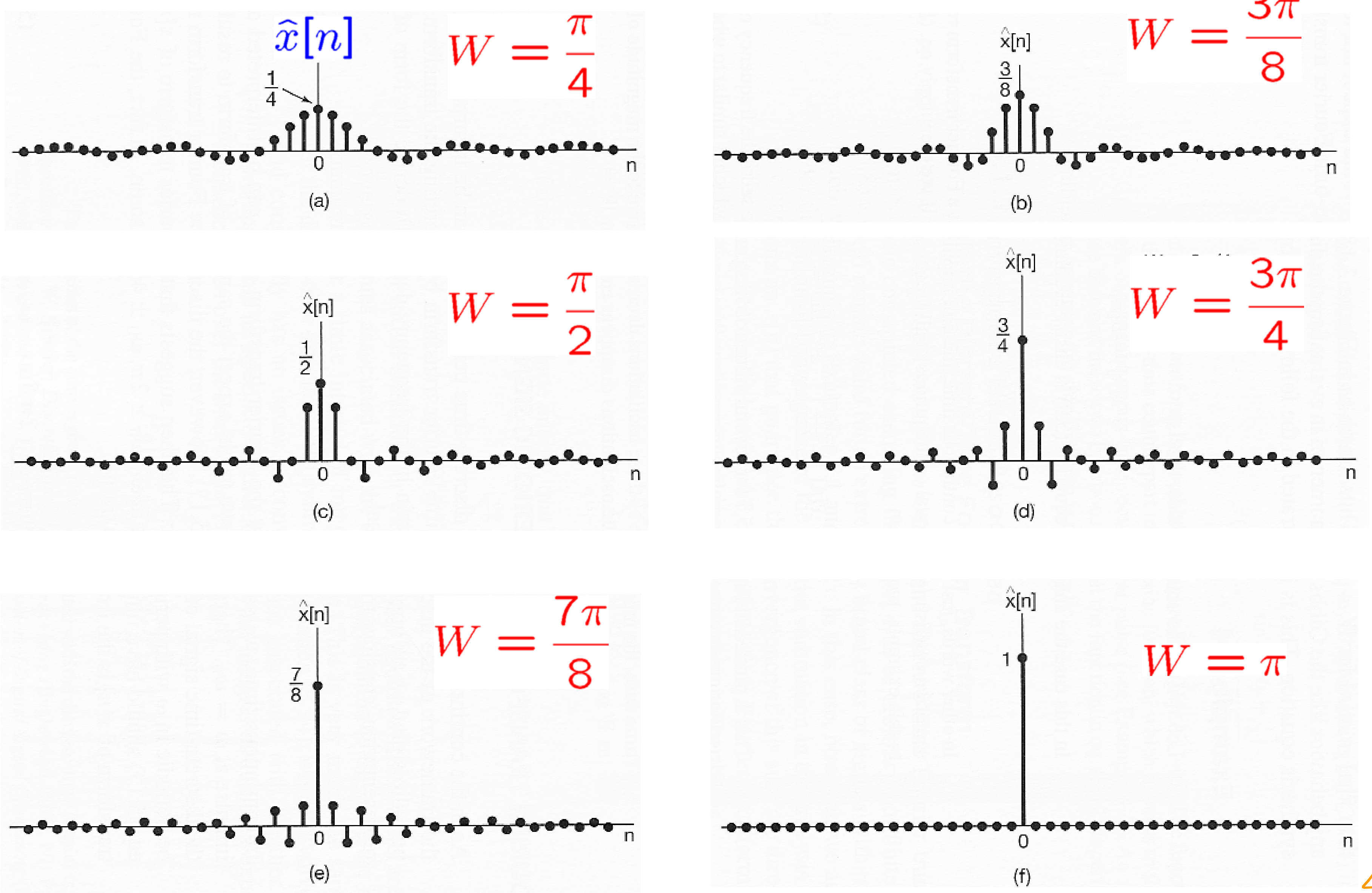
▪ Approximation of an Aperiodic Signal:

$$\hat{x}[n] = \frac{\sin Wn}{\pi n}$$



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- Representation of Aperiodic Signals:
  the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals ss4-21
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
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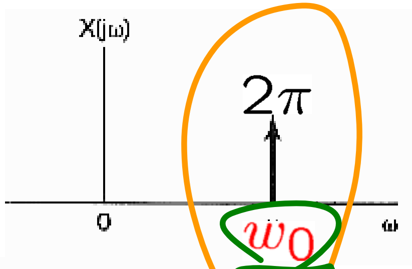
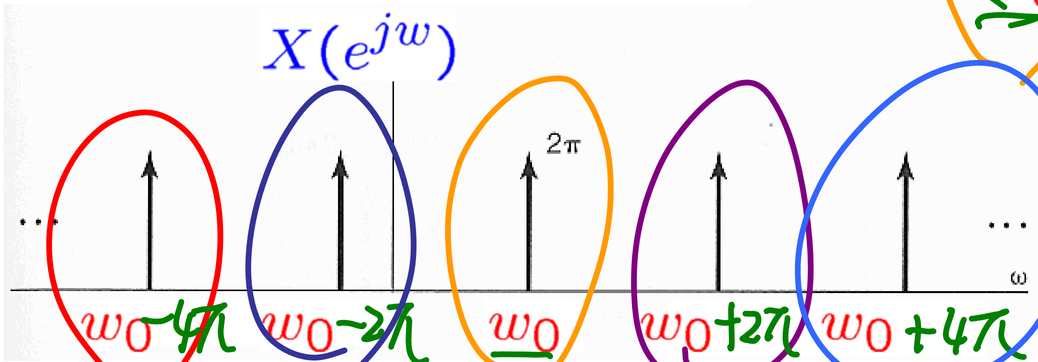
ss4-22

▪ Fourier Transform from Fourier Series:

$$x(t) = e^{jw_0 t} \quad \xleftrightarrow{\mathcal{CTFT}} \quad X(jw) = \boxed{2\pi\delta(w - w_0)}$$



$$x[n] = e^{jw_0 n} \quad \xleftrightarrow{\mathcal{DTFT}}$$



$$X(e^{jw}) = \cdots + 2\pi\delta(w - w_0 + 2\pi) + 2\pi\delta(w - w_0) + 2\pi\delta(w - w_0 - 2\pi) + \cdots$$

$$= \sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)$$

$$\boxed{w_0 = \frac{2\pi}{N}}$$

$$\frac{1}{2\pi}\int_{2\pi} X(e^{jw}) e^{jwn}\, dw = \frac{1}{2\pi}\int_{2\pi} \boxed{\sum_{l=-\infty}^{+\infty} 2\pi\delta(w - w_0 - 2\pi l)\, e^{jwn}}\, dw$$

$$= e^{j(w_0 + 2\pi l)n} = e^{jw_0 n}$$

$w_0 = \frac{2\pi}{N}$

Fourier Transform from Fourier Series:

- more generally,

$$x[n] = \sum_{k=<N>} a_k \, e^{jk(w_0)n} = \sum_{k=<N>} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - kw_0\right) = \sum_{k=-\infty}^{+\infty} 2\pi \, a_k \, \delta\left(w - k\frac{2\pi}{N}\right)$$

N x ∞

- If $k = 0, 1, ..., N-1$

N

$$x[n] = a_0 + a_1 \, e^{j\,1\left(\frac{2\pi}{N}\right)n} + a_2 \, e^{j\,2\left(\frac{2\pi}{N}\right)n} + \cdots + a_{N-1} \, e^{j\,(N-1)\left(\frac{2\pi}{N}\right)n}$$

$$= x_0 + x_1 + x_2 + \cdots + x_{N-1}$$

N

a linear combination of signals with $0, \frac{2\pi}{N}, \frac{2\cdot 2\pi}{N}, \cdots, \frac{(N-1)\cdot 2\pi}{N}$

$0, w_0, 2w_0, \cdots, (N-1)w_0$

$\boxed{\omega_0 = \frac{2\pi}{N}}$

- Fourier Transform from Fourier Series:

$0\omega_0$: $x_0 \overset{\mathcal{F}}{\longleftrightarrow}$

$+$

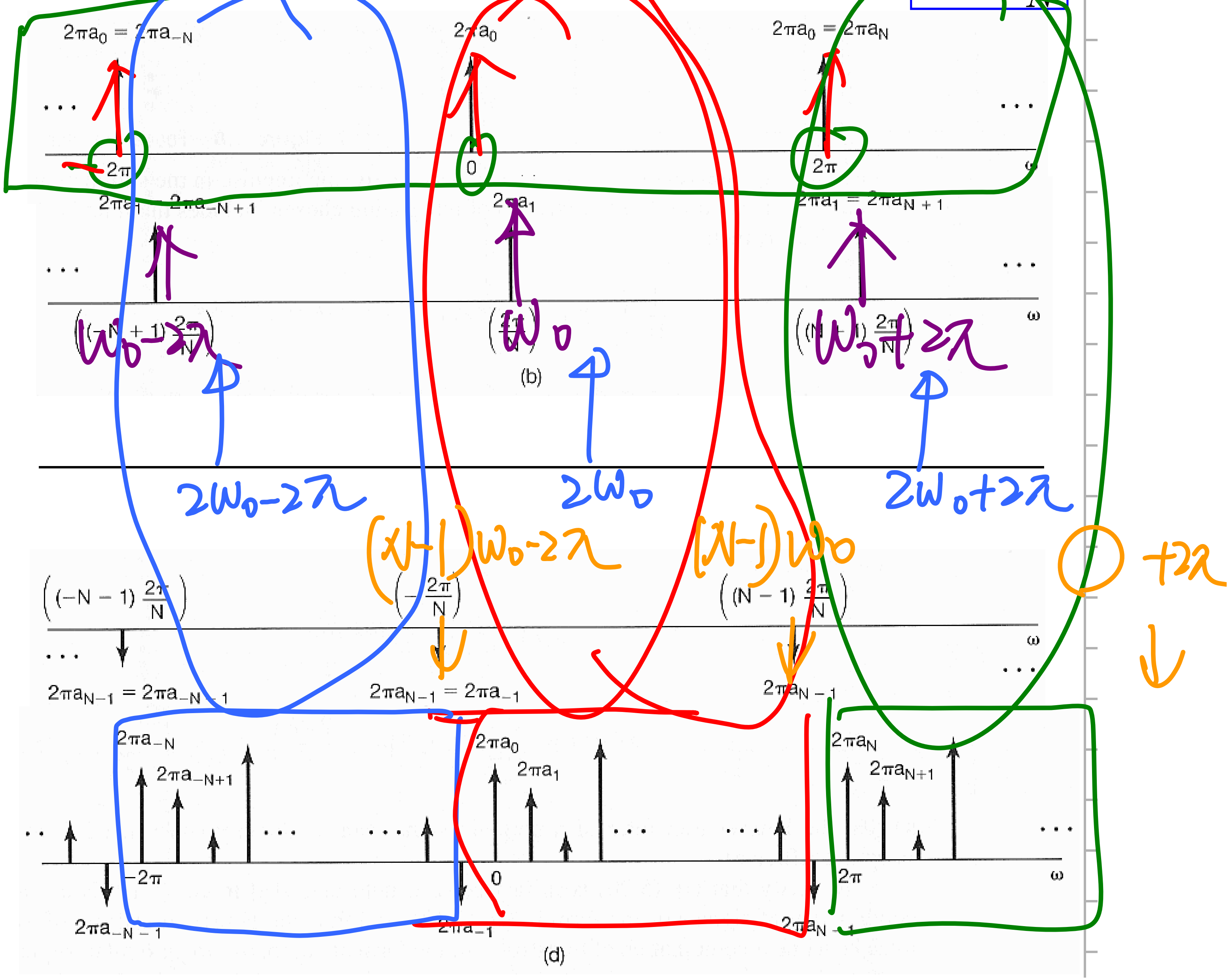
$1\omega_0$: $x_1 \overset{\mathcal{F}}{\longleftrightarrow}$

$2\omega_0$

$(N-1)\omega_0$: $x_{N-1} \overset{\mathcal{F}}{\longleftrightarrow}$

$\parallel$

$\underline{\underline{X(e^{jw})}} \quad =$

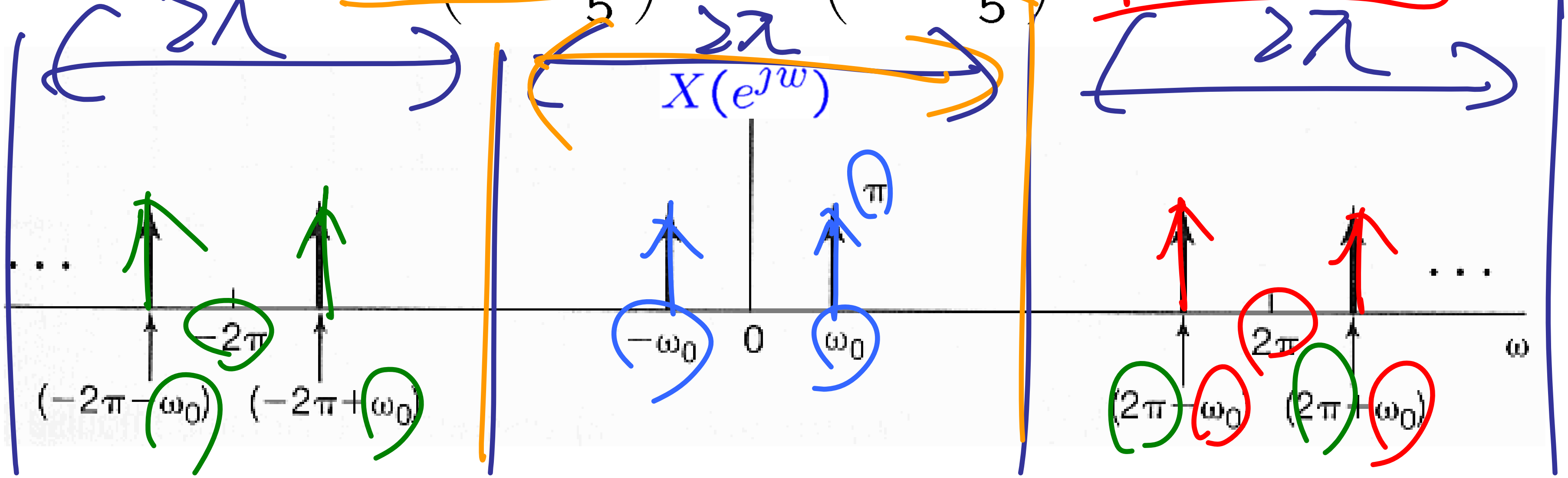


- Example 5.5:

$$x[n] = \cos(w_0 n) = \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} \qquad \text{with } \boxed{w_0 = \frac{2\pi}{5}}$$

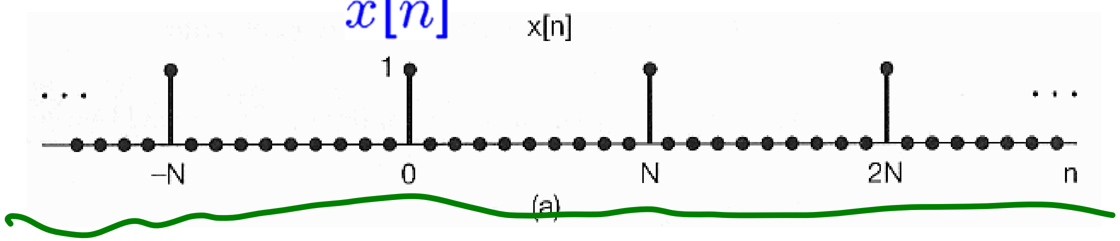
$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\,\delta\left(w + \frac{2\pi}{5} - 2\pi l\right)$$

$$= \pi\,\delta\left(w - \frac{2\pi}{5}\right) + \pi\,\delta\left(w + \frac{2\pi}{5}\right), \quad \boxed{-\pi \le w < \pi}$$



- Example 5.6:

$x[n]$



$$x[n] = \sum_{k=<N>} a_k \, e^{jk(2\pi/N)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(w - \frac{2\pi k}{N}\right)$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

ss4-25

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

choose $0 \le n \le N-1$

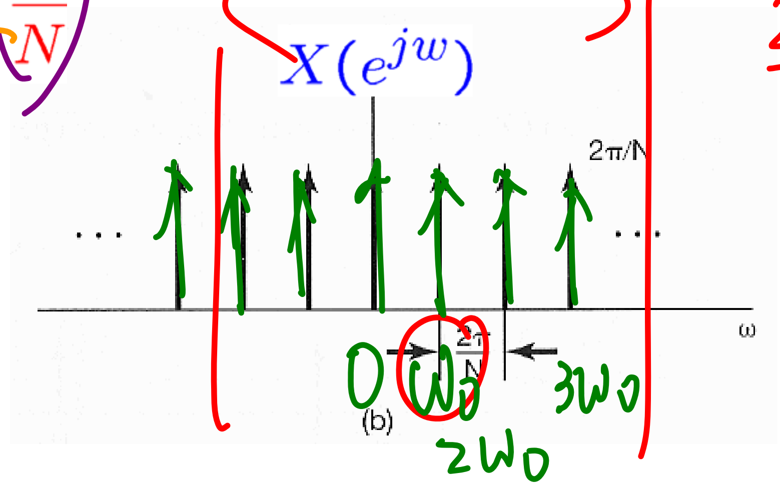
$$\Rightarrow \ a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$\Rightarrow \ X(e^{jw}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(w - k\frac{2\pi}{N}\right)$$

w_0

$N \leftrightarrow 2\pi$

$\frac{2\pi}{N} = w_0$



- Representation of Aperiodic Signals:
  the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform ss4-26
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by
  Linear Constant-Coefficient Difference Equations

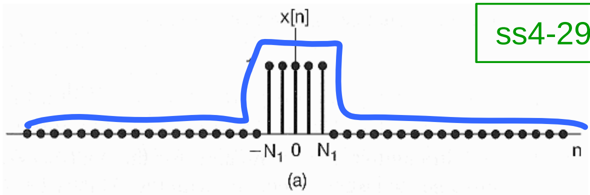
Outline

Section	Property
5.3.2	Linearity
5.3.3	Time Shifting
5.3.3	Frequency Shifting
5.3.4	Conjugation
5.3.6	Time Reversal
5.3.7	Time Expansion
5.4	Convolution
5.5	Multiplication
5.3.5	Differencing in Time
5.3.5	Accumulation
5.3.8	Differentiation in Frequency
5.3.4	Conjugate Symmetry for Real Signals
5.3.4	Symmetry for Real and Even Signals
5.3.4	Symmetry for Real and Odd Signals
5.3.4	Even-Odd Decomposition for Real Signals
5.3.9	Parseval’s Relation for Aperiodic Signals

Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

ss4-29



(a)

▪ Fourier Transform Pair:

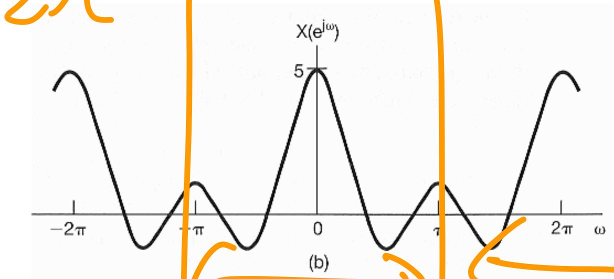
- Synthesis equation:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- Analysis equation:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

2π 2π 2π



(b)

- Notations:

$|a| < 1$

$$X(e^{jw}) = \mathcal{F}\{x[n]\} \qquad \frac{1}{1-ae^{jw}} = \mathcal{F}\{a^n u[n]\}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{jw})\} \qquad a^n u[n] = \mathcal{F}^{-1}\{\frac{1}{1-ae^{jw}}\}$$

$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad a^n u[n] \xleftrightarrow{\mathcal{DTFT}} \frac{1}{1-ae^{jw}}$$

Periodicity of DT Fourier Transform:

ss5-6

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$X(e^{j(w+2\pi)}) = X(e^{jw})$$

Linearity:

ss4-30

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{jw})$$

$$\Rightarrow a\,x[n] + b\,y[n] \stackrel{\mathcal{F}}{\longleftrightarrow} a\,X(e^{jw}) + b\,Y(e^{jw})$$

$w_0 \quad n_0$

Time & Frequency Shifting:

ss4-31

$$\Rightarrow x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-jwn_0}X(e^{jw})$$

$$\Rightarrow e^{jw_0n}x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(w-w_0)})$$

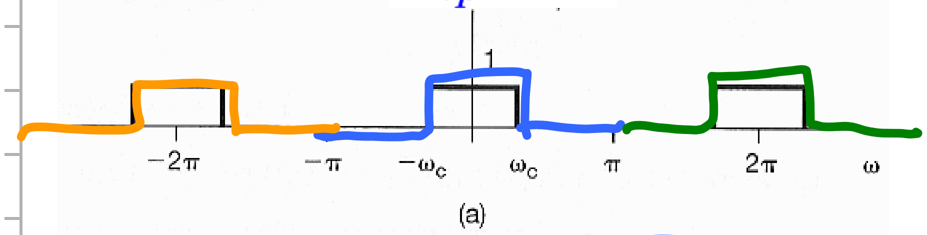
e^{jwn}

ss4-19

ss5-14

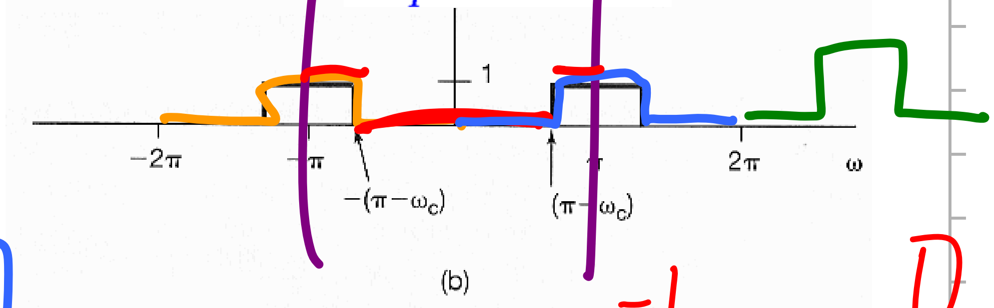
- Example 5.7:

$H_{lp}(e^{jw})$



(a)

$H_{lp}(e^{j(w-\pi)})$



(b)

$$H_{hp}(e^{jw}) = \boxed{H_{lp}(e^{j(w-\pi)})}$$

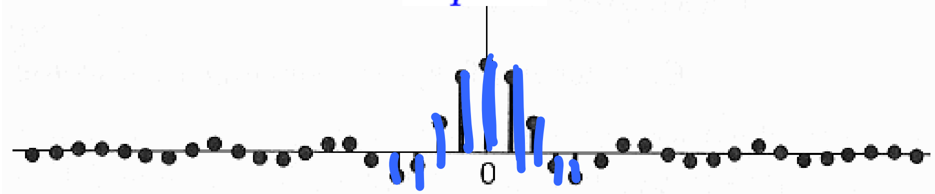
$$\Rightarrow \quad h_{hp}[n] = e^{j\pi n} \, h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

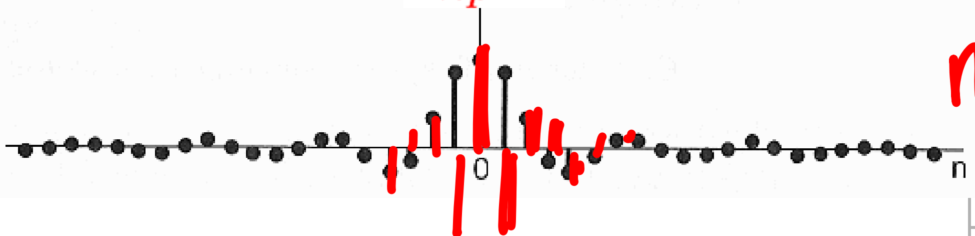
-1 0

$$= (-1)^n h_{lp}[n]$$

$h_{lp}[n]$



$h_{hp}[n]$



n

Conjugation & Conjugate Symmetry:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

ss4-34

$$x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{-jw})$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

- $x[n] = x^*[n] \quad \Rightarrow \quad X(e^{-jw}) = X^*(e^{jw})$

ss4-35

IF $x[n]$ is real $\quad \Rightarrow \quad X(e^{jw})$ is conjugate symmetric

- Conjugation & Conjugate Symmetry:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) \qquad x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-jw})$$

ss4-34

- IF $x[n] = x^*[n]$ & $x[-n] = x[n]$

$\Rightarrow X(e^{-jw}) = X^*(e^{jw})$ & $X(e^{-jw}) = X(e^{jw})$

$\Rightarrow X^*(e^{jw}) = X(e^{jw})$

- IF $x[n]$ is real & even $\Rightarrow X(e^{jw})$ are real & even

ss4-36

- IF $x[n]$ is real & odd $\Rightarrow X(e^{jw})$ are purely imaginary & odd

- IF $x[n] = x^*[n]$ & $x[-n] = -x[n]$

$\Rightarrow X(e^{-jw}) = X^*(e^{jw})$ & $X(e^{-jw}) =$ - $X(e^{jw})$

$\Rightarrow X^*(e^{jw}) = -X(e^{jw})$

- ## Conjugation & Conjugate Symmetry:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

ss4-37

$$\mathcal{E}v\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{jw})\}$$

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\{X(e^{jw})\}$$

4/8/13
10:10 am

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

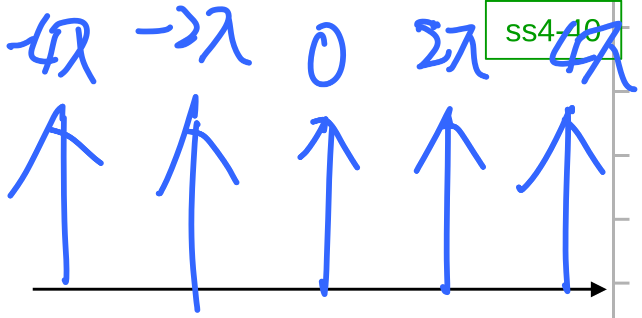
▪ Differencing & Accumulation:

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jw}) - e^{-jw}X(e^{jw})$$

$$= \left(1 - e^{-jw}\right)X(e^{jw})$$

ss4-40



$x[n-1]$ $x[n]$

$$y[n] = \sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-jw}}X(e^{jw}) + \pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(w-2\pi k)$$

dc or average value

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

$$y[n-1] = \sum_{m=-\infty}^{n-1} x[m]$$

$$\Rightarrow\ y[n] - y[n-1] = x[n]$$

$$\Rightarrow\ \left(1 - e^{-jw}\right)Y(e^{jw}) = X(e^{jw})$$

Differentiation in Frequency:

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$\frac{1}{j} n x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{d}{dw} X(e^{jw})$$

$$n x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j \frac{d}{dw} X(e^{jw})$$

$$j \frac{d}{dw} X(e^{jw}) = \frac{d}{dw} \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} (-jn) x[n] e^{-jwn}$$

$$= j (-j) \sum_{n=-\infty}^{+\infty} \left[n x[n] \right] e^{-jwn}$$

$$-j = \frac{1}{j}$$

Time Reversal:

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{-jw})$$

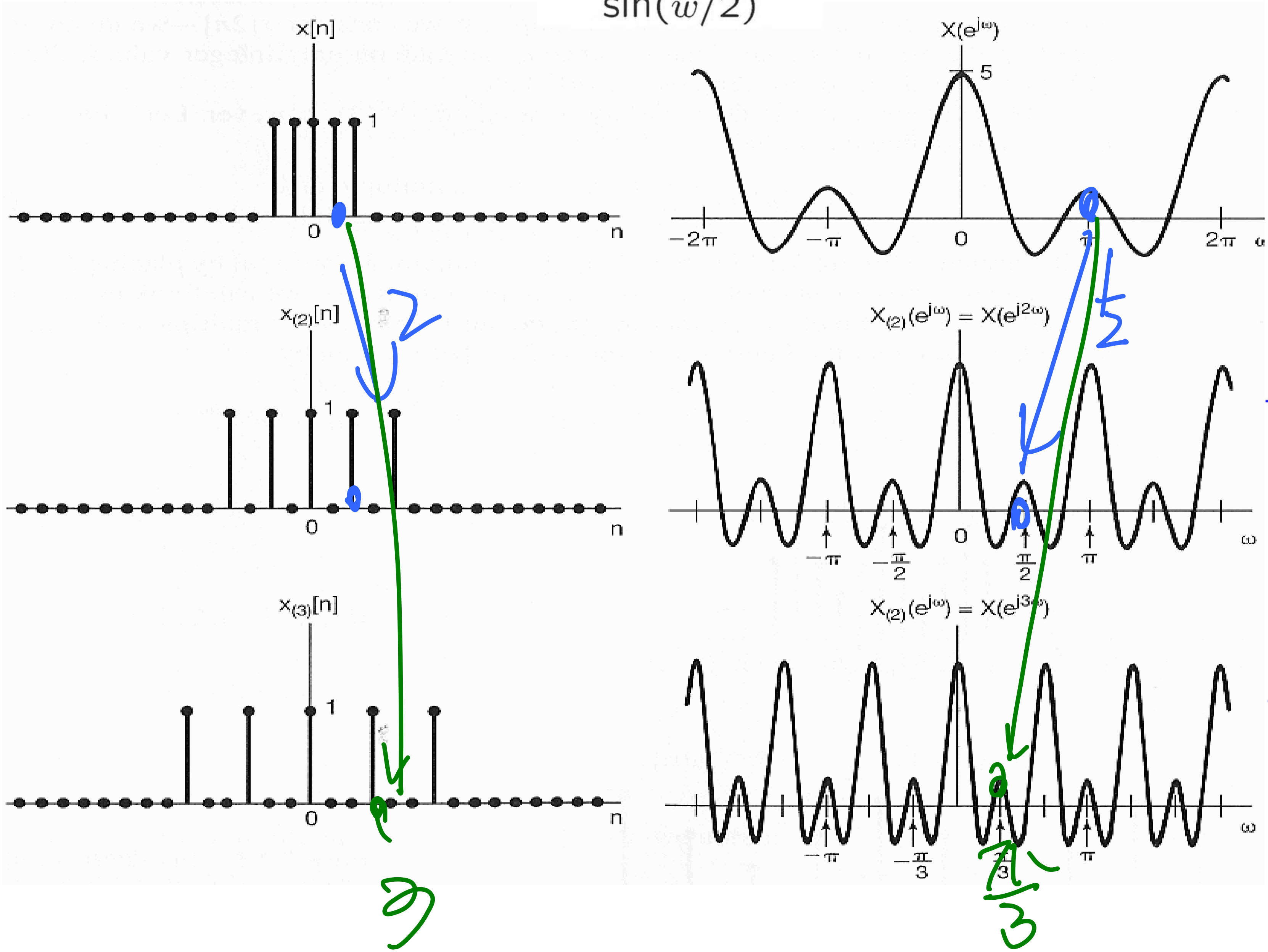
$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$X(e^{j(-w)}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(-w)n}$$

ss4-45

▪ Time Expansion:

$$\frac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$$

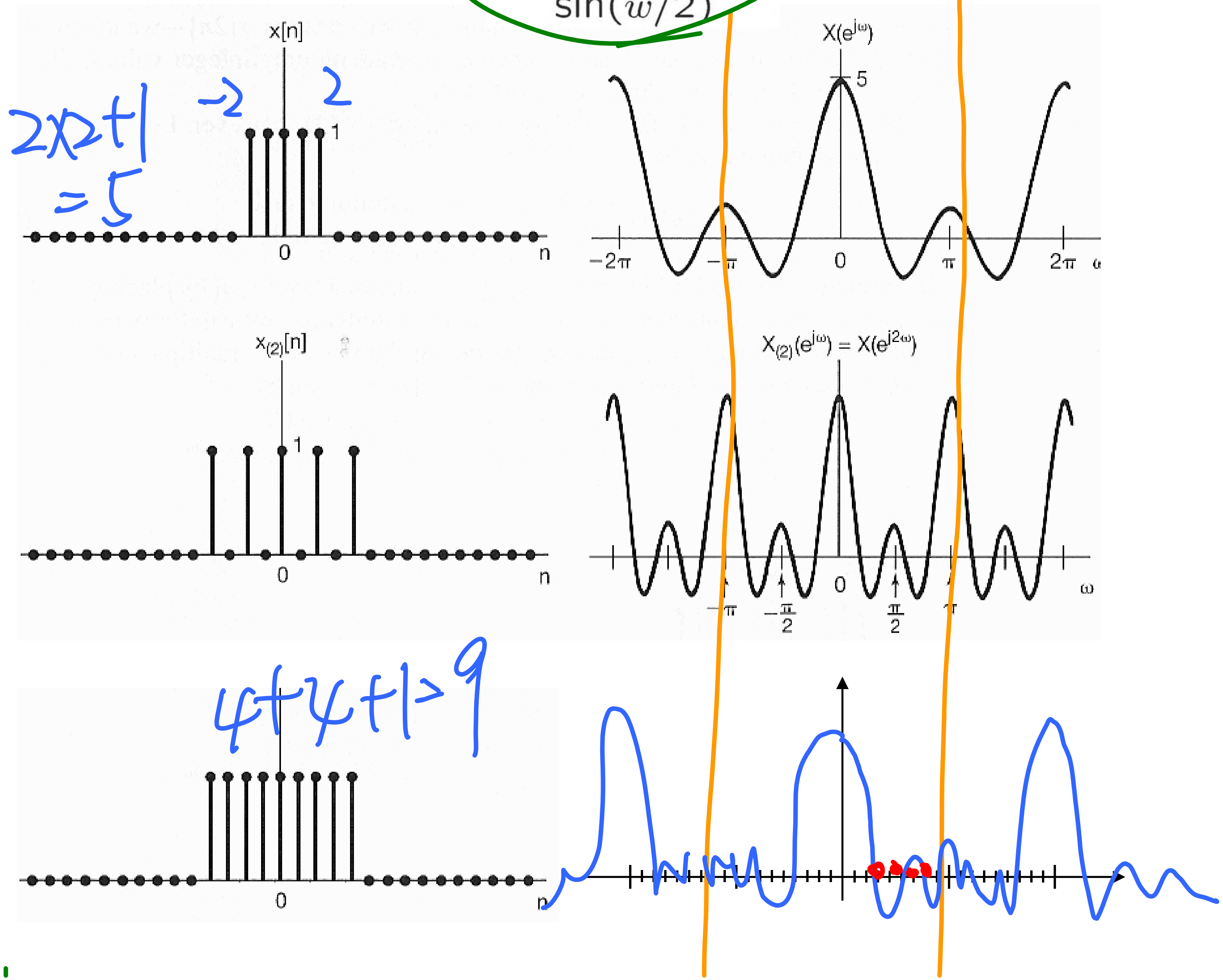


$X(e^{jw})$

$X(e^{j2w})$

$X(e^{j3w})$

- Time Expansion: $\dfrac{\sin\left(w(N_1+\frac{1}{2})\right)}{\sin(w/2)}$



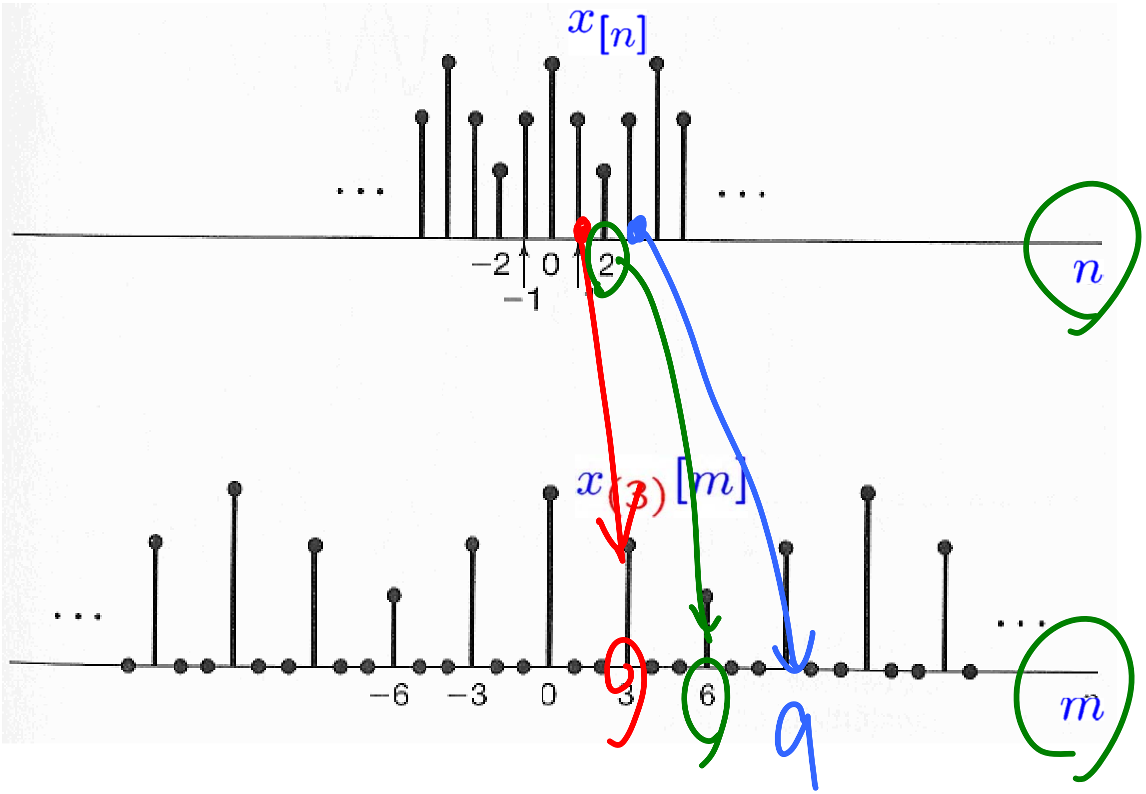
$X(e^{jw})$

$X(e^{j2w})$

- Time Expansion:

$n = \frac{m}{k}$ $\quad$ $k = 3$

$$x_{(k)}[m] = \begin{cases} x[m/k], & \text{if } m \text{ is a multiple of } k \\ 0, & \text{if } m \text{ is not a multiple of } k \end{cases}$$



$x[n]$ $\quad$ X

$x_{(k)}[m]$ $\quad$ $X_{(k)}$

- Time Expansion:

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

$$\Rightarrow X_{(k)}(e^{jw}) = \sum_{m=-\infty}^{+\infty} x_{(k)}[m] e^{-jwm}$$

$$m = nk$$

$$= \sum_{n=-\infty}^{+\infty} x_{(k)}[nk] e^{-jwnk}$$

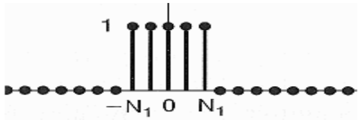
$$x_{(k)}[nk] = x[n]$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j(kw)n}$$

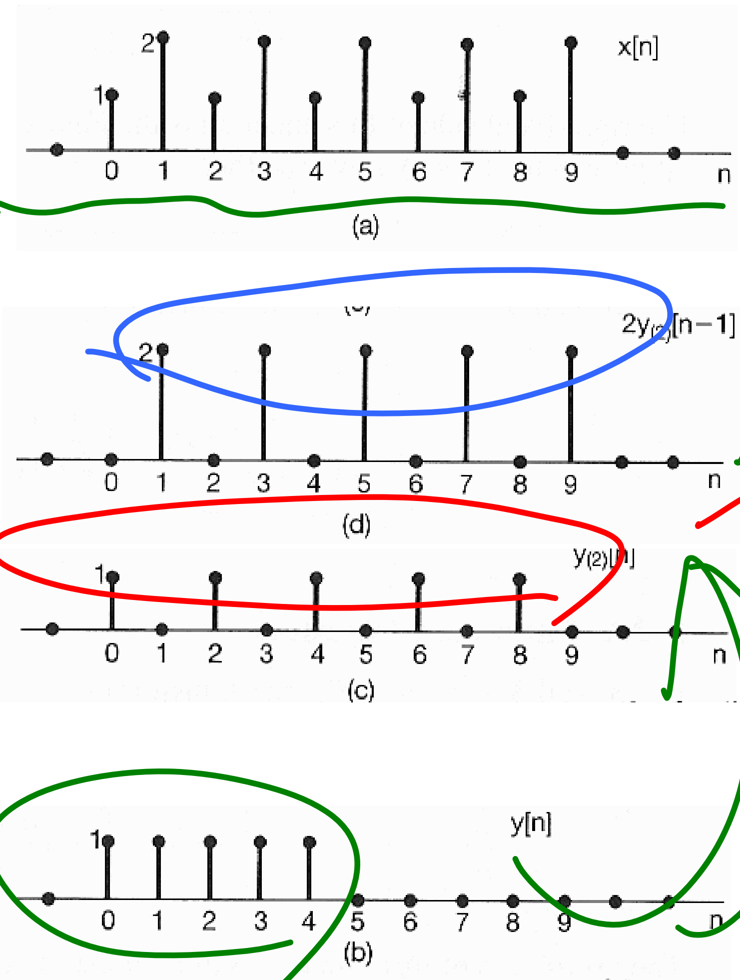
$$= X(e^{jkw})$$

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jkw})$$

- # Example 5.9:



$$\xleftrightarrow{\mathcal{F}} \quad \frac{\sin(5w/2)}{\sin(w/2)}$$



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2e^{-jw}e^{-j4w}\frac{\sin(5w)}{\sin(w)}$$

$$Y(e^{jw}) = (e^{-j2w})\frac{\sin(5w/2)}{\sin(w/2)}$$

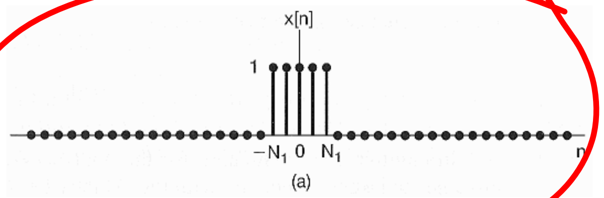
$$X(e^{jw}) = \boxed{(1 + 2e^{-jw}) \cdot e^{-j4w} \cdot \frac{\sin(5w)}{\sin(w)}}$$

ss4-50

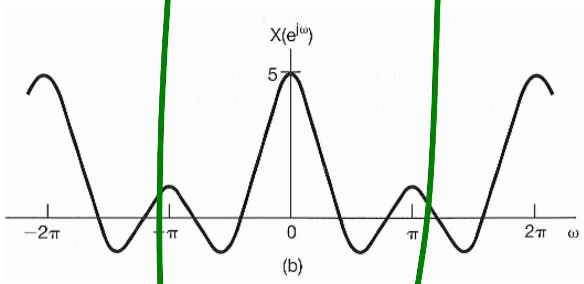
- ## Parseval's relation:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{jw})|^2 dw$$



(a)

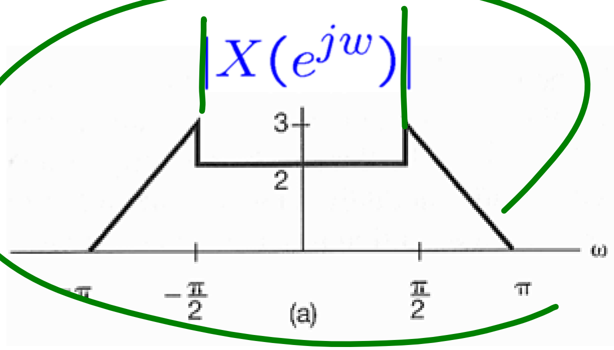


(b)

2π

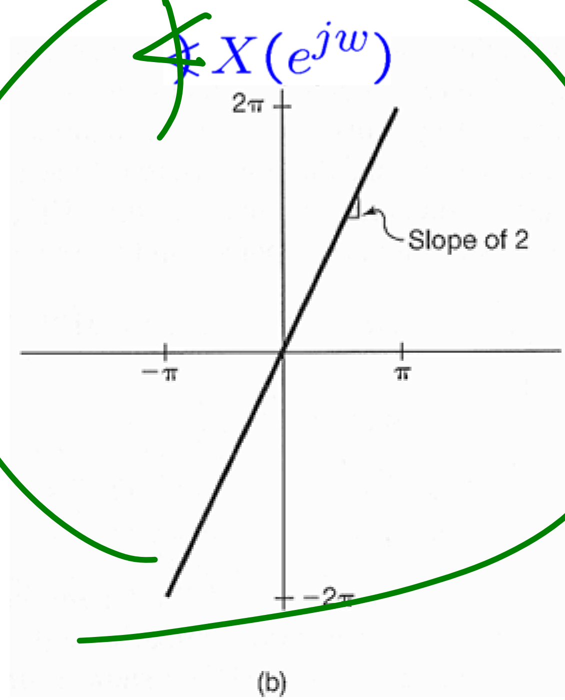
- Example 5.10:

even



(a)

odd



(b)

- $x[n]$ is periodic, real, even, and/or of finite energy?

$\rightarrow$ $X(e^{jw}) \neq 0$ $\Rightarrow$ $x[n]$ is NOT periodic

$\rightarrow$ even magnitude, odd phase $\Rightarrow$ $x[n]$ is real

$\rightarrow$ $X(e^{jw})$ is NOT real $\Rightarrow$ $x[n]$ is NOT even

$\rightarrow$ $X(e^{jw})$ is finite $\Rightarrow$ $x[n]$ is finite

Signals

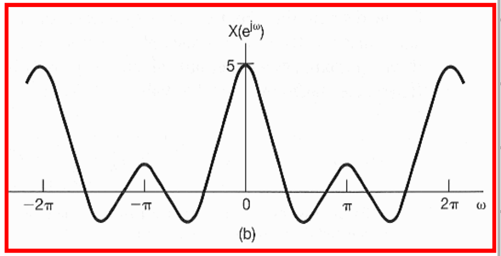
- Representation of Aperiodic Signals:
  the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform

Systems

- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by
  Linear Constant-Coefficient Difference Equations

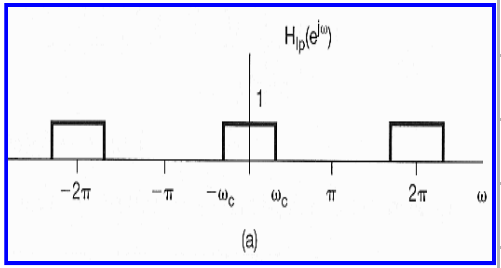
ss4-51

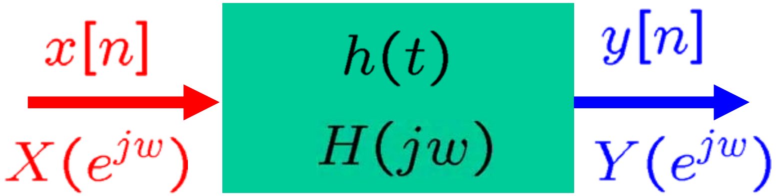
■ Convolution Property:



$$y[n] = x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

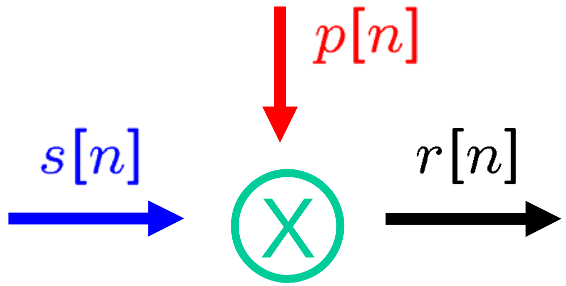
$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$





ss4-52

■ Multiplication Property:



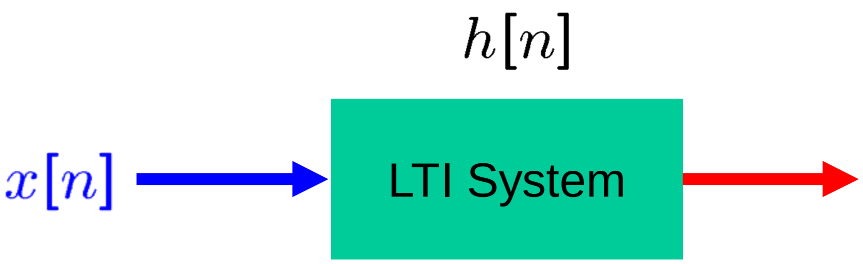
$$r[n] = s[n]p[n] \stackrel{\mathcal{F}}{\longleftrightarrow} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

ss4-57

- Example 5.11:



$$y[n] = x[n] * h[n]$$
$$= x[n] * \delta[n-n_0]$$
$$= x[n-n_0]$$

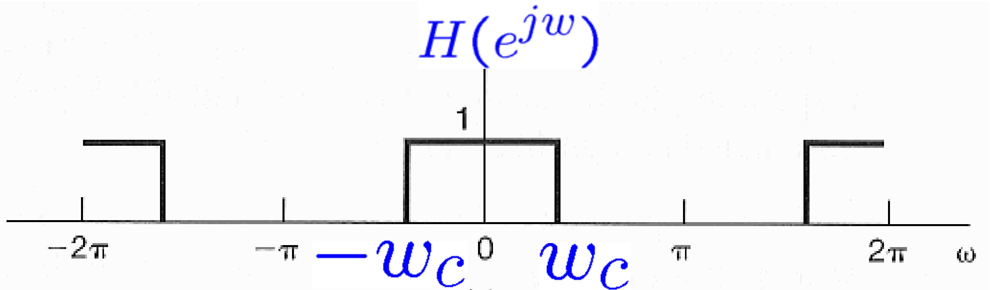
$$h[n] = \delta[n - n_0]$$

$$\Rightarrow H(e^{jw}) = \sum_{n=-\infty}^{+\infty} \delta[n-n_0]e^{-jwn} = e^{-jwn_0}$$

$$\Rightarrow Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

$$= e^{-jwn_0} X(e^{jw}) \qquad \Rightarrow y[n] = x[n - n_0]$$

- Example 5.12:

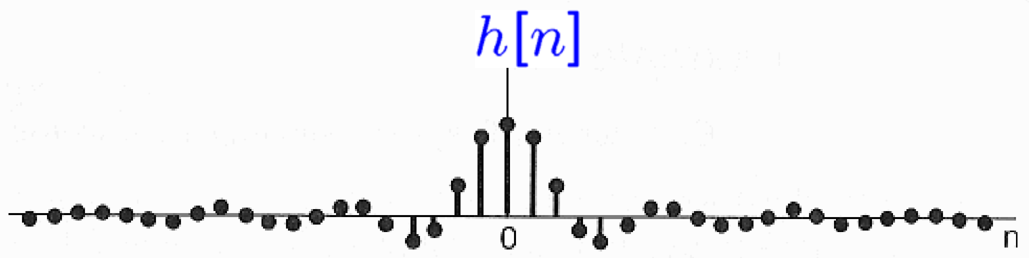


(a)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jwn} dw$$

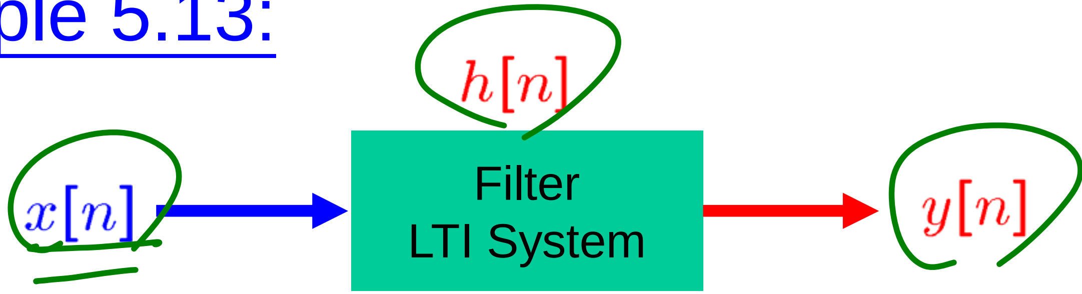
$$= \frac{\sin w_c n}{\pi n}$$



(b)

- not causal
- oscillatory

- Example 5.13:



$= x[n] * h[n]$

$$h[n] = a^n u[n], \quad |a| < 1 \qquad \Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \qquad \Rightarrow \quad X(e^{jw}) = \frac{1}{1 - be^{-jw}}$$

$$\Rightarrow \quad Y(e^{jw}) = H(e^{jw}) X(e^{jw})$$

$$= \frac{1}{1 - ae^{-jw}} \frac{1}{1 - be^{-jw}}$$

- Example 5.13:

if $a \neq b$ $\quad Y(e^{jw}) = \left[\left(\frac{a}{a-b}\right) \frac{1}{1 - ae^{-jw}} + \left(\frac{-b}{a-b}\right) \frac{1}{1 - be^{-jw}} \right]$

$$\Rightarrow y[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]$$

if $a = b$ $\quad Y(jw) = \left(\frac{1}{1 - ae^{-jw}}\right)^2 = \frac{1}{a} j\, e^{jw} \frac{d}{dw} \left(\frac{1}{1 - ae^{-jw}}\right)$

since $\quad a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-jw}}$

and $\quad n\, a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j \frac{d}{dw} \left[\frac{1}{1 - ae^{-jw}}\right]$

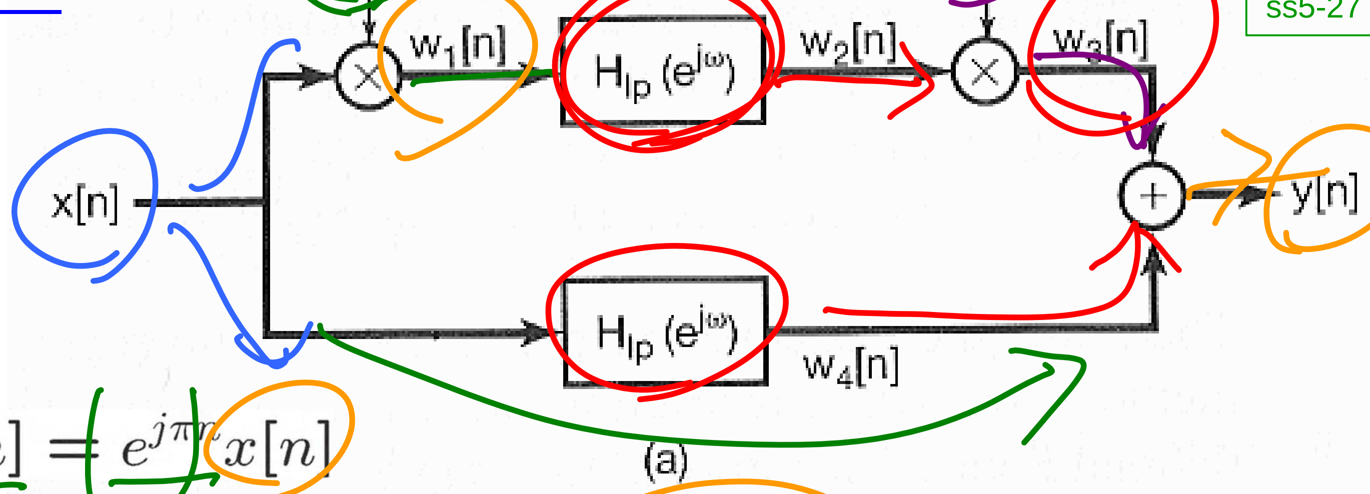
and $\quad (n+1)\, a^{n+1} u[n+1] \stackrel{\mathcal{F}}{\longleftrightarrow} j\, e^{jw} \frac{d}{dw} \left[\frac{1}{1 - ae^{-jw}}\right]$

$$\Rightarrow y[n] = (n+1) a^n u[n+1]$$

ss4-74

ss5-27

- Example 5.14:



$$w_1[n] = (-1)^n x[n] = e^{j\pi n} x[n]$$

$$\Rightarrow \quad W_1(e^{jw}) = X(e^{j(w-\pi)})$$

$$W_4(e^{jw}) = H_{lp}(e^{jw})\ X(e^{jw})$$

$$W_2(e^{jw}) = H_{lp}(e^{jw})\ W_1(e^{jw}) \quad = H_{lp}(e^{jw})\ X(e^{j(w-\pi)})$$

$$w_3[n] = (-1)^n w_2[n] = e^{j\pi n} w_2[n]$$

$$\Rightarrow \quad W_3(e^{jw}) = W_2(e^{j(w-\pi)}) = H_{lp}(e^{j(w-\pi)})\ X(e^{j(w-2\pi)})$$

$$= H_{lp}(e^{j(w-\pi)})\ X(e^{jw})$$

$-\pi$

- Example 5.14:

$$Y(e^{jw}) = W_3(e^{jw}) + W_4(e^{jw})$$

$$= H_{lp}(e^{j(w-\pi)})\, X(e^{jw}) + H_{lp}(e^{jw})\, X(e^{jw})$$

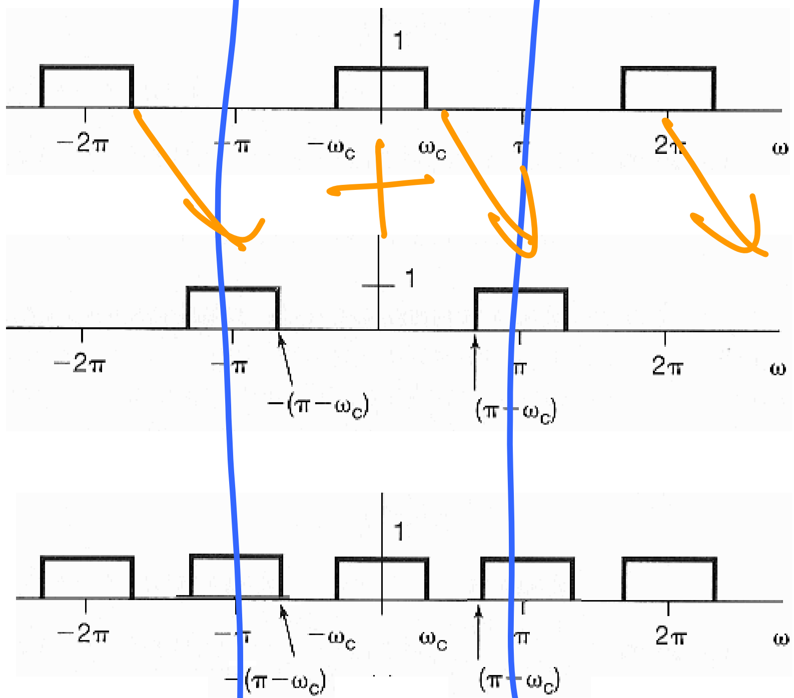
$$= \left[H_{lp}(e^{j(w-\pi)}) + H_{lp}(e^{jw})\right] X(e^{jw})$$

Hep

$$H(e^{jw}) = H_{lp}(e^{j(w-\pi)}) + H_{lp}(e^{jw})$$

highpass + lowpass

$\rightarrow$ bandstop

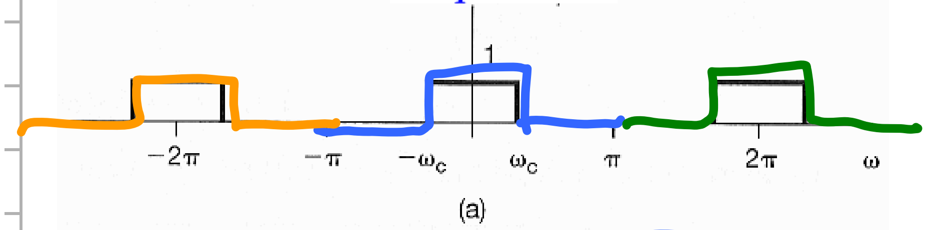


ss4-19

ss5-14

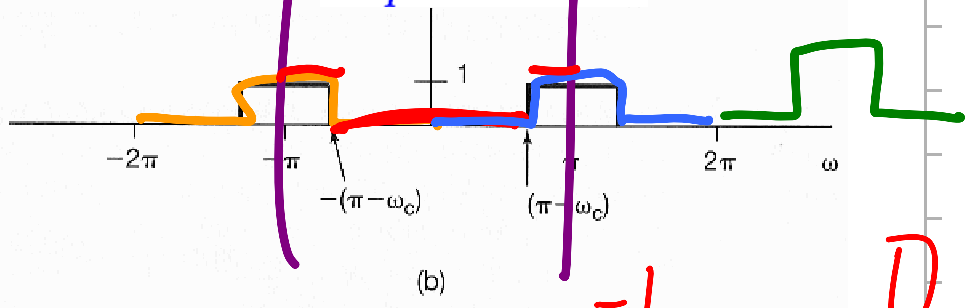
- Example 5.7:

$H_{lp}(e^{jw})$



(a)

$H_{lp}(e^{j(w-\pi)})$



(b)

$$H_{hp}(e^{jw}) = \boxed{H_{lp}(e^{j(w-\pi)})}$$

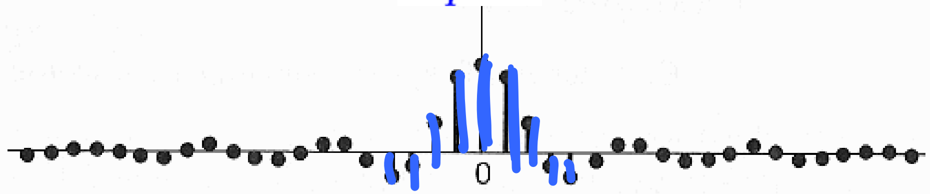
$$\Rightarrow \quad h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

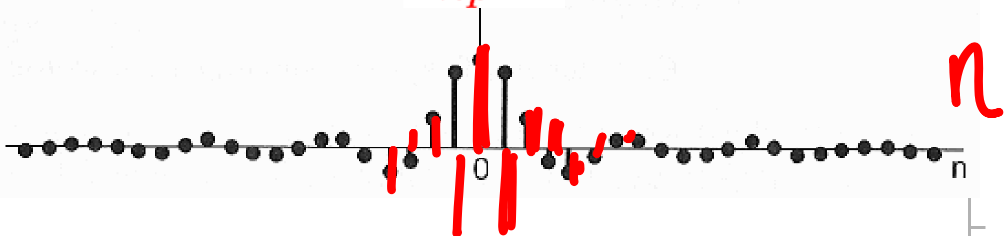
-1 0

$$= (-1)^n h_{lp}[n]$$

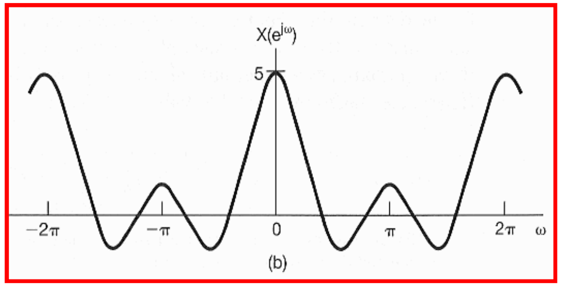
$h_{lp}[n]$



$h_{hp}[n]$

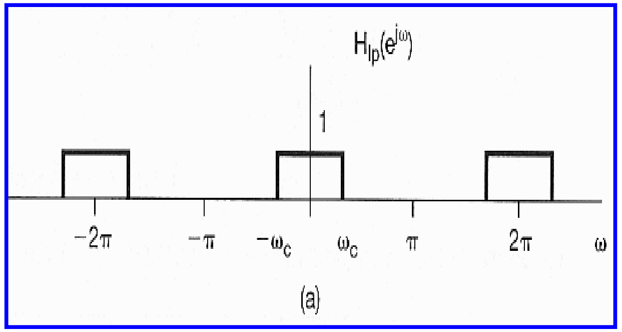


- Convolution Property:



$$y[n] = x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



- Multiplication Property:

ss4-67

$$r[n] = s[n]p[n] \stackrel{\mathcal{F}}{\longleftrightarrow} R(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(w-\theta)})d\theta$$

Multiplication Property

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw})e^{jwn}dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}$$

ss4-68

$$r[n] = s[n]p[n]$$

$$\Rightarrow \quad R(e^{jw}) = \sum_{n=-\infty}^{+\infty} r[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]p[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n]\left\{\frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-jwn}$$

$w-\theta$

$S(e^{jw})$

$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})\underbrace{\left[\sum_{n=-\infty}^{+\infty} s[n]e^{-j(w-\theta)n}\right]}_{S(e^{j(w-\theta)})} d\theta$$

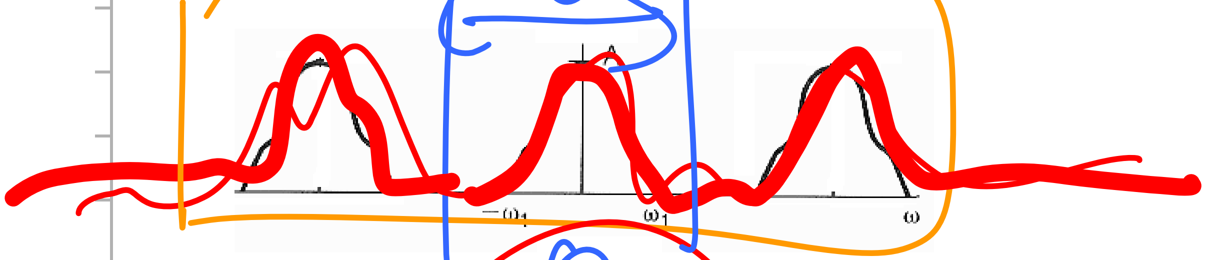
$$= \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})S(e^{j(w-\theta)})d\theta \quad = \frac{1}{2\pi}\int_{2\pi} P(e^{j(w-\theta)})S(e^{j\theta})d\theta$$

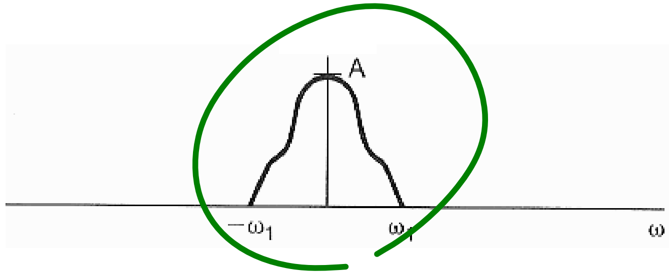
$$y(\theta) = \int_{\Theta} x(\overset{\tau}{\cancel{\theta}})h(\theta - \tau)d\tau$$

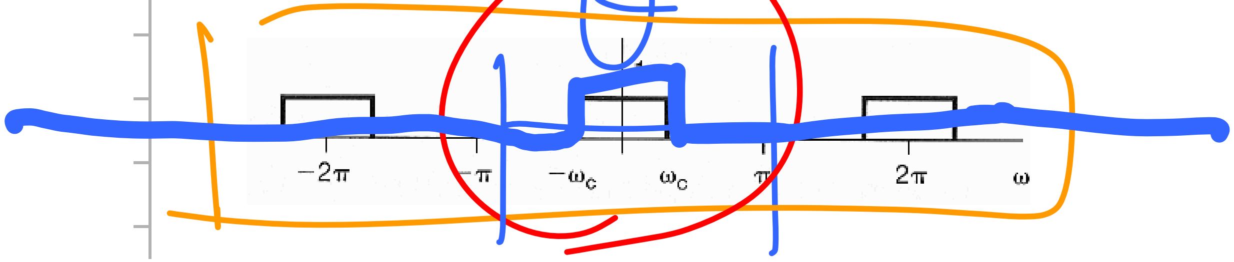
periodic convolution

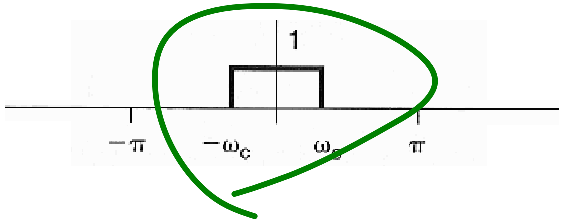
$$y(\theta) = \int_{-\infty}^{+\infty} x(\tau)h(\theta - \tau)d\tau$$

aperiodic convolution







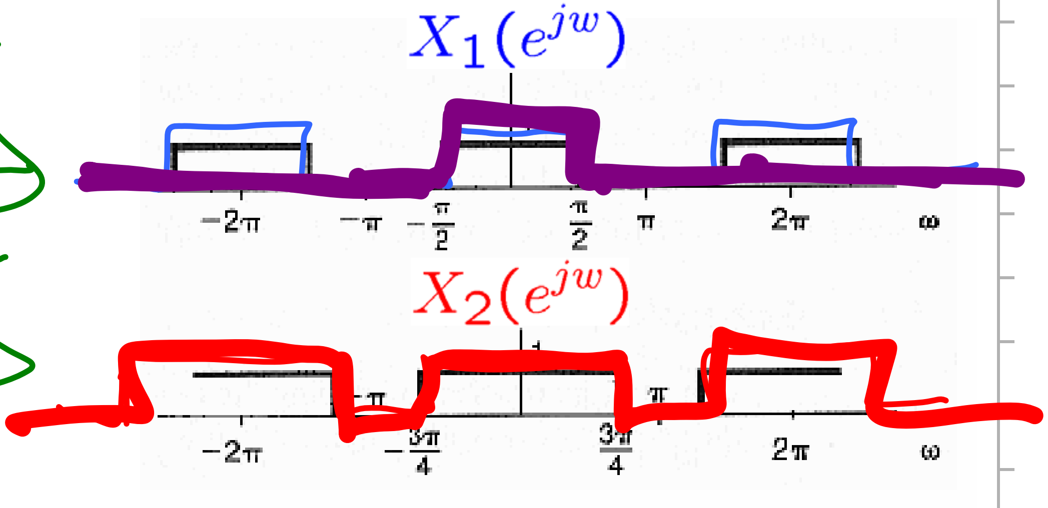


Example 5.15:

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n} \xleftrightarrow{FT}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n} \xleftrightarrow{FT}$$



$$X(e^{jw}) = \frac{1}{2\pi}\boxed{\int_{-\pi}^{+\pi}} X_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

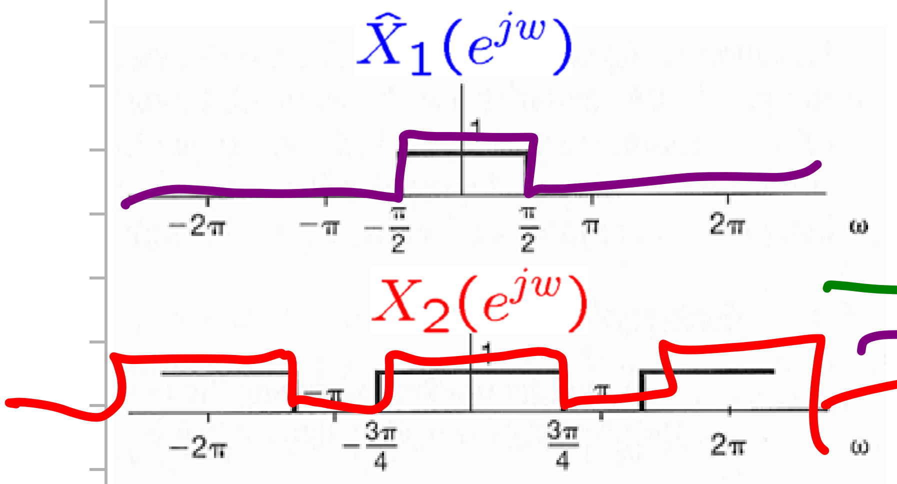
$$\hat{X}_1(e^{jw}) = \begin{cases} X_1(e^{jw}), & \text{for } -\pi < w \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(w-\theta)})d\theta$$

- Example 5.15:

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(w-\theta)}) d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(w-\theta)}) d\theta$$



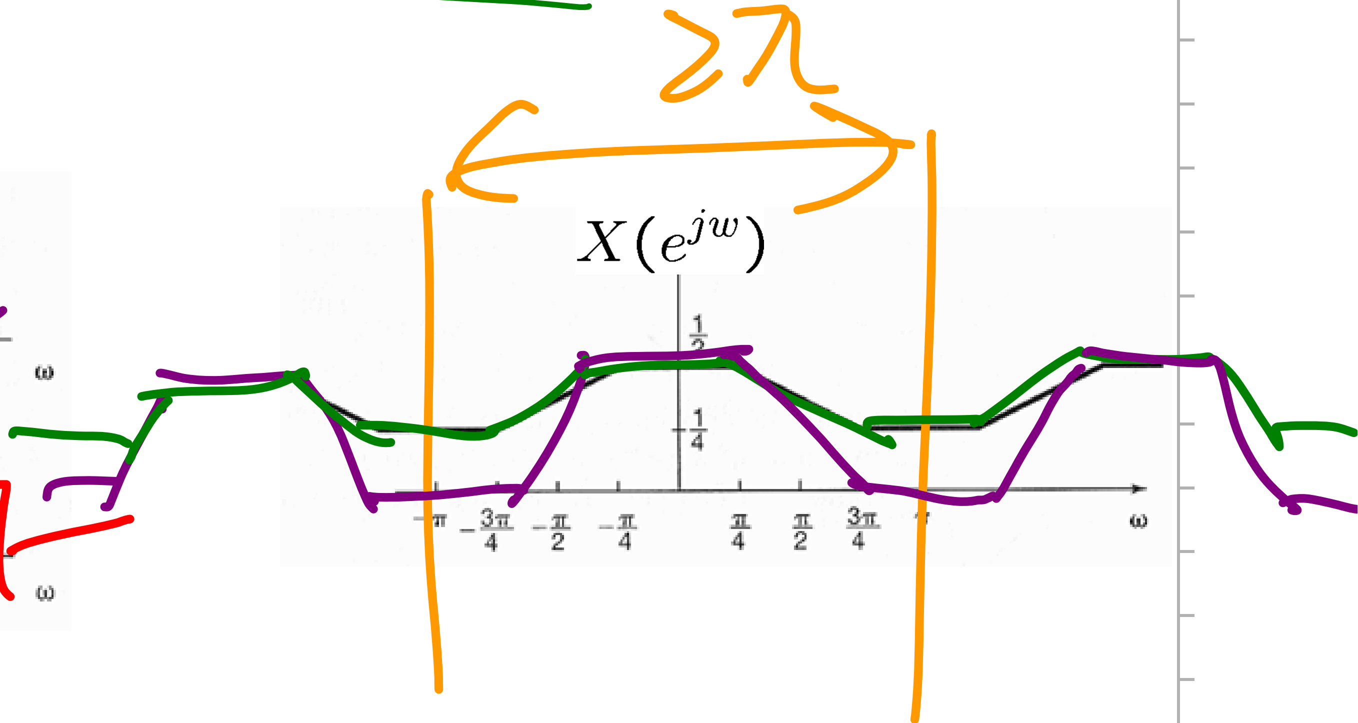


TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

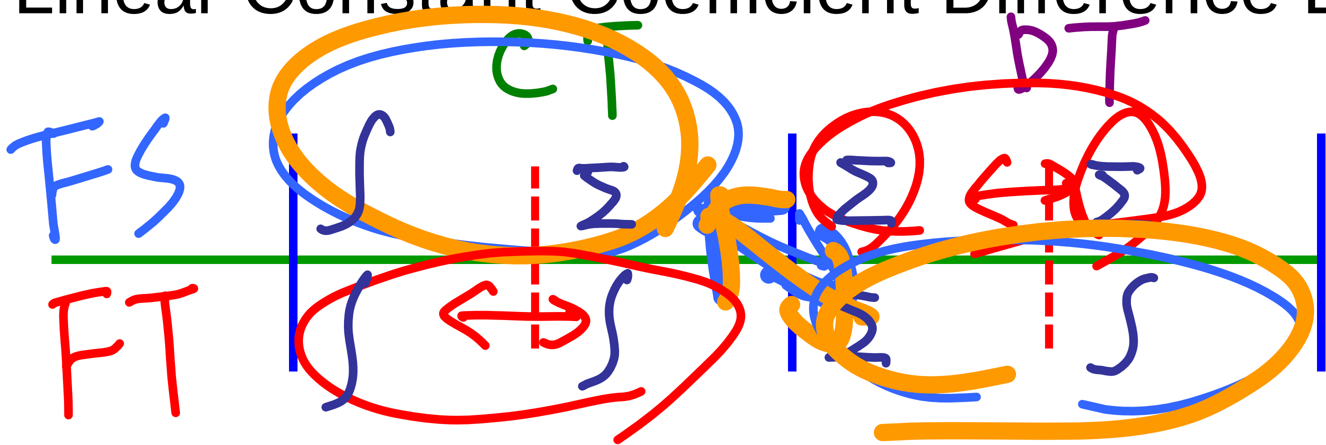
Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k

$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \neq 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

- Representation of Aperiodic Signals: the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality

ss4-47

- Systems Characterized by Linear Constant-Coefficient Difference Equations



DT Fourier Series Pair of Periodic Signals:

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

IF $f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk(2\pi/N)n}$ $\qquad g[n] \overset{\mathcal{FS}}{\longleftrightarrow} f[k]$

LET $k = n, n = -k$ $\qquad f[n] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N} g[-k]$

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}$$

$$a_k \; :=: \; \frac{1}{N} x[-n]$$

- Duality in DT Fourier Series:

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$x[n-n_0] \xleftrightarrow{\mathcal{FS}} a_k \, e^{-jk(2\pi/N)n_0}$$

$$e^{+jm(2\pi/N)n} \, x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m}$$

$$\sum_{r=<N>} x[r] \, y[n-r] \xleftrightarrow{\mathcal{FS}} N a_k \, b_k$$

$$x[n] \, y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=<N>} a_l \, b_{k-l}$$

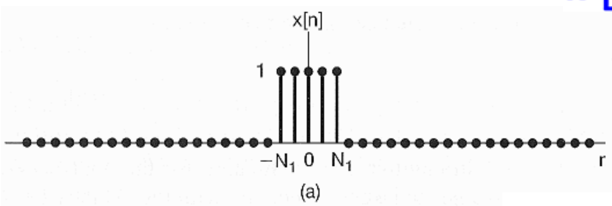
Duality between DT-FT & CT-FS:

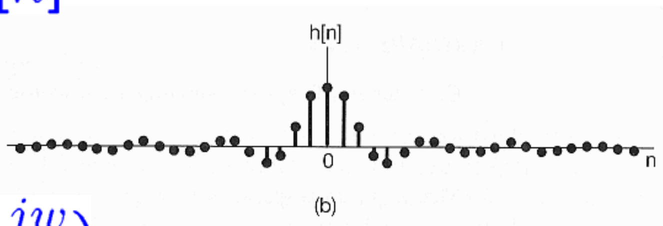
$$x[n] \xleftrightarrow{\mathcal{DTFT}} X(e^{jw}) \qquad x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

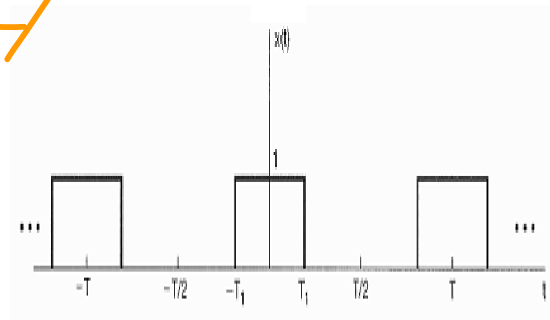
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{jw}) e^{jwn} dw \qquad a_k = \frac{1}{T}\int_{T} x(t) e^{-jkw_0 t} dt$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

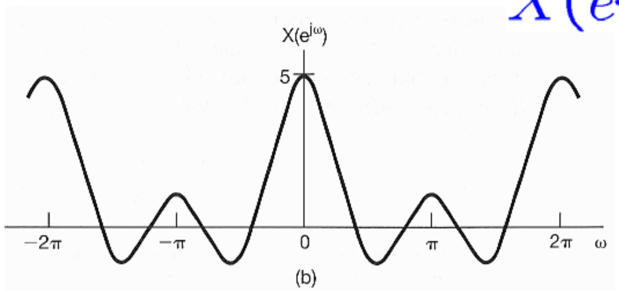
$x[n]$

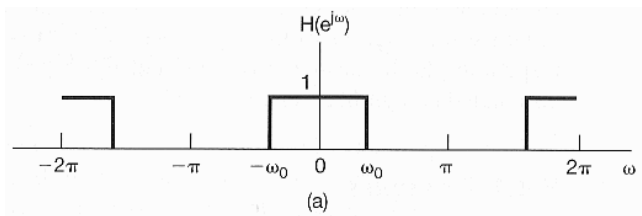


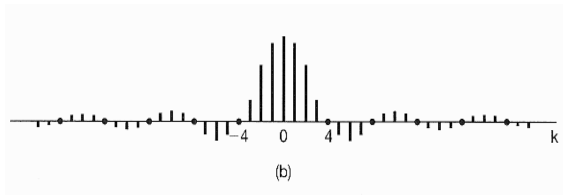




$X(e^{jw})$







Duality

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ **continuous frequency** **aperiodic in frequency**	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}$ **discrete time** **aperiodic in time**	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency

duality

duality

duality

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- Representation of Aperiodic Signals:
  the Discrete-Time Fourier Transform
- The Fourier Transform for Periodic Signals
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by
  Linear Constant-Coefficient Difference Equations

ss4-75

ss4-76

- A useful class of DT LTI systems:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

h[n]

$= x[n] * h[n]$

$$Y(e^{jw}) = X(e^{jw}) H(e^{jw}) \qquad H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$\boxed{x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jwn_0}X(e^{jw})}$$

ss4-79

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{F}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{F}\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k e^{-jkw} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jkw} X(e^{jw})$$

$$\left(e^{\mp jw}\right)^{-M}$$

$$\Rightarrow H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

$$= \frac{b_0 + b_1 e^{-jw} + \cdots + b_M e^{-jMw}}{a_0 + a_1 e^{-jw} + \cdots + a_N e^{-jNw}}$$

- **Examples 5.18 & 5.19:**

$$x[n] \rightarrow \boxed{\text{LTI System}} \rightarrow y[n]$$

ss4-81

$|a| < 1$

$$y[n] - ay[n-1] = x[n] \quad \Leftarrow \quad \Rightarrow \quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$Y(\cdot) - e^{-jw}Y(\cdot)$$

$$\Rightarrow \quad h[n] = a^n u[n]$$

$$\mathcal{F}\left\{ y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] \right\} = \mathcal{F}\left\{ 2x[n] \right\}$$

$$\Rightarrow \quad H(e^{jw}) = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}} = \frac{2}{(1 - \frac{1}{2}e^{-jw})(1 - \frac{1}{4}e^{-jw})}$$

$$= \frac{4}{(1 - \frac{1}{2}e^{-jw})} - \frac{2}{(1 - \frac{1}{4}e^{-jw})}$$

$$\Rightarrow \quad h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

- Example 5.20:

$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H(e^{jw}) = \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \longrightarrow \text{LTI System} \longrightarrow y[n] = ???$$

ss4-82

$$\Rightarrow \quad Y(e^{jw}) = X(e^{jw})H(e^{jw}) \qquad = x[n] * h[n]$$

$$= \left[\frac{1}{1-\frac{1}{4}e^{-jw}}\right]\left[\frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}\right]$$

$$= \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})^2}$$

$$= \frac{8}{(1-\frac{1}{2}e^{-jw})} - \frac{4}{(1-\frac{1}{4}e^{-jw})} - \frac{2}{(1-\frac{1}{4}e^{-jw})^2}$$

ss5-45

$$\Rightarrow \quad y[n] = 8\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n+1]$$

$$= \left\{8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n\right\} u[n]$$

- Representation of Aperiodic Signals: the DT FT
- The FT for Periodic Signals
- Properties of the DT FT
  - Linearity — Time Shifting — Frequency Shifting
  - Conjugation — Time Reversal — Time Expansion
  - Convolution — Multiplication
  - Differencing in Time — Accumulation — Differentiation in Frequency
  - Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals & for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals
  - Parseval's Relation for Aperiodic Signals
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3)
CT
DT

Aperiodic

FT
CT (Chap 4)
DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)