

Spring 2015

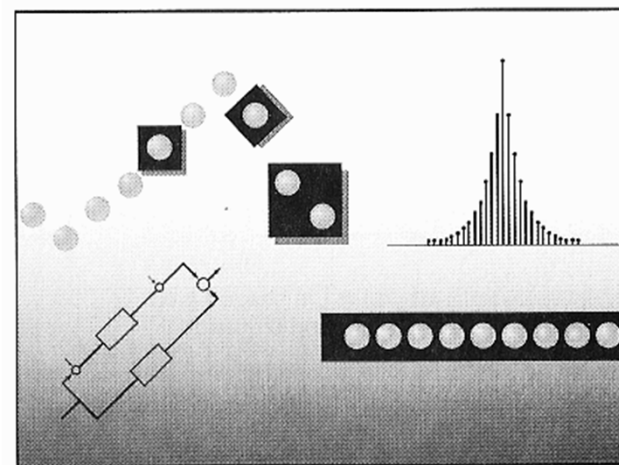
# 信號與系統 Signals and Systems

## Chapter SS-5 The Discrete-Time Fourier Transform

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NTU-EE

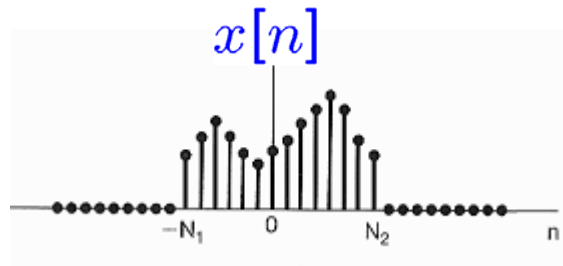
Feb15 – Jun15



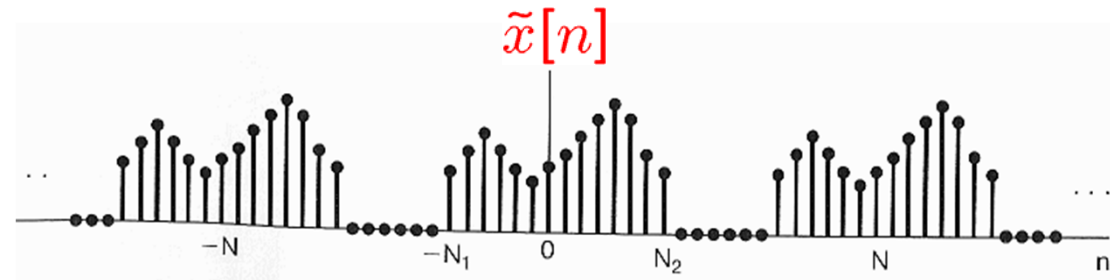
Figures and images used in these lecture notes are adopted from  
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997

- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- The Fourier Transform for **Periodic** Signals
- **Properties** of Discrete-Time Fourier Transform
- The **Convolution** Property
- The **Multiplication** Property
- Duality
- **Systems** Characterized by  
Linear Constant-Coefficient Difference Equations

## DT Fourier Transform of an Aperiodic Signal:



an aperiodic signal



a periodic signal

$$\omega_0 = \frac{2\pi}{N}$$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\omega_0)n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(\omega_0)n} = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

$$\Rightarrow a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(\omega_0)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(\omega_0)n}$$

## DT Fourier Transform of an Aperiodic Signal:

- Define  $X(e^{j\omega})$ :

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

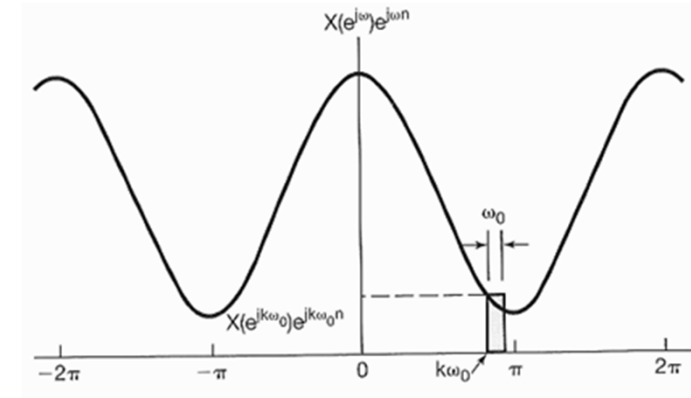
- Then.

$$a_k = \frac{1}{N} X(e^{jk\omega_0}) \quad \omega = k\omega_0$$

- Hence,

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$



$$\omega_0 = \frac{2\pi}{N}$$

$$\frac{1}{N} = \frac{1}{2\pi} \omega_0$$

$$\omega_0 N = 2\pi$$

$$\omega_0, 2\omega_0, 3\omega_0, \dots, N\omega_0,$$

## DT Fourier Transform of an Aperiodic Signal:

- As  $N \rightarrow \infty$ ,  $\tilde{x}[n] \rightarrow x[n]$

$$\omega_0 N = 2\pi$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- inverse Fourier transform eqn

- synthesis eqn

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

-  $X(e^{j\omega})$ : Fourier transform of  $x[n]$   
spectrum

- analysis eqn

$$a_k = \frac{1}{N} X(e^{j\omega}) \Big|_{\omega = k\omega_0}$$

$$\omega_0 = \frac{2\pi}{N}$$

## Periodicity of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

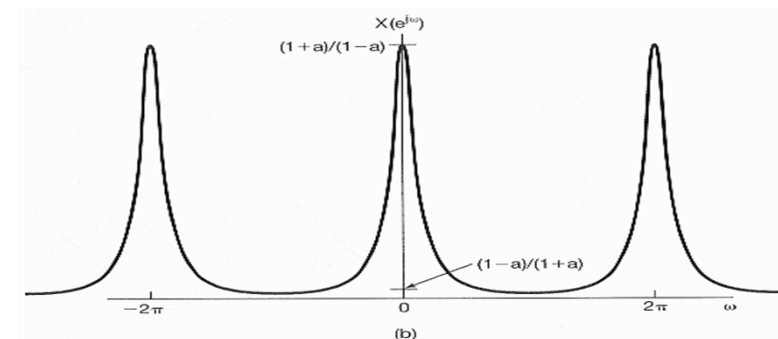
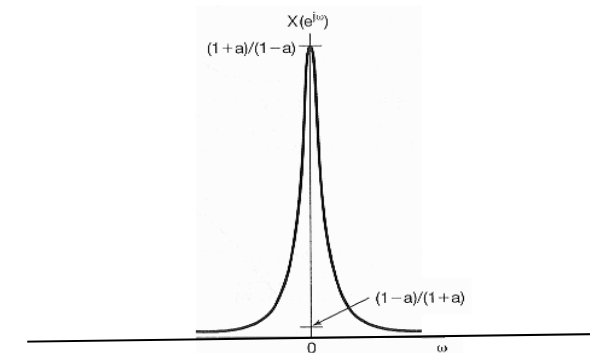
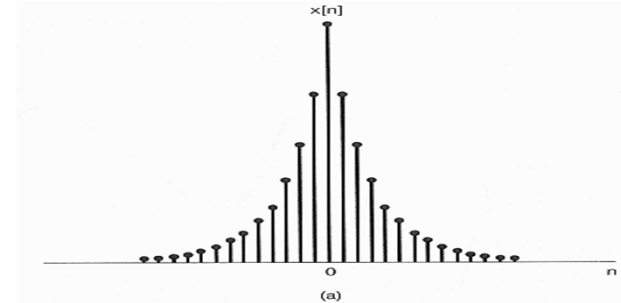
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$\begin{aligned} X(e^{j(\omega+2\pi)}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} e^{-j(2\pi)n} \end{aligned}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

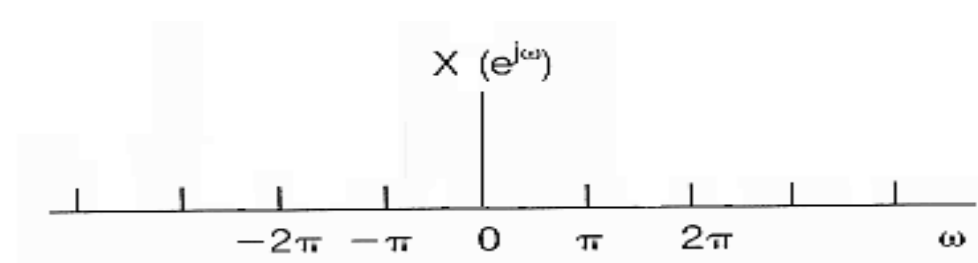
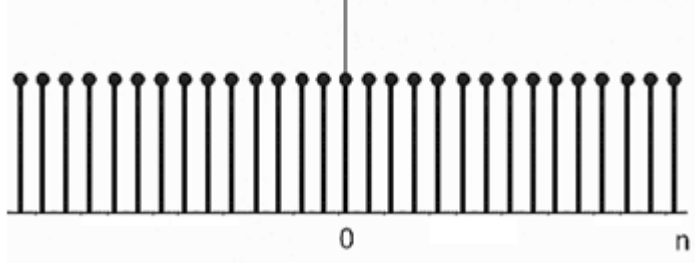
$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$= X(e^{j\omega})$$



High-Frequency & Low-Frequency Signals:

low-frequency signal

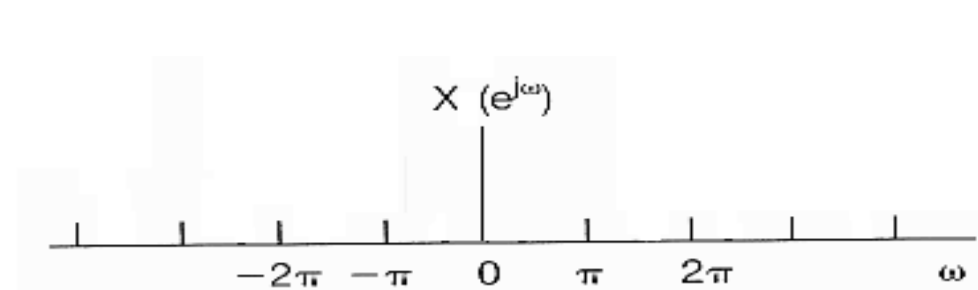
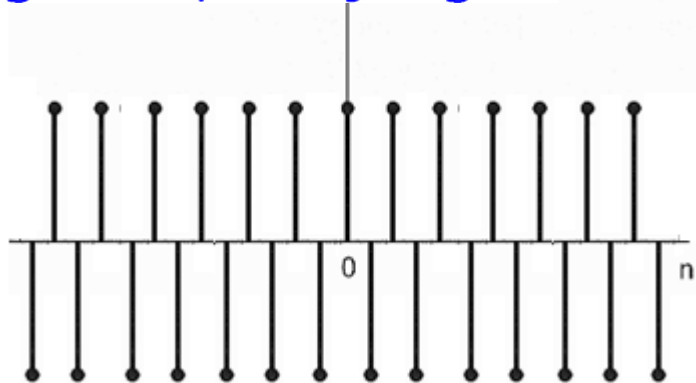


$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

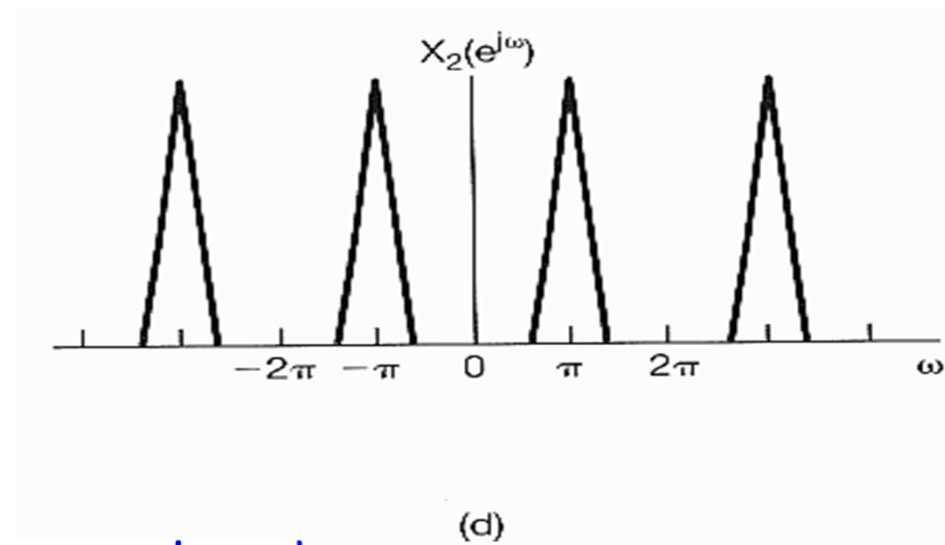
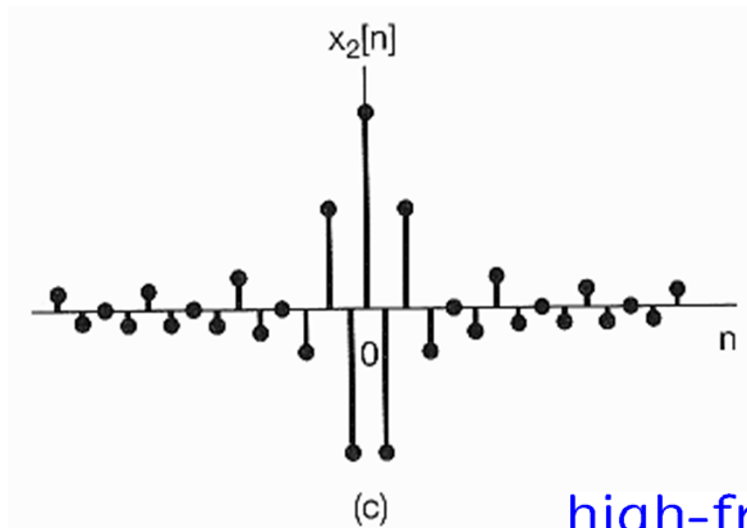
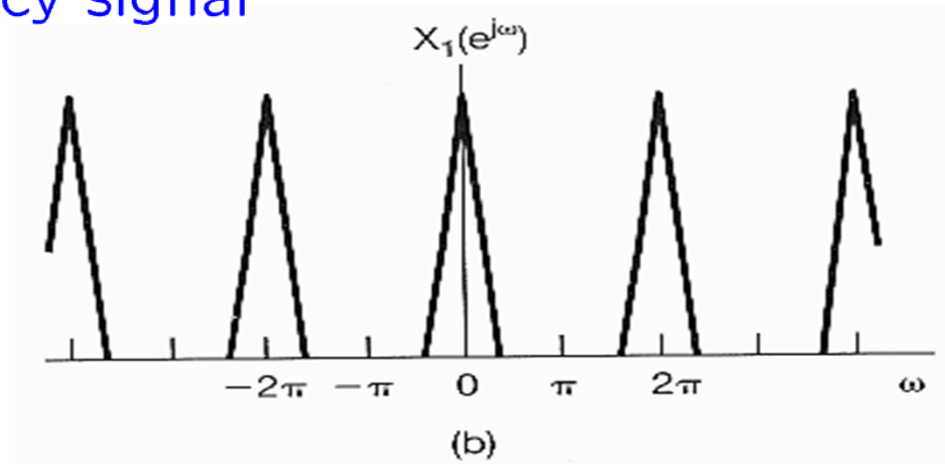
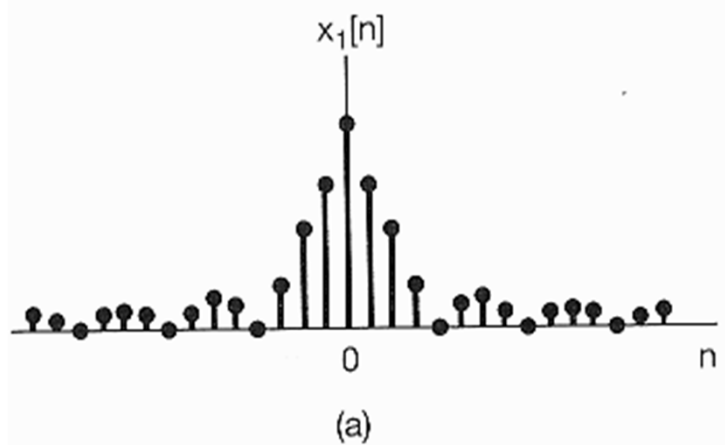
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

high-frequency signal



■ High-Frequency & Low-Frequency Signals:

low-frequency signal



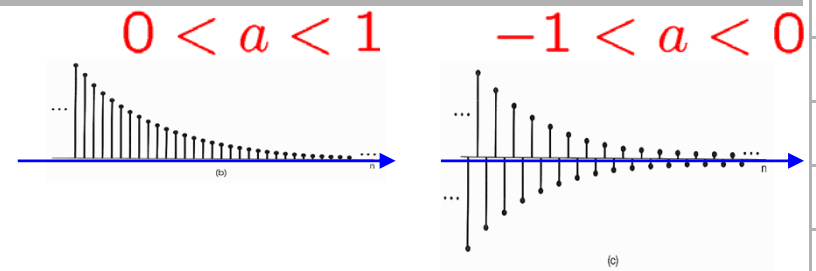
high-frequency signal



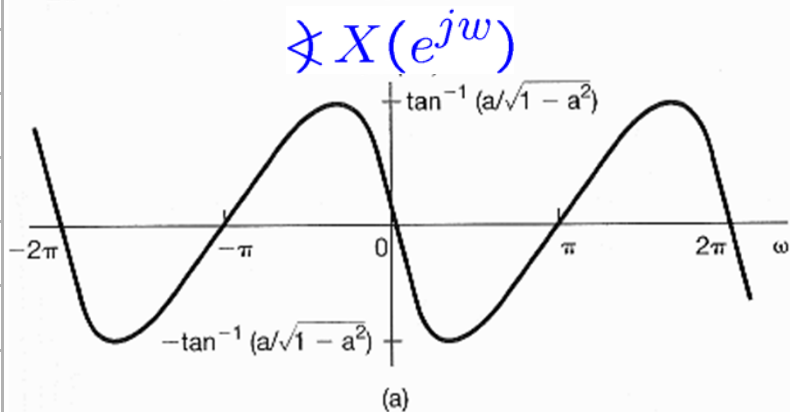
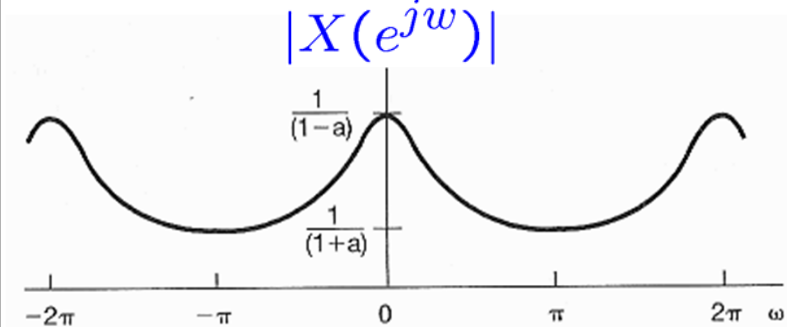
## Example 5.1:

$$x[n] = a^n u[n], \quad |a| < 1$$

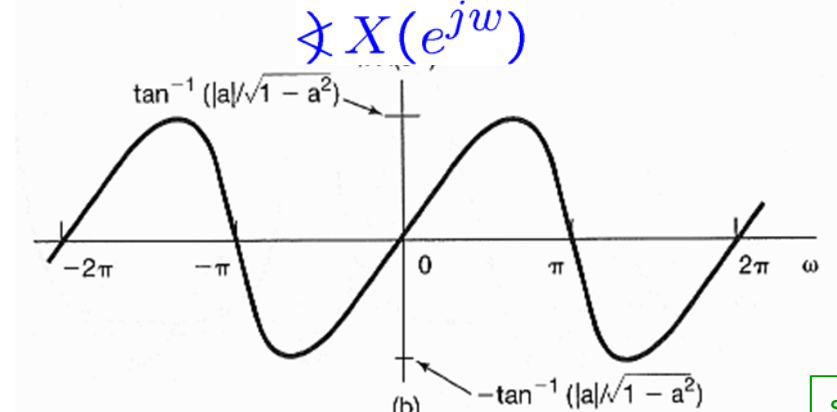
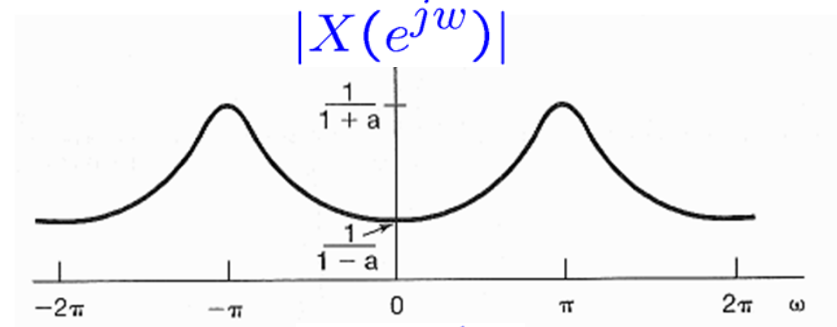
$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$



$0 < a < 1$



$-1 < a < 0$



■ Example 5.2:  $x[n] = a^{|n|}$ ,  $0 < a < 1$

$-1 < a < 0$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

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$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

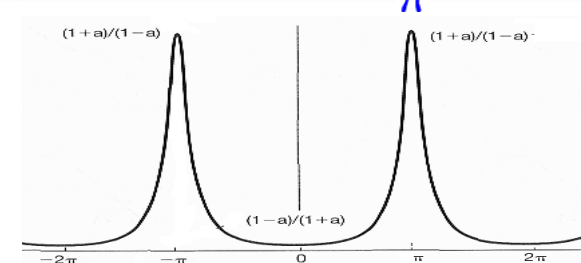
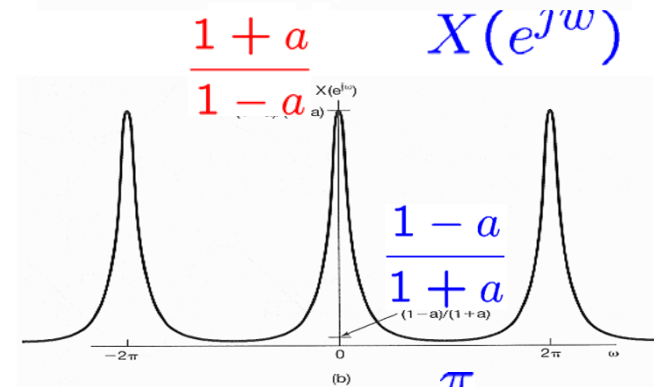
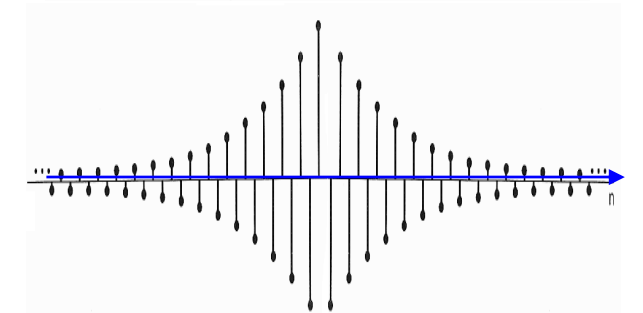
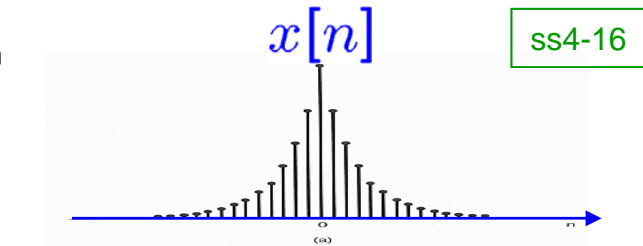
$$= \sum_{n=0}^{+\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a \cos \omega + a^2} = \frac{1 - a^2}{(1 - a)^2} = \frac{1 + a}{1 - a}$$

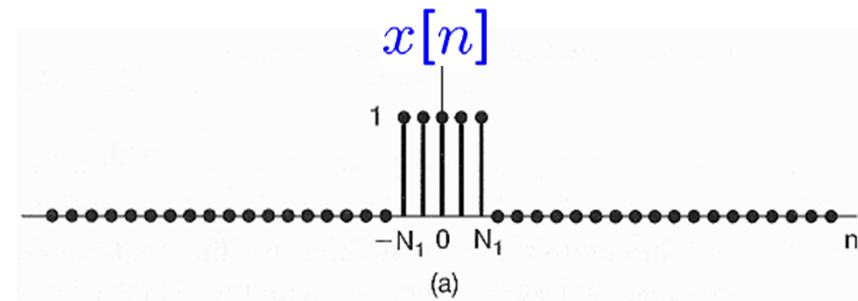
$$= \frac{1 - a^2}{(1 + a)^2} = \frac{1 - a}{1 + a}$$



### Example 5.3:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$



$$\Rightarrow X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= e^{-j\omega(-N_1)} + \dots + e^{-j\omega(N_1)}$$

$$= e^{-j\omega(-N_1)} \left( \frac{1 - (e^{-j\omega})^{2N_1+1}}{1 - (e^{-j\omega})} \right)$$

$$= e^{j\omega(N_1)} \left( \frac{(e^{-j\omega})^{N_1+1/2} \left( (e^{j\omega})^{N_1+1/2} - (e^{-j\omega})^{N_1+1/2} \right)}{(e^{-j\omega/2}) \left( (e^{j\omega/2}) - (e^{-j\omega/2}) \right)} \right)$$

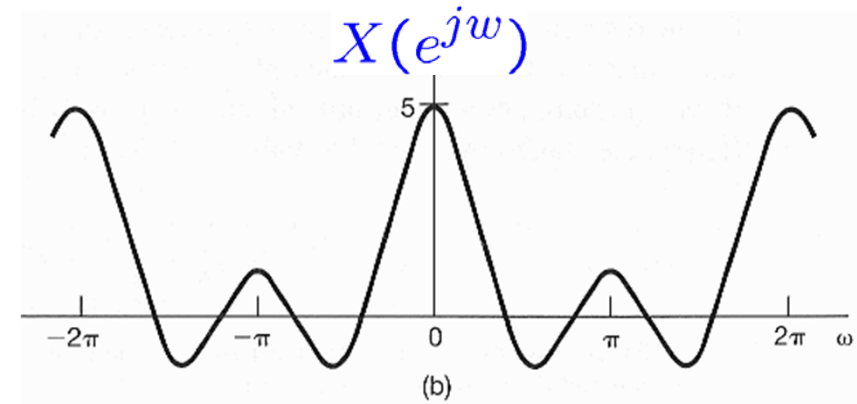
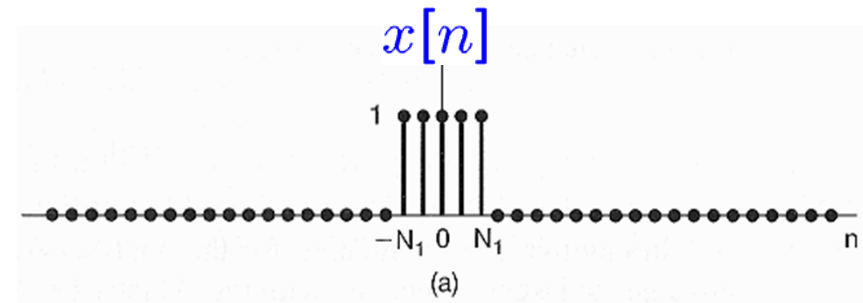
$$= \frac{\sin \left( \omega \left( N_1 + \frac{1}{2} \right) \right)}{\sin(\omega/2)}$$

$$\begin{aligned} & 1 - e^{-j\theta} \\ &= e^{-j\frac{\theta}{2}} e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} e^{-j\frac{\theta}{2}} \\ &= e^{-j\frac{\theta}{2}} \left( e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} \right) \end{aligned}$$

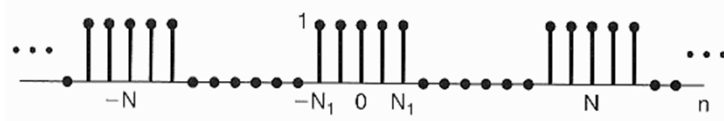
### Example 5.3:

$$\begin{aligned} \Rightarrow X(e^{j\omega}) &= \sum_{n=-N_1}^{N_1} e^{-j\omega n} \\ &= \frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin(\omega/2)} \\ &= \frac{\sin\left(\omega\left(\quad\right)\right)}{\sin(\omega/2)} \end{aligned}$$

$$N_1 = 2$$

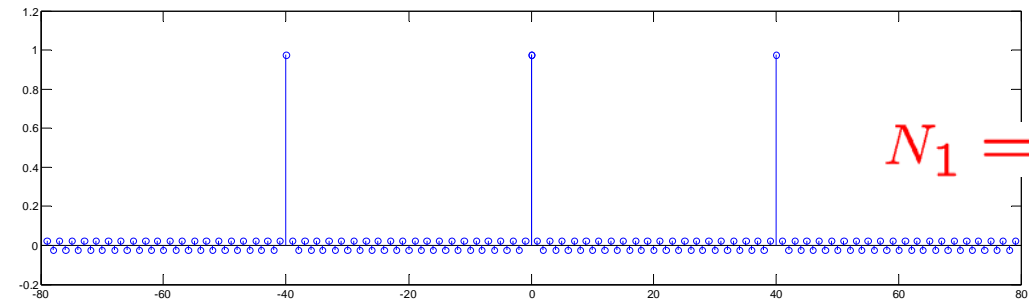
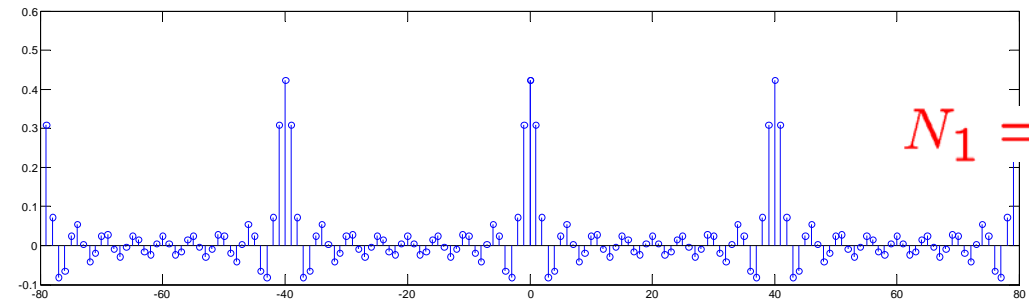
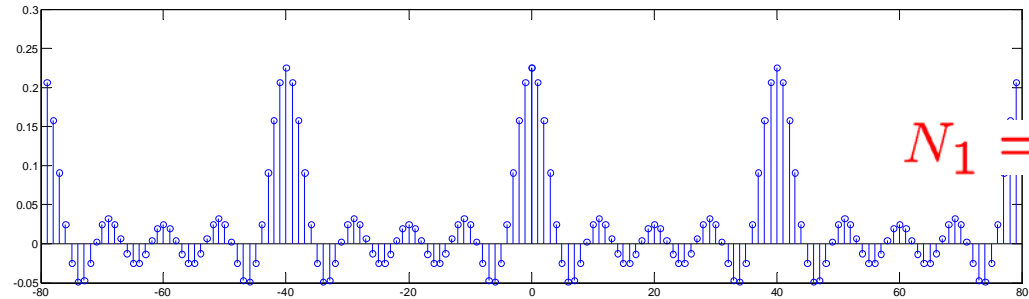
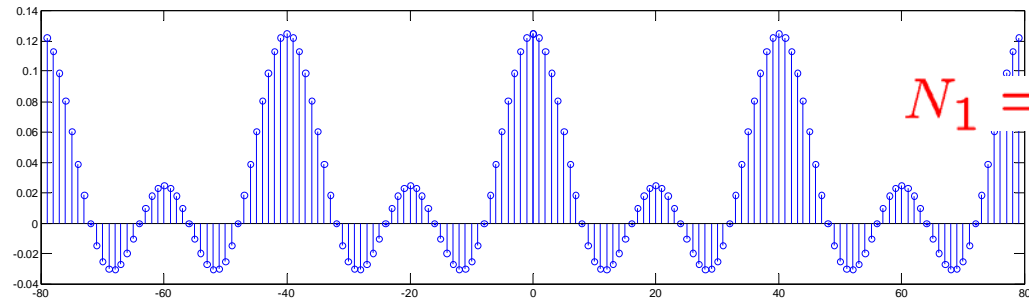
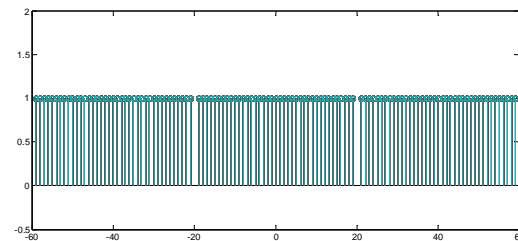
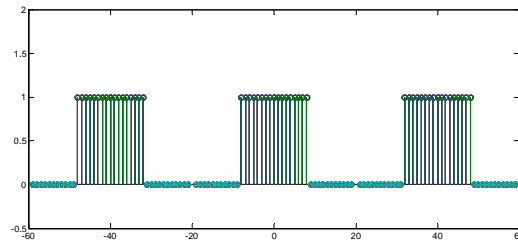
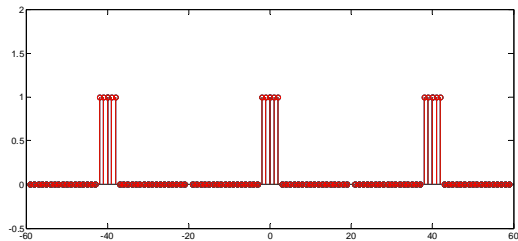


## Example 3.12:



•  $N = 40$

$$a_k = \frac{1}{N} \frac{\sin \left[ \left( \frac{2\pi}{N} \right) k \left( N_1 + \frac{1}{2} \right) \right]}{\sin \left[ \left( \frac{\pi}{N} \right) k \right]}$$



## ■ Convergence of DT Fourier Transform:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

## ■ The analysis equation will converge:

- Either if  $x[n]$  is **absolutely summable**, that is,

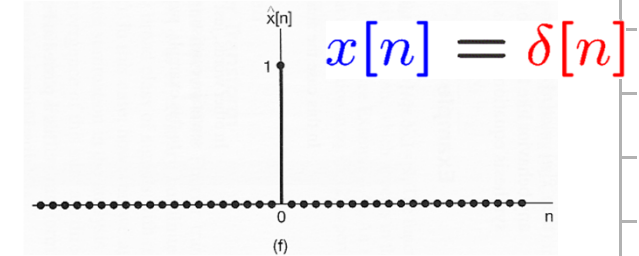
$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

- Or, if  $x[n]$  has **finite energy**, that is,

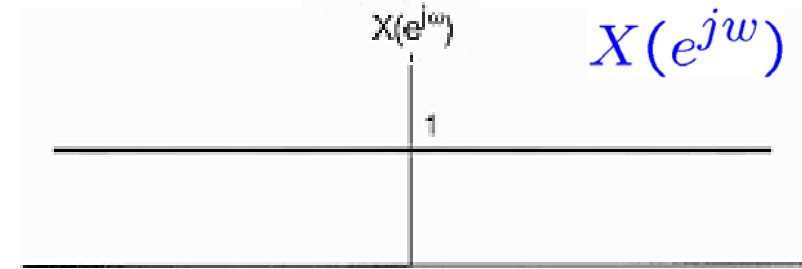
$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- Example 5.4:

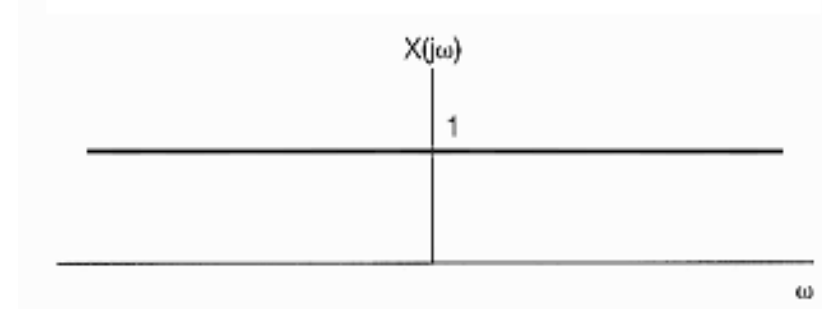
$x[n] = \delta[n]$ , i.e., unit impulse



$$\begin{aligned} \Rightarrow X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\omega n} = 1 \end{aligned}$$

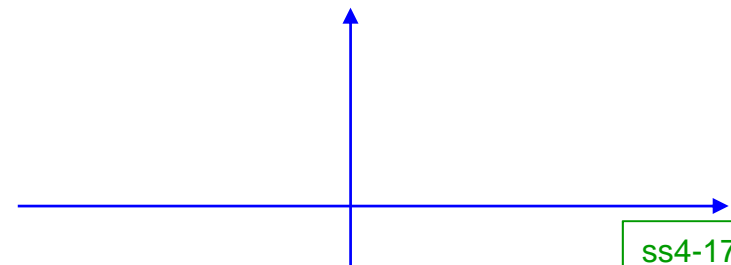


$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{2\pi} e^{j\omega n} d\omega \end{aligned}$$



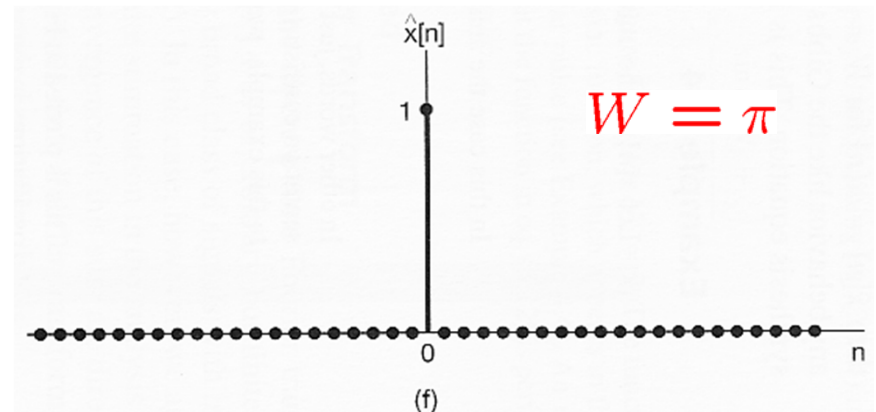
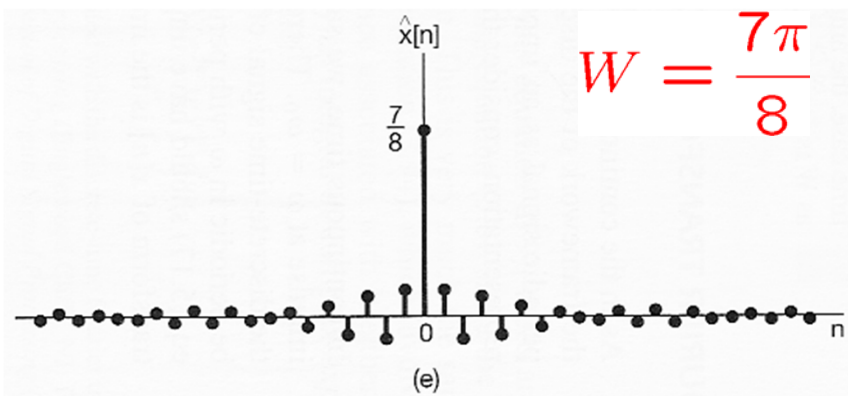
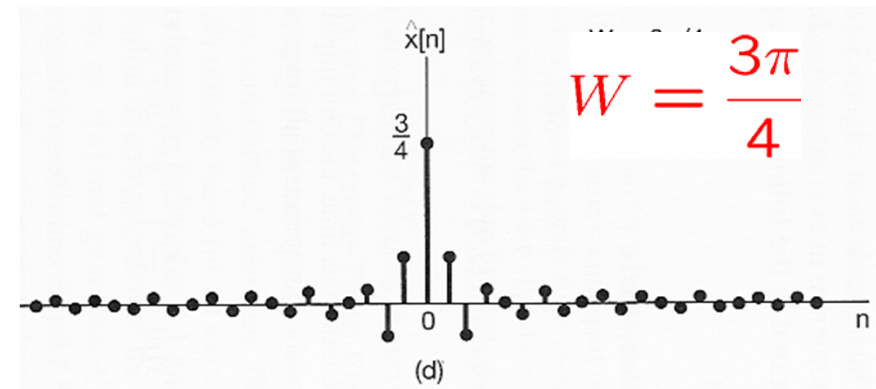
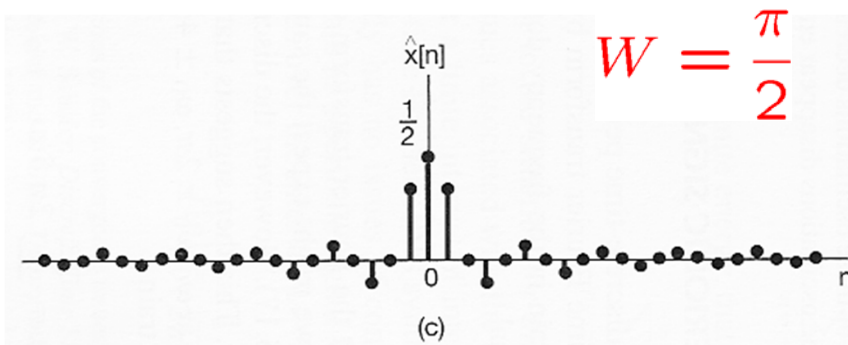
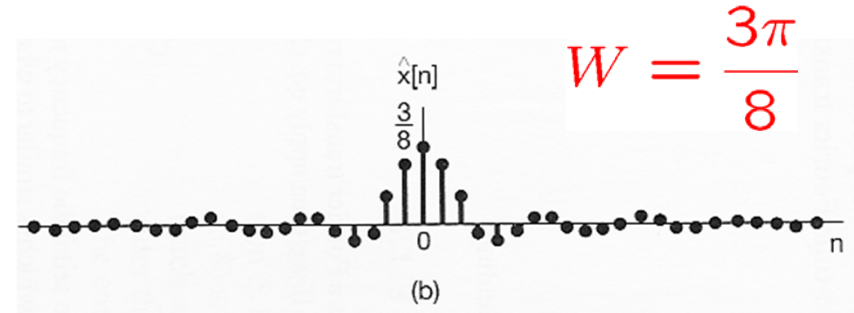
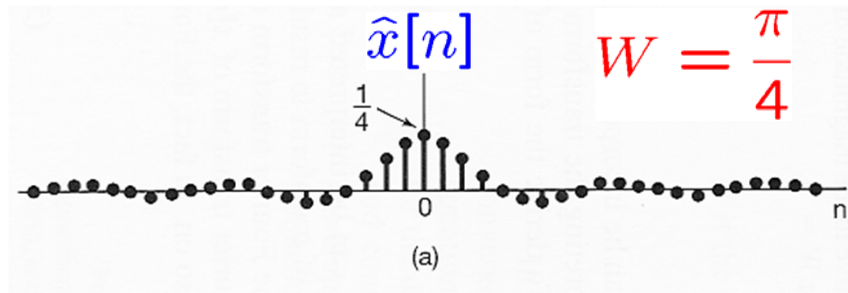
- Approximation

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-W}^{+W} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$



## Approximation of an Aperiodic Signal:

$$\hat{x}[n] = \frac{\sin Wn}{\pi n}$$

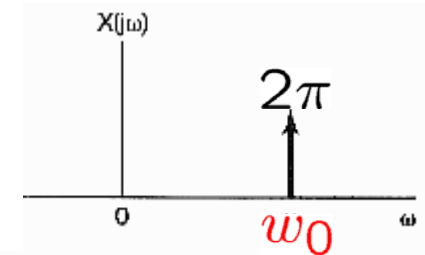




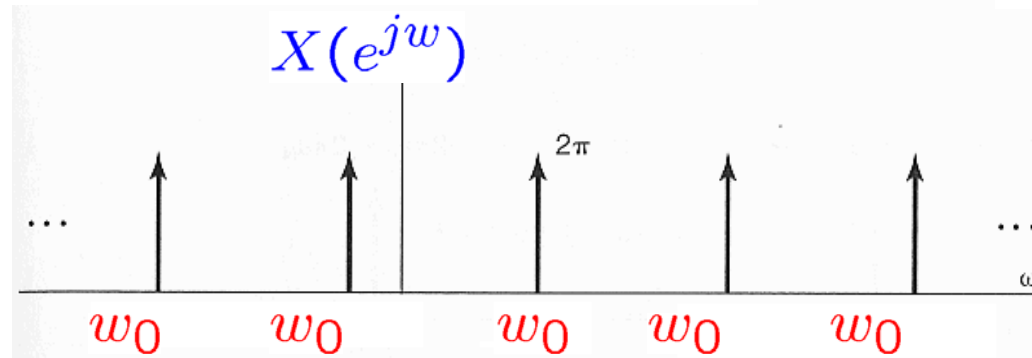
- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- **The Fourier Transform for Periodic Signals**
- Properties of Discrete-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by  
Linear Constant-Coefficient Difference Equations

## Fourier Transform from Fourier Series:

$$x(t) = e^{j\omega_0 t} \xleftrightarrow{CTFT} X(j\omega) = 2\pi\delta(\omega - \omega_0)$$



$$x[n] = e^{j\omega_0 n} \xleftrightarrow{DTFT}$$



$$\begin{aligned} X(e^{j\omega}) &= \dots + 2\pi\delta(\omega - \omega_0 + 2\pi) + 2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega - \omega_0 - 2\pi) + \dots \\ &= \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) \end{aligned}$$

$$\omega_0 = \frac{2\pi}{N}$$

$$\begin{aligned} \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega &= \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l) e^{j\omega n} d\omega \\ &= e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n} \end{aligned}$$

$$w_0 = \frac{2\pi}{N}$$

## Fourier Transform from Fourier Series:

- more generally,

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(w_0)n} = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(w - k\frac{2\pi}{N}\right)$$

- If  $k = 0, 1, \dots, N - 1$

$$\begin{aligned} x[n] &= a_0 + a_1 e^{j1\left(\frac{2\pi}{N}\right)n} + a_2 e^{j2\left(\frac{2\pi}{N}\right)n} + \dots + a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n} \\ &= x_0 + x_1 + x_2 + \dots + x_{N-1} \end{aligned}$$

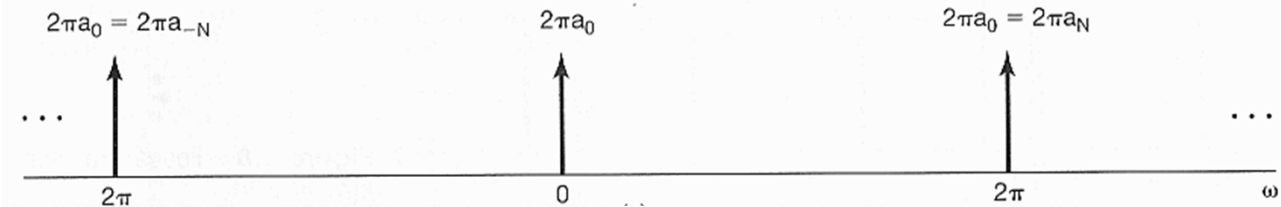
a linear combination of signals with  $0, \frac{2\pi}{N}, \frac{2 \cdot 2\pi}{N}, \dots, \frac{(N-1) \cdot 2\pi}{N}$

$0, w_0, 2w_0, \dots, (N-1)w_0$

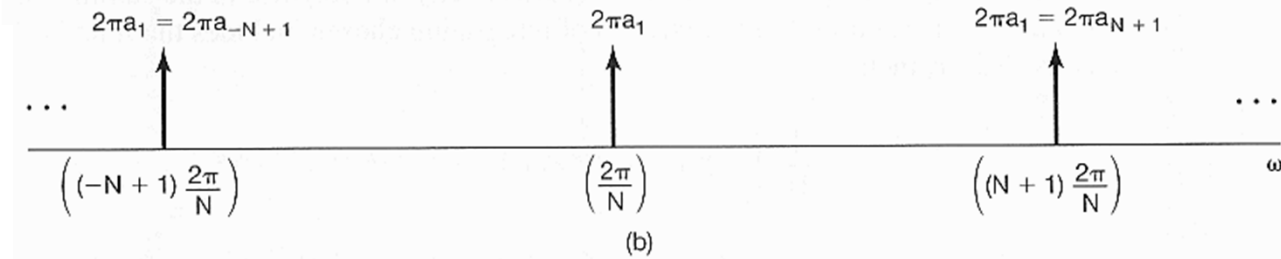
$$w_0 = \frac{2\pi}{N}$$

## Fourier Transform from Fourier Series:

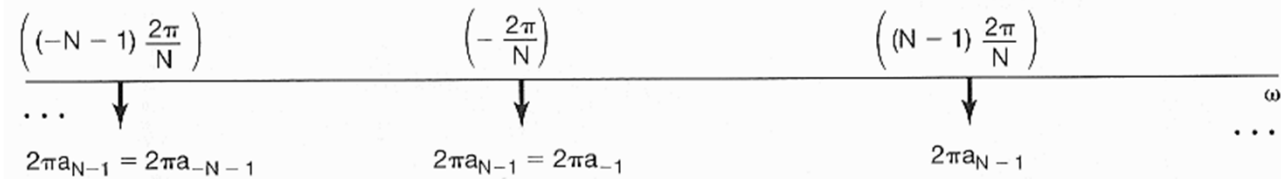
$x_0$   $\longleftrightarrow$   $\mathcal{F}$



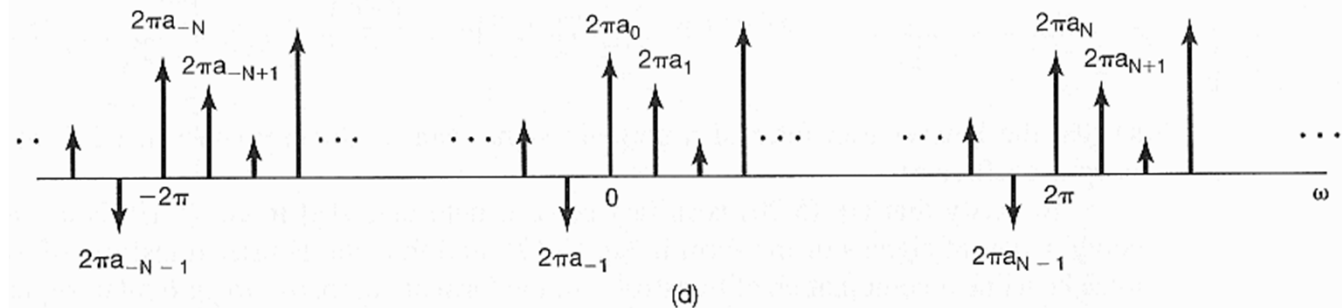
$x_1$   $\longleftrightarrow$   $\mathcal{F}$



$x_{N-1}$   $\longleftrightarrow$   $\mathcal{F}$



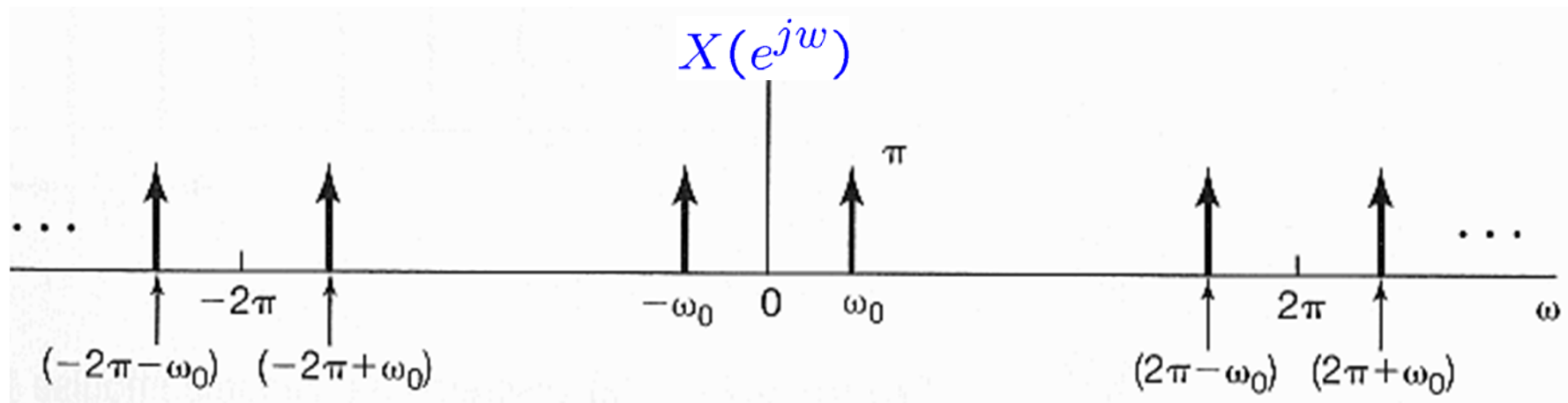
$X(e^{jw}) =$



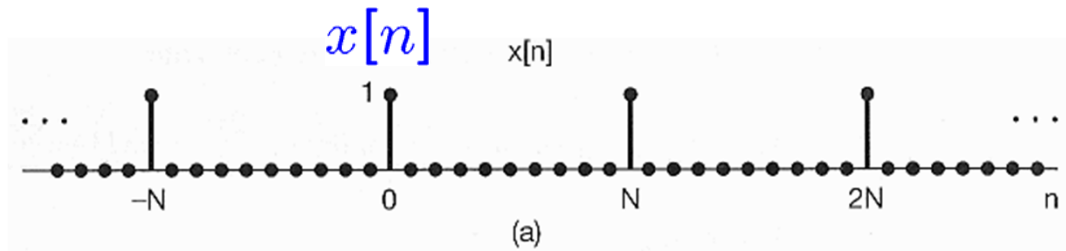
■ Example 5.5:

$$x[n] = \cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \quad \text{with } \omega_0 = \frac{2\pi}{5}$$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{l=-\infty}^{+\infty} \pi \delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi \delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right) \\ &= \pi \delta\left(\omega - \frac{2\pi}{5}\right) + \pi \delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \leq \omega < \pi \end{aligned}$$



## Example 5.6:



$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

choose  $0 \leq n \leq N - 1$

$$\Rightarrow a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

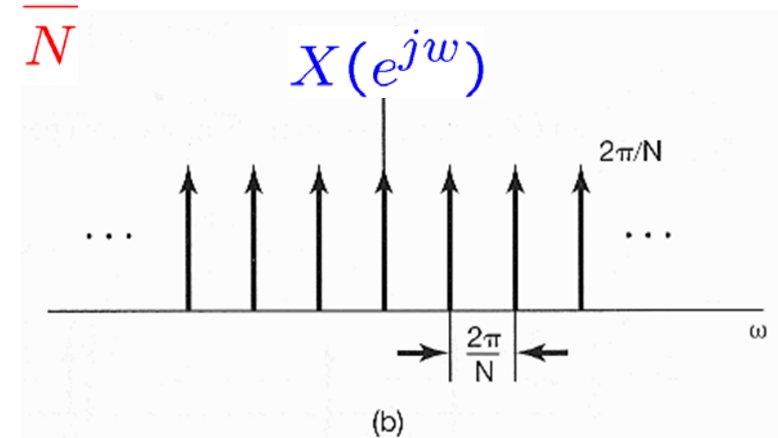
$$\Rightarrow X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{N})$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

ss4-25



- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- The Fourier Transform for **Periodic** Signals
- **Properties of Discrete-Time Fourier Transform** ss4-26
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by  
Linear Constant-Coefficient Difference Equations

<b>Section</b>	<b>Property</b>
5.3.2	Linearity
5.3.3	Time Shifting
5.3.3	Frequency Shifting
5.3.4	Conjugation
5.3.6	Time Reversal
5.3.7	Time Expansion
5.4	Convolution
5.5	Multiplication
5.3.5	Differencing in Time
5.3.5	Accumulation
5.3.8	Differentiation in Frequency
5.3.4	Conjugate Symmetry for Real Signals
5.3.4	Symmetry for Real and Even Signals
5.3.4	Symmetry for Real and Odd Signals
5.3.4	Even-Odd Decomposition for Real Signals
5.3.9	Parseval's Relation for Aperiodic Signals



<b>Property</b>	<b>CTFS</b>	<b>DTFS</b>	<b>CTFT</b>	<b>DTFT</b>	<b>LT</b>	<b>zT</b>
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

■ Fourier Transform Pair:

- Synthesis equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Analysis equation:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

- Notations:

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\}$$

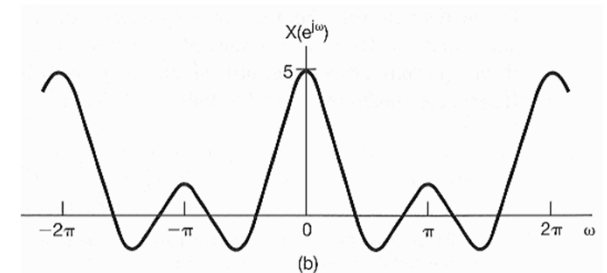
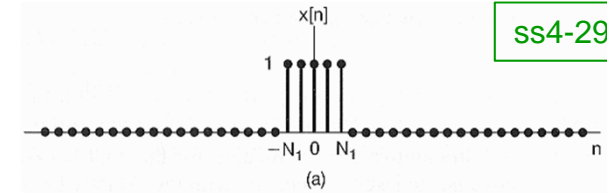
$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$$

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

$$\frac{1}{1 - ae^{j\omega}} = \mathcal{F}\{a^n u[n]\}$$

$$a^n u[n] = \mathcal{F}^{-1}\left\{\frac{1}{1 - ae^{j\omega}}\right\}$$

$$a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{1 - ae^{j\omega}}$$



$$|a| < 1$$

ss5-6

■ Periodicity of DT Fourier Transform:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

ss4-30

■ Linearity:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$y[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega})$$

$$\Rightarrow a x[n] + b y[n] \xleftrightarrow{\mathcal{F}} a X(e^{j\omega}) + b Y(e^{j\omega})$$

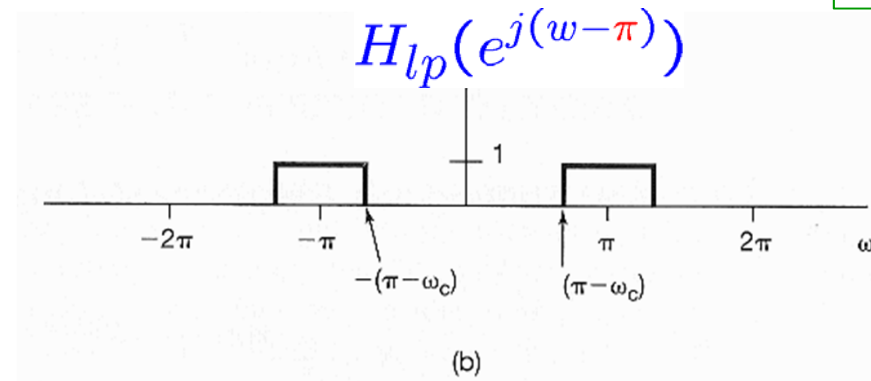
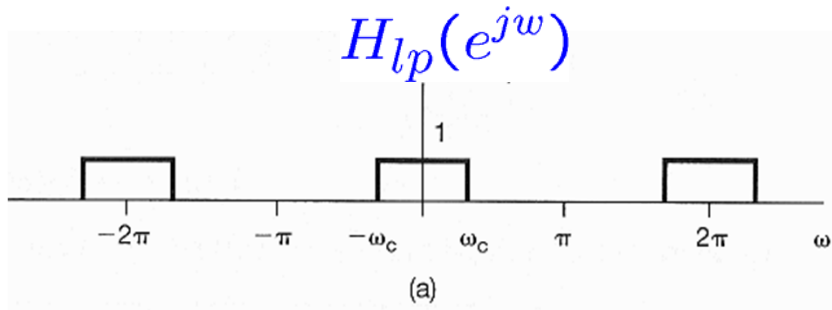
■ Time & Frequency Shifting:

ss4-31

$$\Rightarrow x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega - \omega_0)})$$

■ Example 5.7:

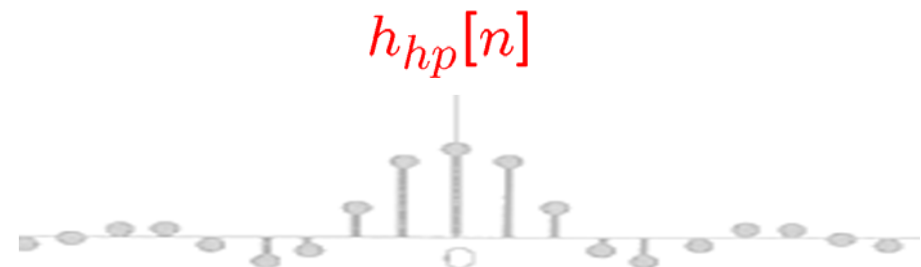
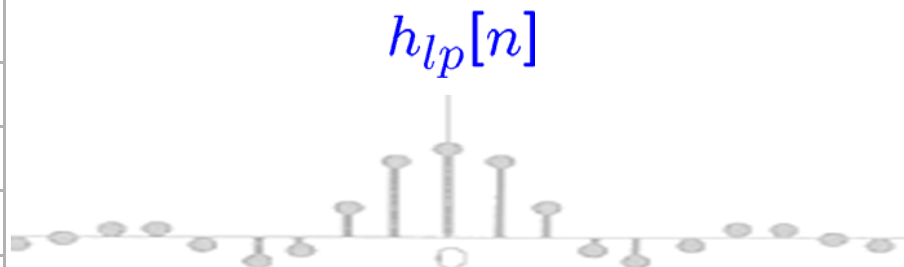


$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

$$\Rightarrow h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$

$$e^{j\pi n} = \cos(\pi n) + j \sin(\pi n)$$

$$= (-1)^n h_{lp}[n]$$



■ Conjugation & Conjugate Symmetry:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

ss4-34

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

•  $x[n] = x^*[n] \Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$

ss4-35

IF  $x[n]$  is real  $\Rightarrow X(e^{j\omega})$  is conjugate symmetric

▪ Conjugation & Conjugate Symmetry:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \quad x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

ss4-34

• IF  $x[n] = x^*[n]$  &  $x[-n] = x[n]$

$$\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega}) \quad \& \quad X(e^{-j\omega}) = X(e^{j\omega})$$

$$\Rightarrow X^*(e^{j\omega}) = X(e^{j\omega})$$

• IF  $x[n]$  is real & even  $\Rightarrow X(e^{j\omega})$  are real & even

ss4-36

• IF  $x[n]$  is real & odd  $\Rightarrow X(e^{j\omega})$  are purely imaginary & odd

• IF  $x[n] = x^*[n]$  &  $x[-n] = -x[n]$

$$\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega}) \quad \& \quad X(e^{-j\omega}) = -X(e^{j\omega})$$

$$\Rightarrow X^*(e^{j\omega}) = -X(e^{j\omega})$$

- Conjugation & Conjugate Symmetry:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

ss4-37

$$\mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(e^{j\omega})\}$$

$$\mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j \mathcal{I}m\{X(e^{j\omega})\}$$

■ Differencing & Accumulation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

ss4-40

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) e^{-j\omega} X(e^{j\omega})$$

$$(1 - e^{-j\omega}) X(e^{j\omega})$$

$$\sum_{m=-\infty}^n x[m] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

dc or average value

$$y[n] = \sum_{m=-\infty}^n x[m] \Rightarrow y[n] - y[n-1] = x[n]$$

$$y[n-1] = \sum_{m=-\infty}^{n-1} x[m] \Rightarrow (1 - e^{-j\omega}) Y(e^{j\omega}) = X(e^{j\omega})$$



■ Differentiation in Frequency:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\frac{d}{d\omega} X(e^{j\omega}) = \frac{d}{d\omega} \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$\frac{1}{j} n x[n] \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} X(e^{j\omega})$$

$$= \sum_{n=-\infty}^{+\infty} (-jn) x[n] e^{-j\omega n}$$

$$n x[n] \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} X(e^{j\omega})$$

$$= (-j) \sum_{n=-\infty}^{+\infty} [n x[n]] e^{-j\omega n}$$

■ Time Reversal:

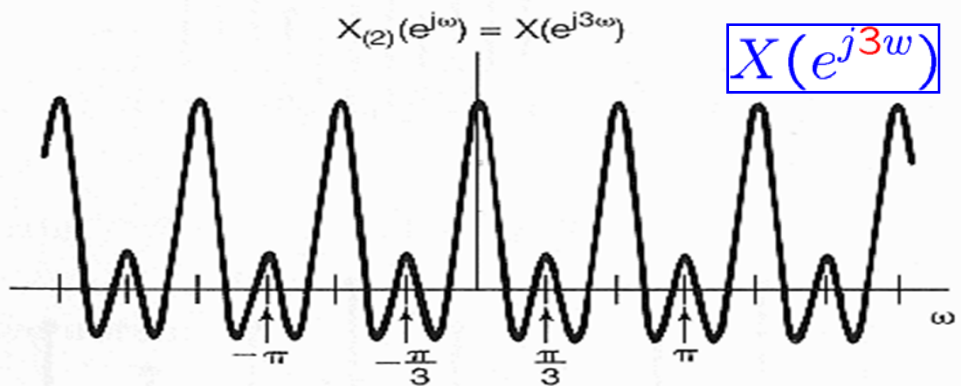
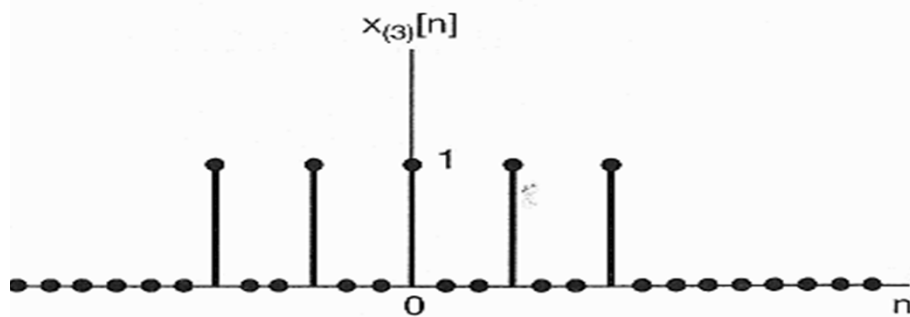
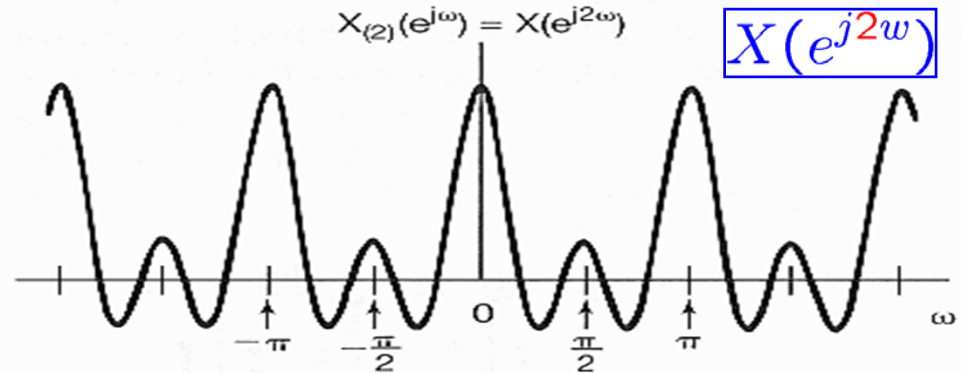
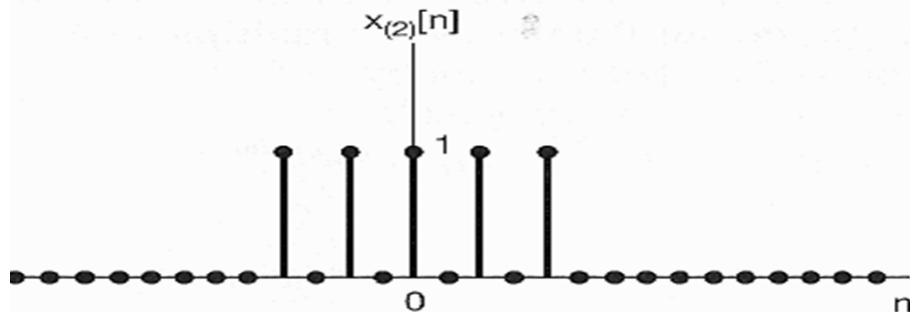
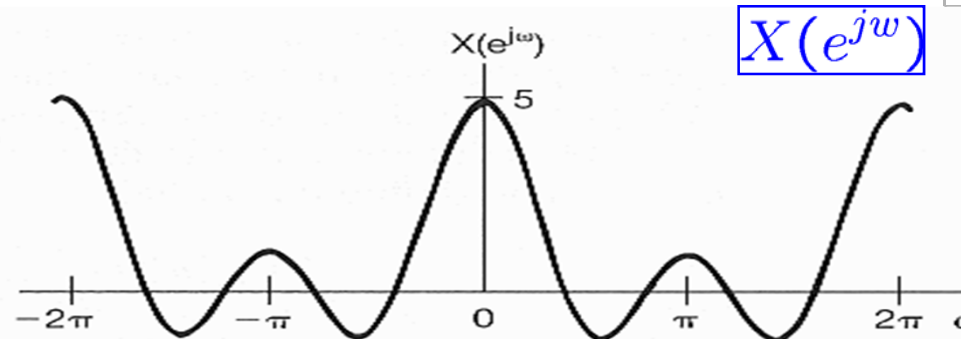
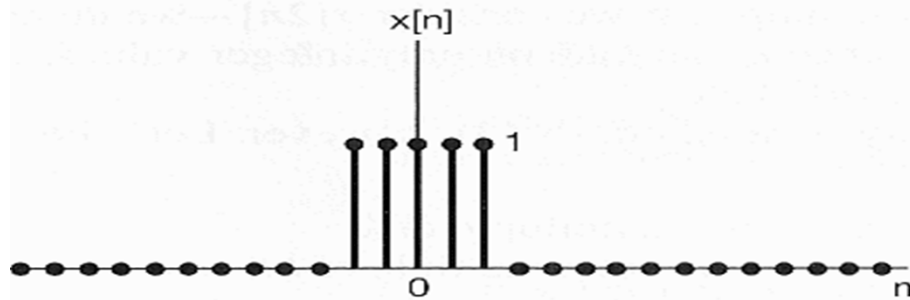
$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

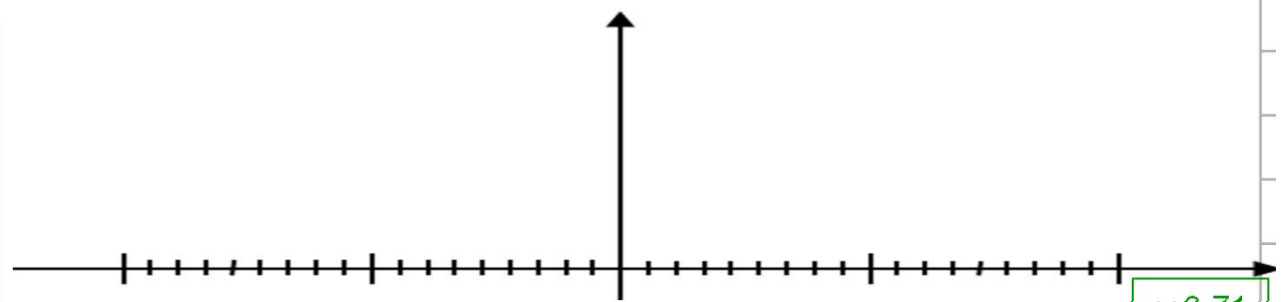
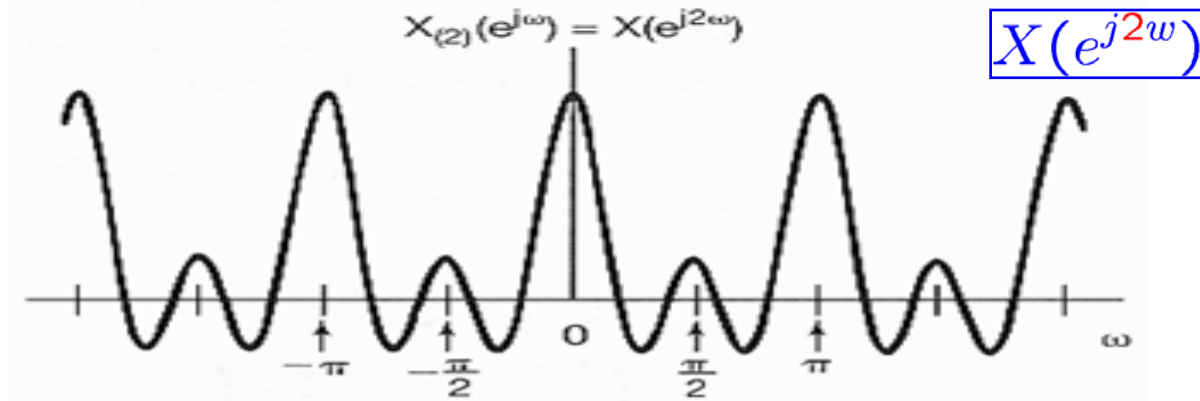
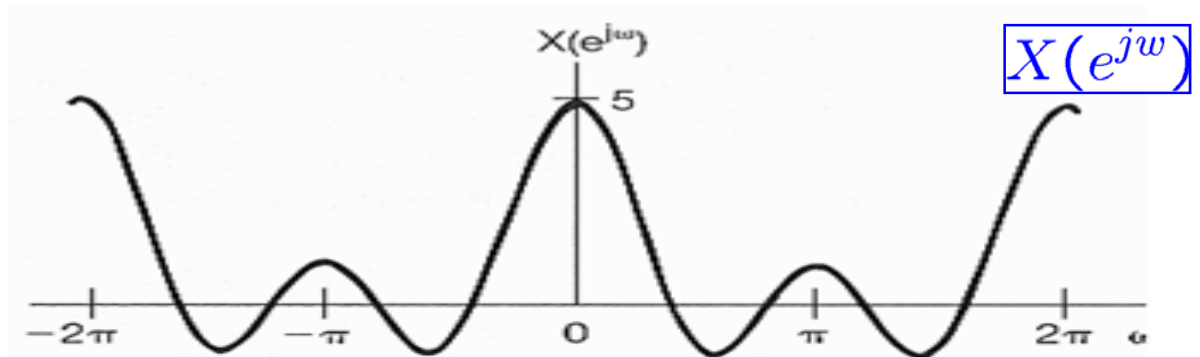
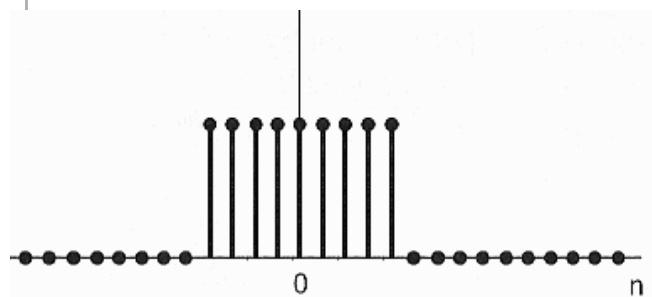
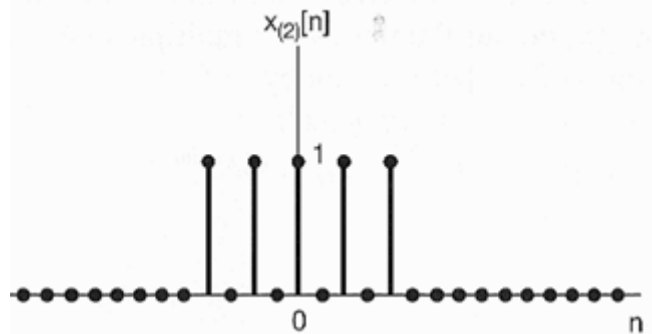
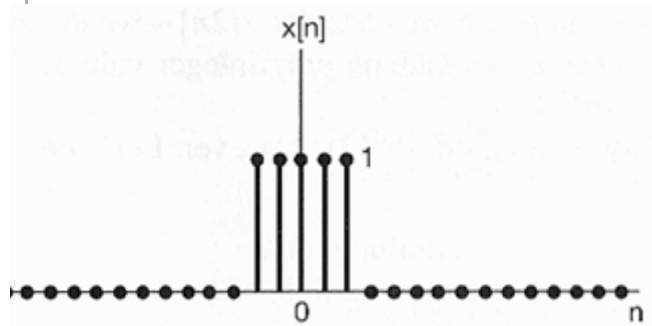
$$X(e^{j(-\omega)}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(-\omega)n}$$

- Time Expansion:  $\frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin(\omega/2)}$



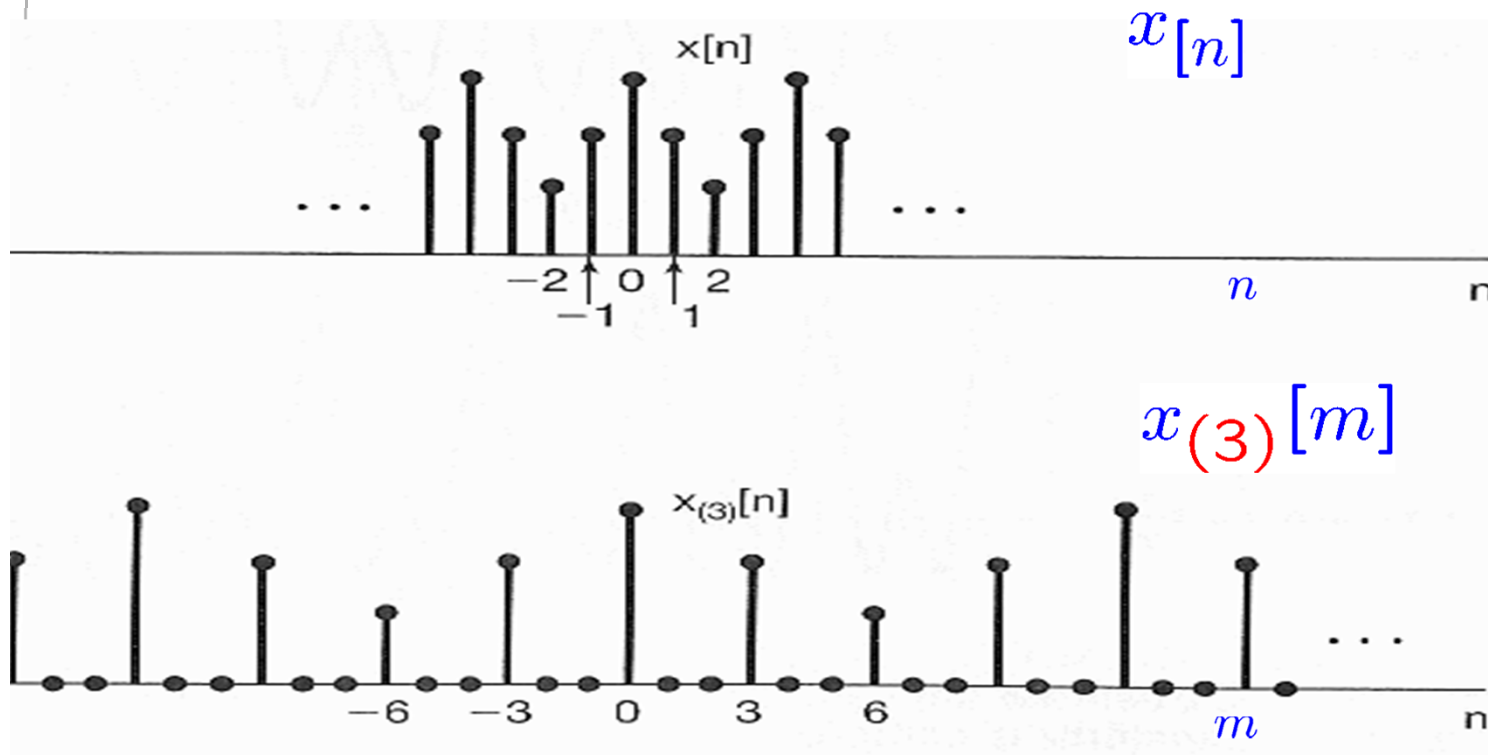
- Time Expansion:

$$\frac{\sin\left(\omega\left(N_1 + \frac{1}{2}\right)\right)}{\sin(\omega/2)}$$



■ Time Expansion:

$$x_{(k)}[m] = \begin{cases} x[m/k], & \text{if } m \text{ is a multiple of } k \\ 0, & \text{if } m \text{ is not a multiple of } k \end{cases}$$



■ Time Expansion:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$\Rightarrow X_{(k)}(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x_{(k)}[m]e^{-j\omega m}$$

$$m = nk$$

$$= \sum_{n=-\infty}^{+\infty} x_{(k)}[nk]e^{-j\omega nk}$$

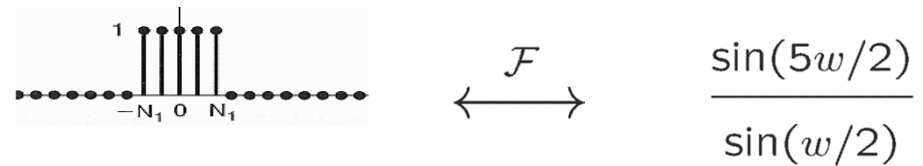
$$= \sum_{n=-\infty}^{+\infty} x[n]e^{-j(k\omega)n}$$

$$x_{(k)}[nk] = x[n]$$

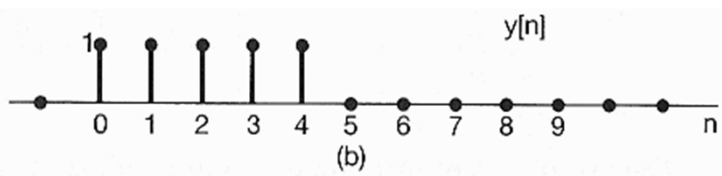
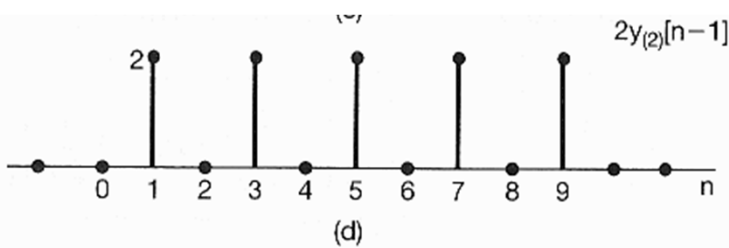
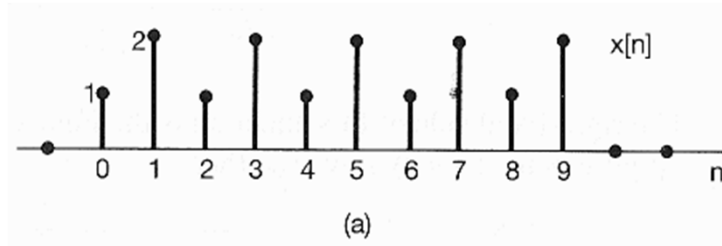
$$= X(e^{jk\omega})$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X(e^{jk\omega})$$

## Example 5.9:



$$\begin{matrix} 1 \\ \uparrow \uparrow \uparrow \uparrow \uparrow \\ \bullet \bullet \bullet \bullet \bullet \\ \leftarrow -N_1 \quad 0 \quad N_1 \rightarrow \end{matrix} \quad \xleftrightarrow{\mathcal{F}} \quad \frac{\sin(5w/2)}{\sin(w/2)}$$



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n - 1]$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} e^{-j4w} \frac{\sin(5w)}{\sin(w)}$$

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2e^{-jw} e^{-j4w} \frac{\sin(5w)}{\sin(w)}$$

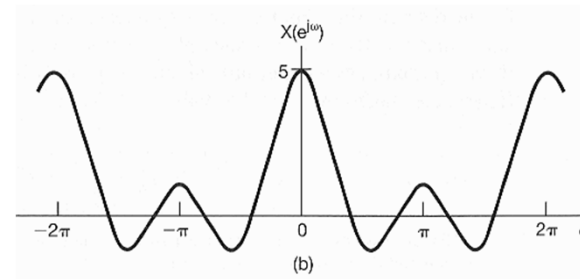
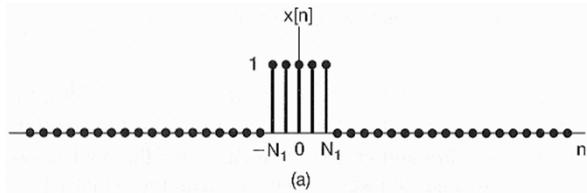
$$Y(e^{jw}) = (e^{-j2w}) \frac{\sin(5w/2)}{\sin(w/2)}$$

$$X(e^{jw}) = (1 + 2e^{-jw}) \cdot e^{-j4w} \cdot \frac{\sin(5w)}{\sin(w)}$$

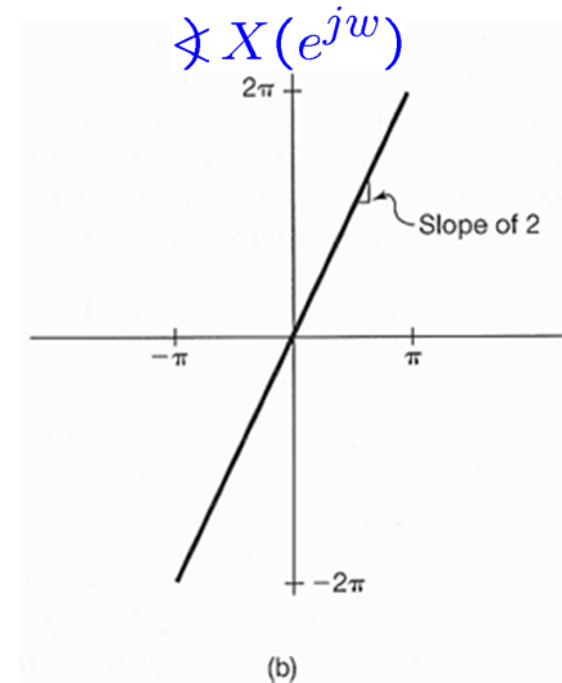
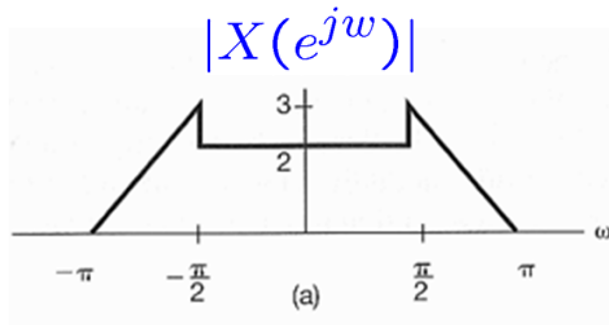
■ Parseval's relation:

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$$



■ Example 5.10:



- $x[n]$  is **periodic**, **real**, **even**,  
and/or of **finite energy**?

→  $X(e^{j\omega}) \neq 0$

→ even magnitude, odd phase

→  $X(e^{j\omega})$  is NOT real

→  $X(e^{j\omega})$  is finite

⇒  $x[n]$  is **NOT** periodic

⇒  $x[n]$  is **real**

⇒  $x[n]$  is **NOT** even

⇒  $x[n]$  is **finite**

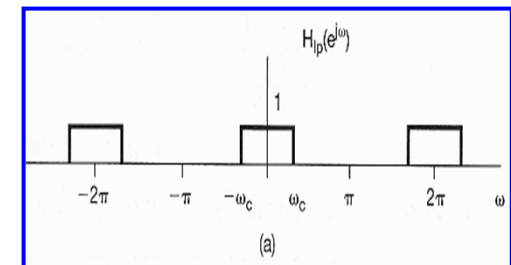
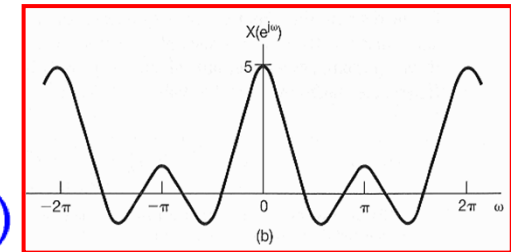
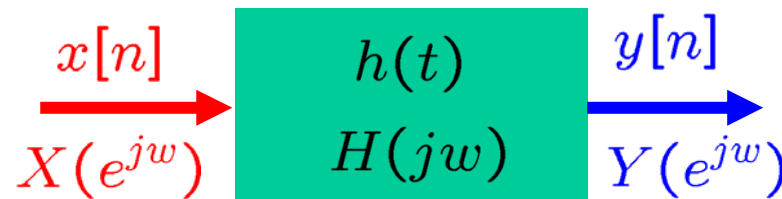


- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- The Fourier Transform for **Periodic** Signals
- **Properties** of Discrete-Time Fourier Transform
- **The Convolution Property**
- **The Multiplication Property**
- Duality
- Systems Characterized by  
Linear Constant-Coefficient Difference Equations

## Convolution Property:

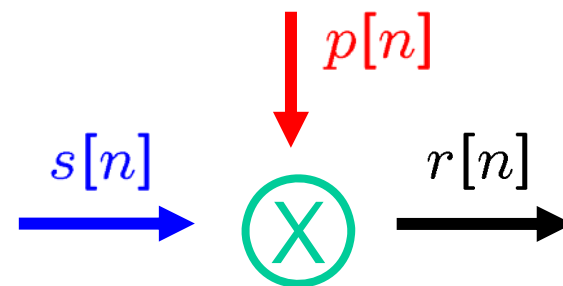
$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

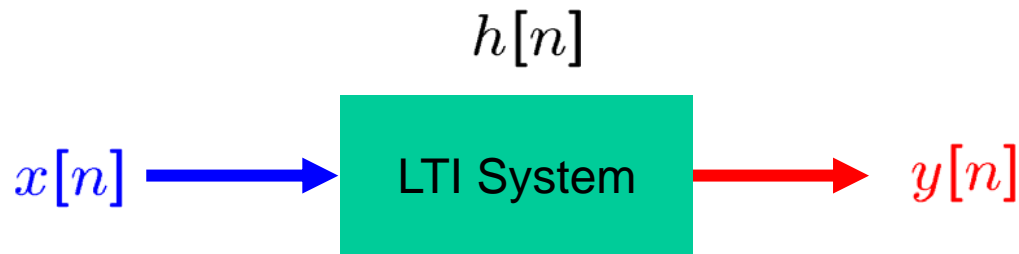


ss4-52

## Multiplication Property:



$$r[n] = s[n]p[n] \xleftrightarrow{\mathcal{F}} R(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} S(e^{j\theta})P(e^{j(\omega-\theta)})d\theta$$

■ Example 5.11:

$$h[n] = \delta[n - n_0]$$

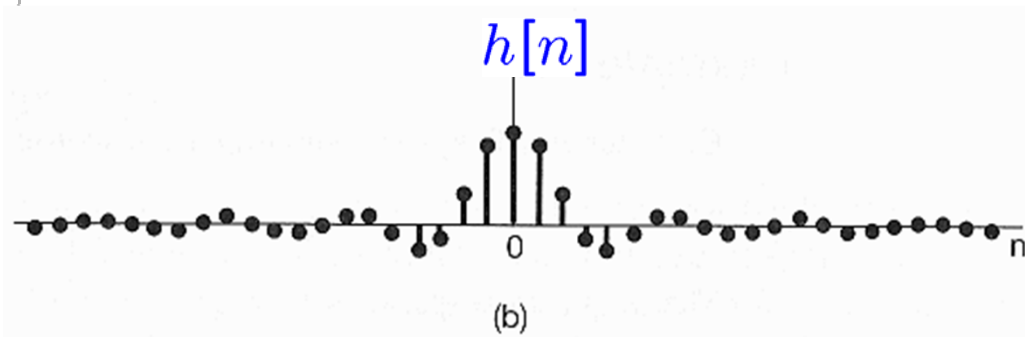
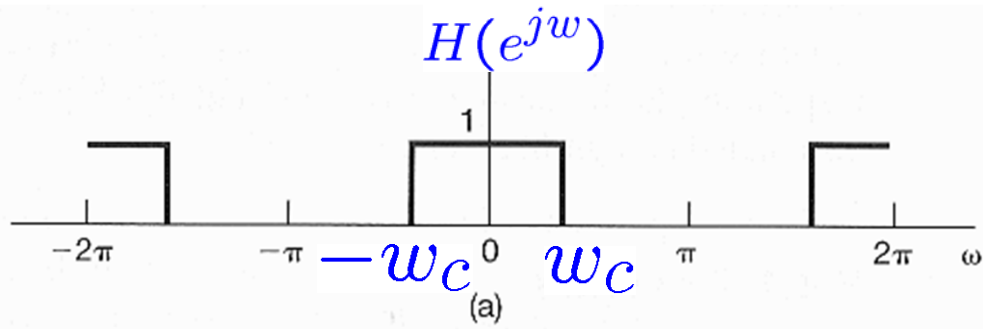
$$\Rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$= e^{-j\omega n_0} X(e^{j\omega}) \quad \Rightarrow y[n] = x[n - n_0]$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

## Example 5.12:

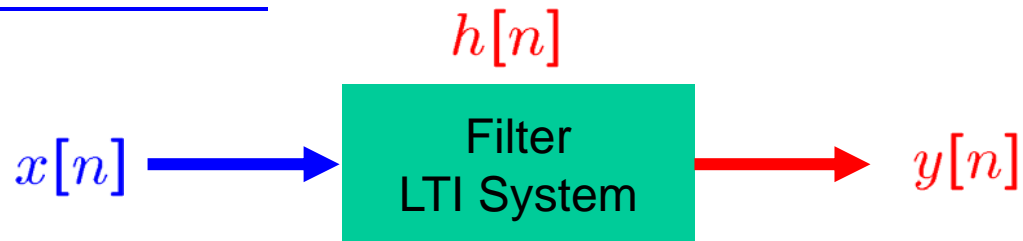


$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$

- not causal
- oscillatory

■ Example 5.13:

$$h[n] = a^n u[n], \quad |a| < 1 \quad \Rightarrow \quad H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x[n] = b^n u[n], \quad |b| < 1 \quad \Rightarrow \quad X(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}}$$

$$\begin{aligned} \Rightarrow Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{1}{1 - ae^{-j\omega}} \frac{1}{1 - be^{-j\omega}} \end{aligned}$$

■ Example 5.13:

$$\text{if } a \neq b \quad Y(e^{j\omega}) = \left[ \left( \frac{a}{a-b} \right) \frac{1}{1 - ae^{-j\omega}} + \left( \frac{-b}{a-b} \right) \frac{1}{1 - be^{-j\omega}} \right]$$

$$\Rightarrow y[n] = \left( \frac{a}{a-b} \right) a^n u[n] - \left( \frac{b}{a-b} \right) b^n u[n]$$

$$\text{if } a = b \quad Y(j\omega) = \left( \frac{1}{1 - ae^{-j\omega}} \right)^2 = \frac{1}{a} j e^{j\omega} \frac{d}{d\omega} \left( \frac{1}{1 - ae^{-j\omega}} \right)$$

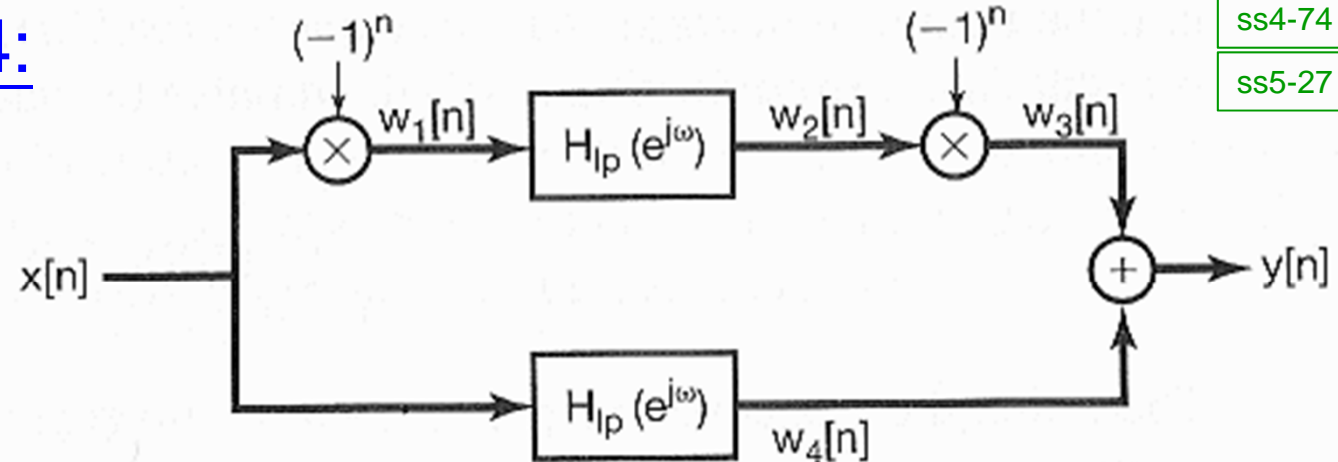
$$\text{since } a^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - ae^{-j\omega}}$$

$$\text{and } n a^n u[n] \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} \left[ \frac{1}{1 - ae^{-j\omega}} \right]$$

$$\text{and } (n+1) a^{n+1} u[n+1] \xleftrightarrow{\mathcal{F}} j e^{j\omega} \frac{d}{d\omega} \left[ \frac{1}{1 - ae^{-j\omega}} \right]$$

$$\Rightarrow y[n] = (n+1) a^n u[n+1]$$

■ Example 5.14:



$$w_1[n] = (-1)^n x[n] = e^{j\pi n} x[n]$$

(a)

$$\Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j\omega})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega}) W_1(e^{j\omega}) = H_{lp}(e^{j\omega}) X(e^{j(\omega-\pi)})$$

$$w_3[n] = (-1)^n w_2[n] = e^{j\pi n} w_2[n]$$

$$\Rightarrow W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = H_{lp}(e^{j(\omega-\pi)}) X(e^{j(\omega-2\pi)})$$

$$= H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega})$$

■ Example 5.14:

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega})$$

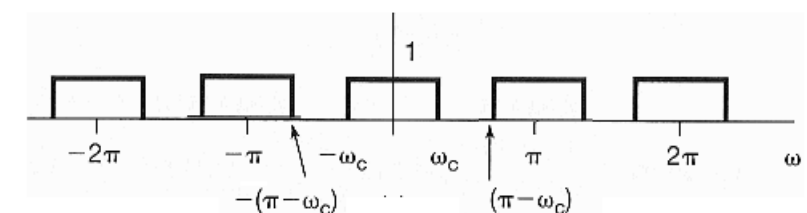
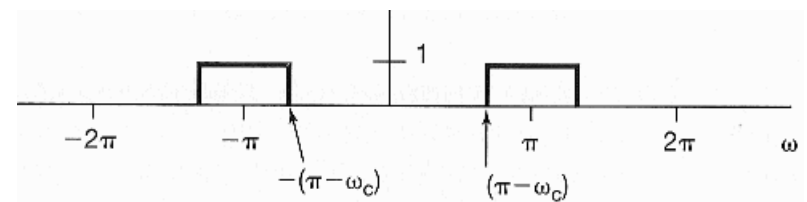
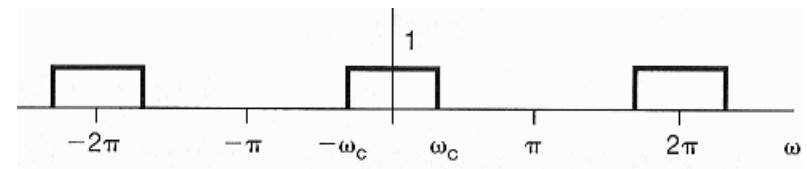
$$= H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega}) + H_{lp}(e^{j\omega}) X(e^{j\omega})$$

$$= [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})] X(e^{j\omega})$$

$$H(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})$$

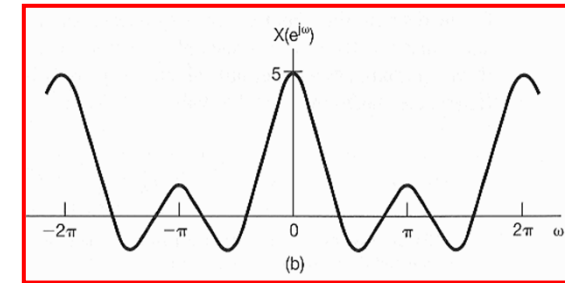
highpass + lowpass

→ bandstop



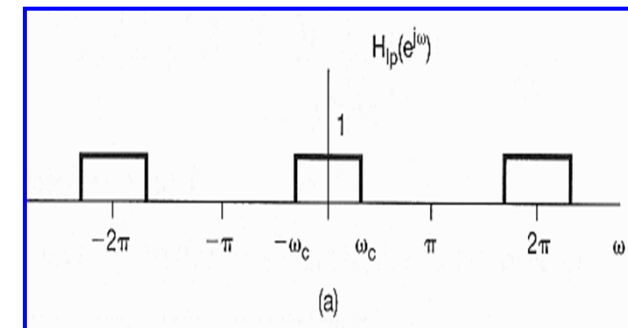


## Convolution Property:



$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



## Multiplication Property:

ss4-67

$$r[n] = s[n]p[n] \xleftrightarrow{\mathcal{F}} R(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} S(e^{j\theta})P(e^{j(\omega-\theta)})d\theta$$

# Multiplication Property

$$r[n] = s[n]p[n]$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$
$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

ss4-68

$$\Rightarrow R(e^{jw}) = \sum_{n=-\infty}^{+\infty} r[n] e^{-jwn}$$

$$= \sum_{n=-\infty}^{+\infty} s[n] p[n] e^{-jwn}$$

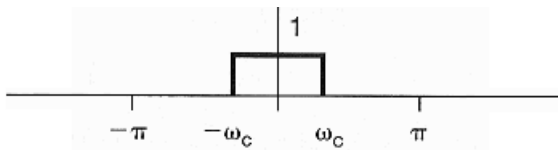
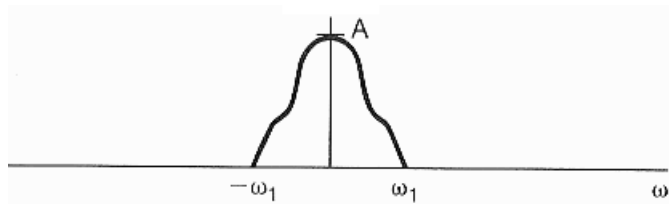
$$= \sum_{n=-\infty}^{+\infty} s[n] \left\{ \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) e^{j\theta n} d\theta \right\} e^{-jwn}$$

$$= \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) \left[ \sum_{n=-\infty}^{+\infty} s[n] e^{-j(w-\theta)n} \right] d\theta$$

$$= \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) S(e^{j(w-\theta)}) d\theta = \frac{1}{2\pi} \int_{2\pi} P(e^{j(w-\theta)}) S(e^{j\theta}) d\theta$$

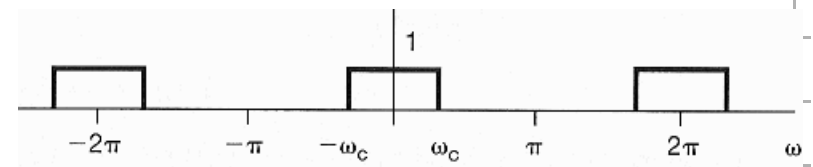
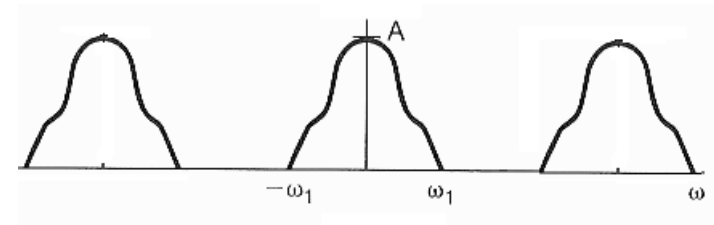
## aperiodic convolution

$$y(\theta) = \int_{-\infty}^{+\infty} x(\tau)h(\theta - \tau)d\tau$$



## periodic convolution

$$y(\theta) = \int_{\Theta} x(\tau)h(\theta - \tau)d\tau$$

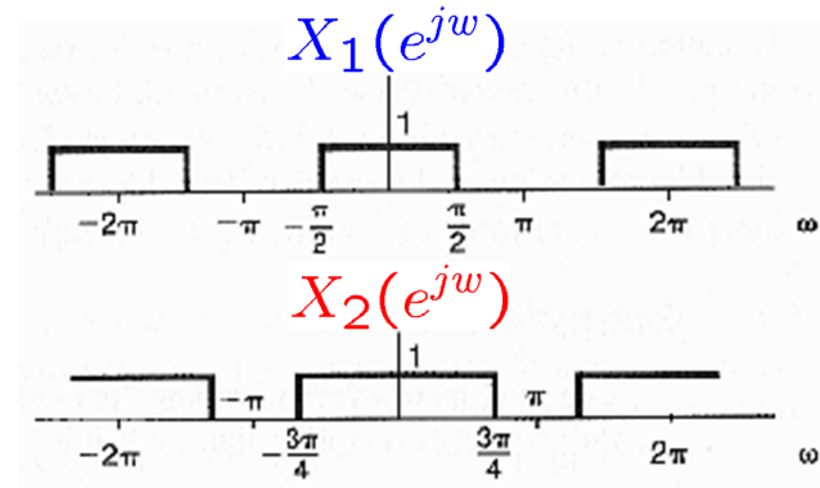


■ Example 5.15:

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(\frac{\pi}{2}n)}{\pi n}$$

$$x_2[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$$



$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

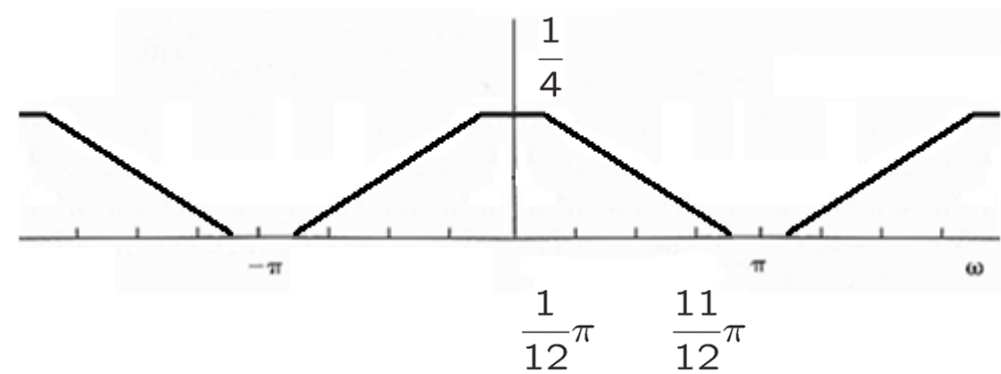
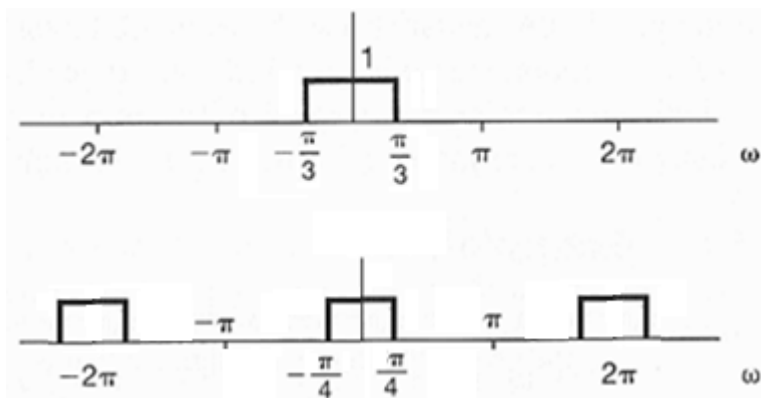
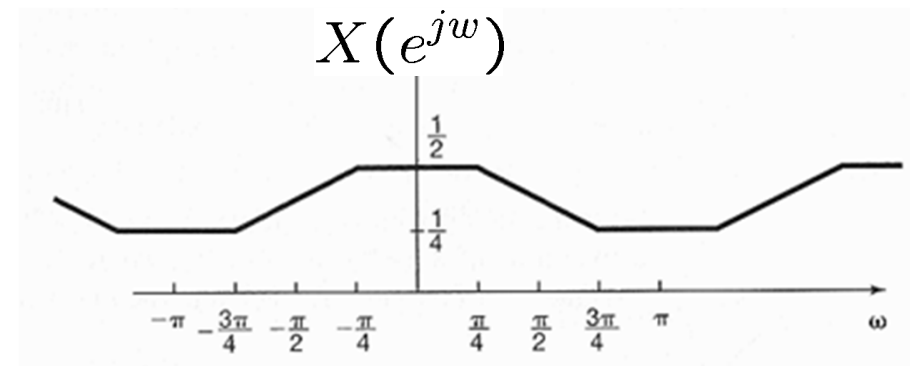
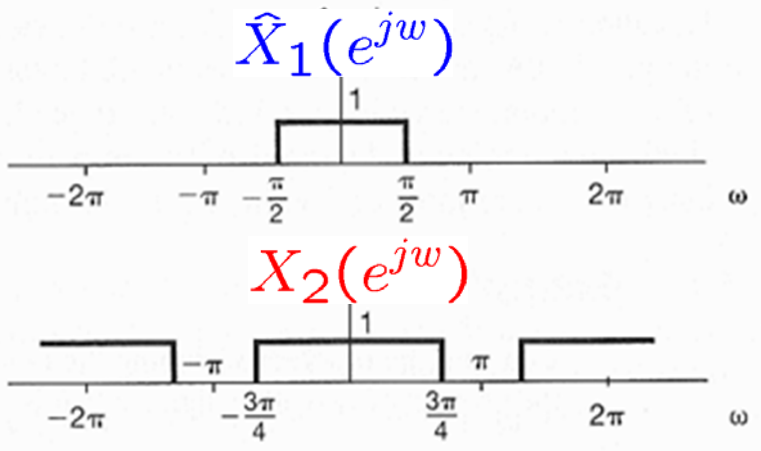
$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), & \text{for } -\pi < \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

## Example 5.15:

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta$$



**TABLE 5.1** PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period $2\pi$
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$
			$+ \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re\{X(e^{j\omega})\} = \Re\{X(e^{-j\omega})\} \\ \Im\{X(e^{j\omega})\} = -\Im\{X(e^{-j\omega})\} \\  X(e^{j\omega})  =  X(e^{-j\omega})  \\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}\{x[n]\}$ [ $x[n]$ real] $x_o[n] = \mathcal{O}\{x[n]\}$ [ $x[n]$ real]	$\Re\{X(e^{j\omega})\}$ $j\Im\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		
		$\sum_{n=-\infty}^{+\infty}  x[n] ^2 = \frac{1}{2\pi} \int_{2\pi}  X(e^{j\omega}) ^2 d\omega$	

**TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS**

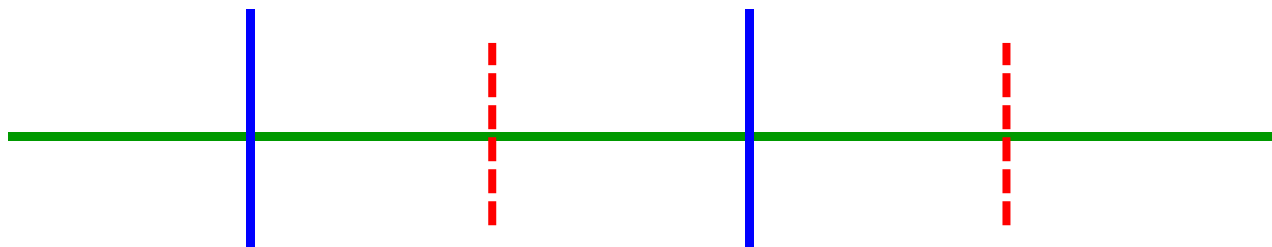
Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k \in \langle N \rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k$
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, &  n  \leq N_1 \\ 0, & N_1 <  n  \leq N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, k = 0, \pm N, \pm 2N, \dots$
$\sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$2\pi \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = 1$ for all $k$

$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
<p>Periodic square wave</p> $x[n] = \begin{cases} 1, &  n  \leq N_1 \\ 0, & N_1 <  n  \leq N/2 \end{cases}$ <p>and</p> $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all $k$
$a^n u[n], \quad  a  < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] \begin{cases} 1, &  n  \leq N_1 \\ 0, &  n  > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \text{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq  \omega  \leq W \\ 0, & W <  \omega  \leq \pi \end{cases}$ $X(\omega) \text{ periodic with period } 2\pi$	—
$\delta[n]$	<b>1</b>	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad  a  < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad  a  < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—



- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- The Fourier Transform for **Periodic** Signals
- **Properties** of Discrete-Time Fourier Transform
- The **Convolution** Property
- The **Multiplication** Property
- **Duality**
- Systems Characterized by  
Linear Constant-Coefficient Difference Equations

ss4-47



## DT Fourier Series Pair of Periodic Signals:

- $x[n] \xleftrightarrow{\mathcal{FS}} a_k$  : DT Fourier series pair

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

$$\text{IF } f[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk(2\pi/N)n}$$

$$g[n] \xleftrightarrow{\mathcal{FS}} f[k]$$

$$\text{LET } k = n, n = -k$$

$$f[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N} g[-k]$$

$$f[n] = \sum_{k=\langle N \rangle} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}$$

$$a_k := \frac{1}{N} x[-n]$$

## ■ Duality in DT Fourier Series:

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

$$x[n-n_0] \xleftrightarrow{\mathcal{FS}} a_k e^{-jk(2\pi/N)n_0}$$

$$e^{+jm(2\pi/N)n} x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m}$$

$$\sum_{r=\langle N \rangle} x[r] y[n-r] \xleftrightarrow{\mathcal{FS}} N a_k b_k$$

$$x[n] y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=\langle N \rangle} a_l b_{k-l}$$

■ Duality between DT-FT & CT-FS:

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$x(t) \xleftrightarrow{FS} a_k$$

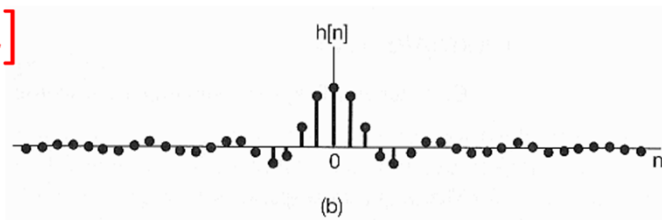
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

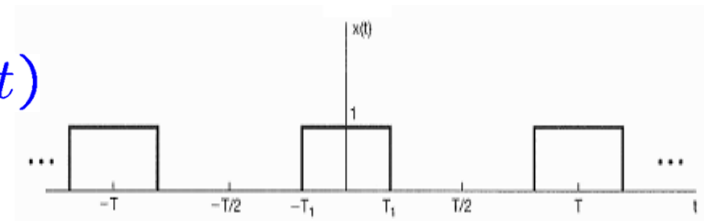
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

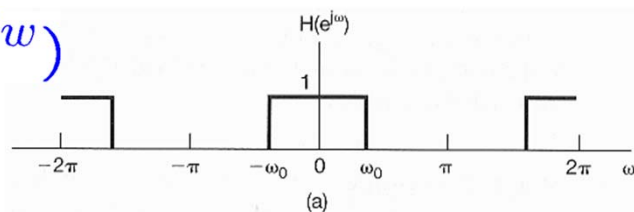
$x[n]$



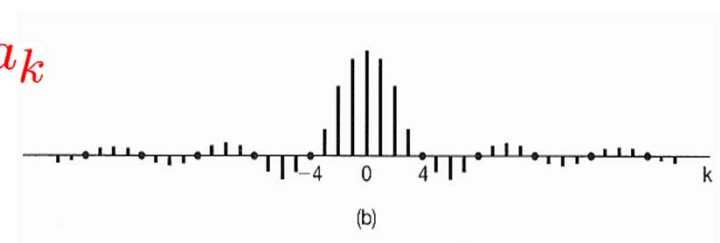
$x(t)$



$X(e^{j\omega})$

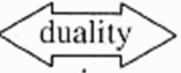
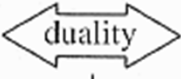


$a_k$



**TABLE 5.3** SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	Frequency domain	Time domain	Frequency domain
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$ continuous frequency periodic in frequency



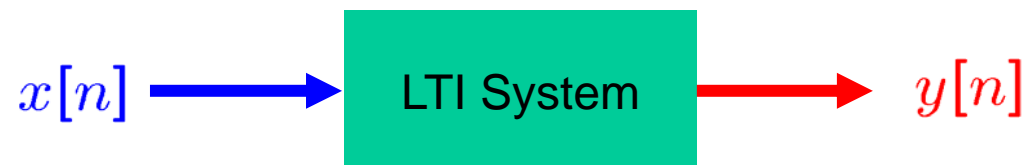
- Representation of **Aperiodic** Signals:  
the Discrete-Time Fourier Transform
- The Fourier Transform for **Periodic** Signals
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- The **Convolution** Property
- The **Multiplication** Property
- **Duality**
- **Systems** Characterized by  
Linear Constant-Coefficient Difference Equations

- A useful class of DT LTI systems:

$$a_0y[n] + a_1y[n-1] + \cdots + a_{N-1}y[n-N+1] + a_Ny[n-N]$$

$$= b_0x[n] + b_1x[n-1] + \cdots + b_{M-1}x[n-M+1] + b_Mx[n-M]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$



$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$x[n - n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X(e^{j\omega})$$

ss4-79

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k]$$

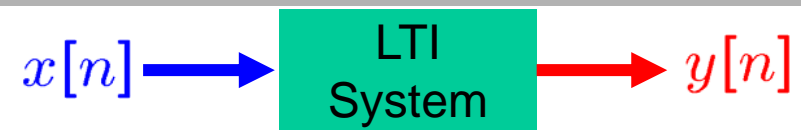
$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k]$$

$$\sum_{k=0}^N a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^M b_k e^{-jk\omega} X(e^{j\omega})$$

$$\begin{aligned} \Rightarrow H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^M b_k e^{-jk\omega}}{\sum_{k=0}^N a_k e^{-jk\omega}} \\ &= \frac{b_0 + b_1 e^{-j\omega} + \dots + b_M e^{-jM\omega}}{a_0 + a_1 e^{-j\omega} + \dots + a_N e^{-jN\omega}} \end{aligned}$$



■ Examples 5.18 & 5.19:



$$|a| < 1$$

ss4-81

$$y[n] - ay[n-1] = x[n] \quad \Rightarrow \quad H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$Y(\cdot) - e^{-j\omega}Y(\cdot) \quad \Rightarrow \quad h[n] = a^n u[n]$$

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

$$\Rightarrow H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

$$= \frac{4}{(1 - \frac{1}{2}e^{-j\omega})} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})}$$

$$\Rightarrow h[n] = 4 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{4}\right)^n u[n]$$

■ Example 5.20:

$$h[n] \xleftrightarrow{\mathcal{F}} H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$



$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = x[n] * h[n]$$

$$= \left[ \frac{1}{1 - \frac{1}{4}e^{-j\omega}} \right] \left[ \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} \right]$$

$$= \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})^2}$$

$$= \frac{8}{(1 - \frac{1}{2}e^{-j\omega})} - \frac{4}{(1 - \frac{1}{4}e^{-j\omega})} - \frac{2}{(1 - \frac{1}{4}e^{-j\omega})^2}$$
ss5-45

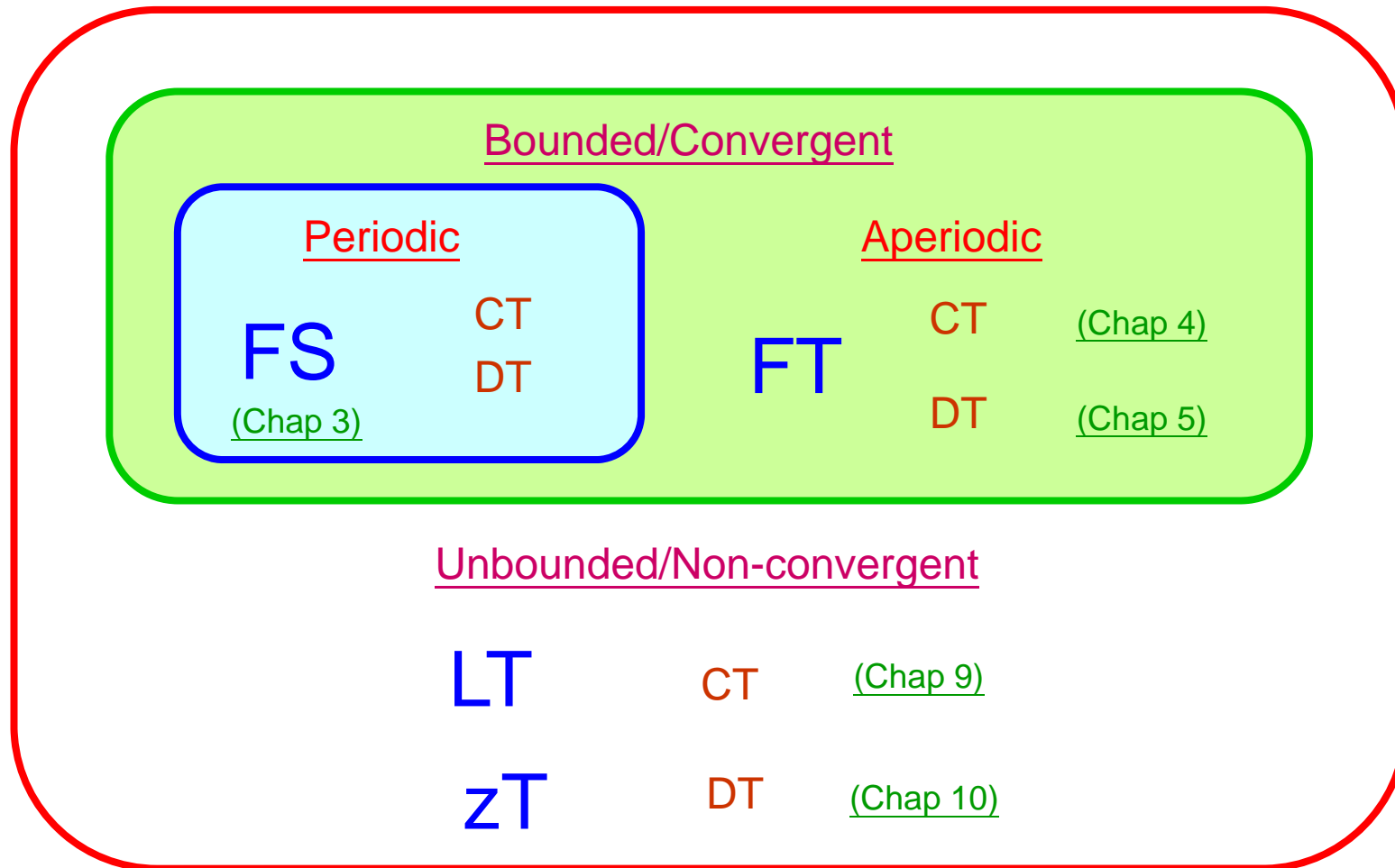
$$\Rightarrow y[n] = 8 \left(\frac{1}{2}\right)^n u[n] - 4 \left(\frac{1}{4}\right)^n u[n] - 2(n+1) \left(\frac{1}{4}\right)^n u[n+1]$$

$$= \left\{ 8 \left(\frac{1}{2}\right)^n - 4 \left(\frac{1}{4}\right)^n - 2(n+1) \left(\frac{1}{4}\right)^n \right\} u[n]$$

- Representation of Aperiodic Signals: the DT FT
- The FT for Periodic Signals
- Properties of the DT FT
  - Linearity
  - Conjugation
  - Convolution
  - Differencing in Time
  - Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals & for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals
  - Parseval's Relation for Aperiodic Signals
  - Time Shifting
  - Time Reversal
  - Multiplication
  - Accumulation
  - Frequency Shifting
  - Time Expansion
  - Differentiation in Frequency
- The Convolution Property
- The Multiplication Property
- Duality
- Systems Characterized by Linear Constant-Coefficient Difference Equations

Signals & Systems [\(Chap 1\)](#)

LTI & Convolution [\(Chap 2\)](#)



Time-Frequency [\(Chap 6\)](#)

Communication [\(Chap 8\)](#)

CT-DT [\(Chap 7\)](#)

Control [\(Chap 11\)](#)

- In discrete time,  
what kind of signals are **low frequency** or **high frequency**?
  - Chap 1, pages 47-49
  - Chap 5, pages 7-9
  
- **Time expansion** and **double the width** in time domain
  - Chap 5, pages 33-34
  - Chap 3, pages 70-71
  - Chap 5, page 11
  
- **Frequency shaping filter** on **low-freq** and **high-freq** signals
  - Chap 3, pages 92-95