Spring 2012

信號與系統
Signals and Systems

Chapter SS-3
Fourier Series Representation of Periodic Signals

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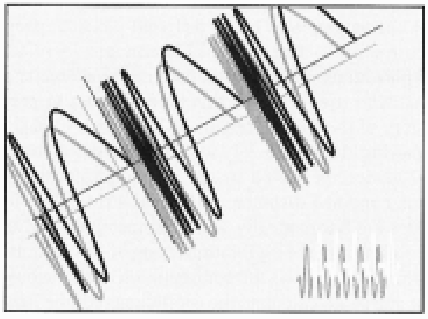
NTU-EE
Feb12 – Jun12

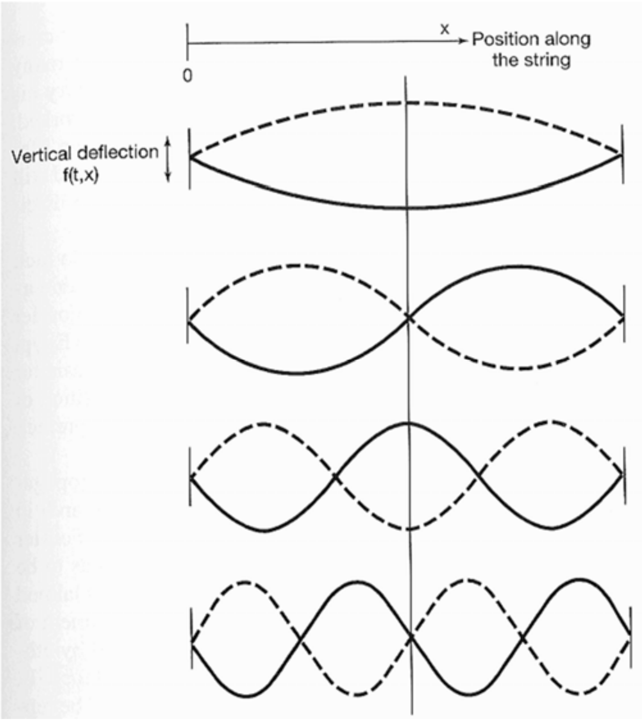


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Outline

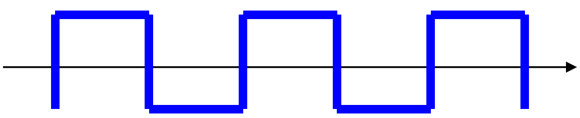
- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- L. Euler's study on the motion of a vibrating string in 1748

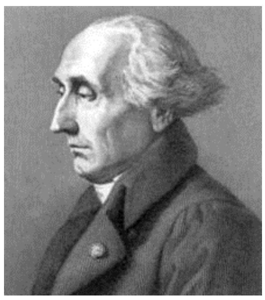


Leonhard Euler
1707-1783
Born in Switzerland
Photo from wikipedia

- L. Euler showed (in 1748)
  - The configuration of a vibrating string at some point in time is a linear combination of these normal modes
- D. Bernoulli argued (in 1753)
  - All physical motions of a string could be represented by linear combinations of normal modes
  - But, he did not pursue this mathematically
- J.L. Lagrange strongly criticized (in 1759)
  - The use of trigonometric series in examination of vibrating strings
  - Impossible to represent signals with corners using trigonometric series



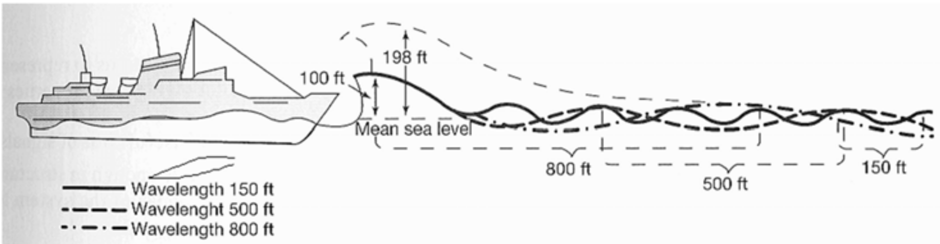
Daniel Bernoulli
1700-1782
Born in Dutch
Photo from wikipedia



Joseph-Louis Lagrange
1736-1813
Born in Italy
Photo from wikipedia

A Historical Perspective

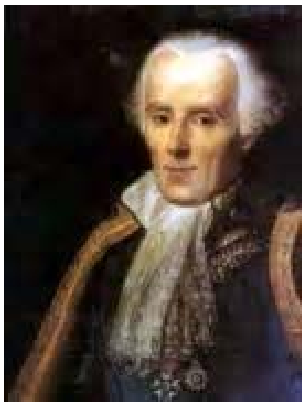
- In 1807, Jean Baptiste Joseph Fourier
  - Submitted a paper of using trigonometric series to represent "any" periodic signal
  - It is examined by S.F. Lacroix, G. Monge, P.S. de Laplace, and J.L. Lagrange,
  - But Lagrange rejected it!
- In 1822, Fourier published a book "Theorie analytique de la chaleur"
  - "The Analytical Theory of Heat"



Jean Baptiste Joseph Fourier
1768-1830
Born in France
Photo from wikipedia

A Historical Perspective

Figure 1.2: A medallion by David d'Angers, the only known portrait of Lacroix, made two years prior to his death. [Académie des Sciences de l'Institut de France]



Sylvestre François de Lacroix
1765-1843
Born in France
Photo from
A short biography of Silvestre-François Lacroix
In Science Networks. Historical Studies, V35,
Lacroix and the Calculus, Birkhäuser Basel
2008, ISBN 978-3-7643-8638-2

Gaspard Monge, Comte de Péluse
1746-1818
Born in France
Photo from wikipedia

Pierre-Simon, Marquis de Laplace
1749-1827
Born in France
Photo from wikipedia

A Historical Perspective

- Fourier's main contributions:
  - Studied vibration, heat diffusion, etc.
  - Found series of harmonically related sinusoids to be useful in representing the temperature distribution through a body
  - Claimed that "any" periodic signal could be represented by such a series (i.e., Fourier series discussed in Chap 3)
  - Obtained a representation for aperiodic signals (i.e., Fourier integral or transform discussed in Chap 4 & 5)
  - (Fourier did not actually contribute to the mathematical theory of Fourier series)

A Historical Perspective

- Impact from Fourier's work:
  - Theory of integration, point-set topology, eigenfunction expansions, etc.
  - Motion of planets, periodic behavior of the earth's climate, wave in the ocean, radio & television stations
  - Harmonic time series in the 18th & 19th centuries
    - > Gauss etc. on discrete-time signals and systems
  - Faster Fourier transform (FFT) in the mid-1960s
    - > Cooley (IBM) & Tukey (Princeton) reinvented in 1965
    - > Can be found in Gauss's notebooks (in 1805)

Carl Friedrich Gauss (Gauß)
1777-1855
Born in Germany
Photo from wikipedia

James W. Cooley & John W. Tukey (1965):
"An algorithm for the machine calculation of complex Fourier series",
Math. Comput. 19, 297–301.

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3) CT DT

Aperiodic

FT CT (Chap 4) DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)

Outline

$\sum \infty$ $\delta(t)$ $\delta[n]$

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- **Basic Idea:**
  - To represent signals as linear combinations of basic signals

$\phi_i(t)$

- **Key Properties:**

1. The set of basic signals can be used to construct a broad and useful class of signals

$x(t)$

2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signals constructed as linear combination of basic signals

$h(t)$

$x(t) \rightarrow$ LTI $\rightarrow y(t)$

- **One of Choices:**
  - The set of complex exponential signals

$$\begin{cases} \text{signals of form } e^{st} \text{ in CT} & t \\ \text{signals of form } z^{n} \text{ in DT} & n \end{cases}$$

$y = Av = \lambda v$

- **The Response of an LTI System:**

input $\rightarrow$ LTI $\rightarrow$ output

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$x(t)$ $h(t)$ $y(t)$

$$\begin{cases} \text{CT: } e^{st} \longrightarrow \boxed{H(s)}\, e^{st} \\ \text{DT: } z^{n} \longrightarrow \boxed{H(z)}\, z^{n} \end{cases}$$

eigenfunction
eigenvalue

Let $x(t) = e^{st}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$\Rightarrow y(t) = H(s)x(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Let $x[n] = z^n$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$\Rightarrow y[n] = H(z)x[n] = H(z)z^n$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

- Eigenfunctions and Superposition Properties:

$$e^{s_k t} \rightarrow \boxed{\text{LTI}} \rightarrow H(s_k)\ e^{s_k t}$$

$k = 1, 2, 3$

$$a_1 e^{s_1 t} \longrightarrow a_1 H(s_1)\ e^{s_1 t}$$
$$a_2 e^{s_2 t} \longrightarrow a_2 H(s_2)\ e^{s_2 t}$$
$$a_3 e^{s_3 t} \longrightarrow a_3 H(s_3)\ e^{s_3 t}$$

$$x(t) = a_1\ e^{s_1 t} + a_2\ e^{s_2 t} + a_3\ e^{s_3 t}$$

$$y(t) = a_1\ H(s_1)\ e^{s_1 t} + a_2\ H(s_2)\ e^{s_2 t} + a_3\ H(s_3)\ e^{s_3 t}$$

CT: $\Rightarrow x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$

DT: $\Rightarrow x[n] = \sum_k a_k z_k{}^n \longrightarrow y[n] = \sum_k a_k H(z_k) z_k{}^n$

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Fourier Series Representation of CT Periodic Signals

- Harmonically related complex exponentials

e^{st}

$s = jkw_0 t$

$$w_0 = \frac{2\pi}{T}$$

$$\phi_k(t) = e^{jkw_0 t} = e^{jk\left(\frac{2\pi}{T}\right)t}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

- The Fourier Series Representation:

$k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad k=2$

$$x(t) = \cdots a_{-2}\,\phi_{-2}(t) + a_{-1}\,\phi_{-1}(t) + a_0\,\phi_0(t) + a_1\,\phi_1(t) + a_2\,\phi_2(t) + \cdots$$

$$= \sum_{k=-\infty}^{+\infty} a_k\,\phi_k(t) = \sum_{k=-\infty}^{+\infty} a_k\,e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k\,e^{jk\left(\frac{2\pi}{T}\right)t}$$

$k = +1, -1$: the first harmonic components
or, the fundamental components

$k = +2, -2$: the second harmonic components

$\cdots$ etc.

- Example 3.2:

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk(2\pi)t}$$

$a_0 = 1$

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

k=0 k=1 cos 2πt k=2 cos 4πt

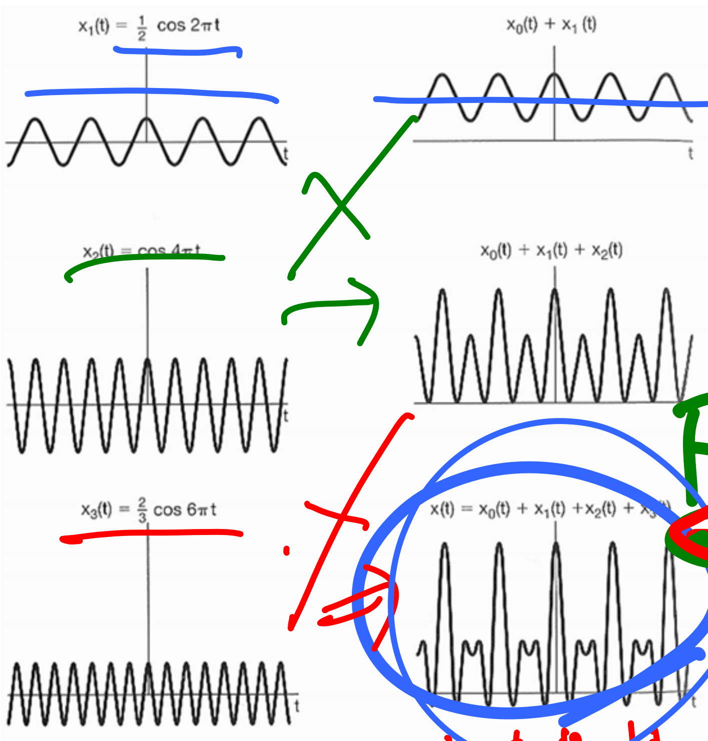
$$\Rightarrow x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})$$
$$+ \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

k=3 cos 6πt

$$\Rightarrow x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$
$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$



F.S.

$a_k = ?$

-3 -1 0 1 2

k

δ(t) ⇒

- **Procedure of Determining the Coefficients:**

$$w_0 = \frac{2\pi}{T}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$x(t) e^{-jnw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t}$$

$-n \in \mathbb{Z}$

k n

$$\boxed{\int_0^T x(t) e^{-jnw_0 t} dt} = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} e^{-jnw_0 t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \boxed{\left[\int_0^T e^{j(k-n)w_0 t} dt \right]}$$

$k=n$

n

$$\int_0^T e^{j(k-n)w_0 t} dt = \int_0^T \cos\left((k-n)w_0 t\right) dt + j \int_0^T \sin\left((k-n)w_0 t\right) dt$$

- **Procedure of Determining the Coefficients:**

$k \neq n \to 0$

0

$$\int_0^T e^{j(k-n)w_0 t} dt = \int_0^T \cos\left((k-n)w_0 t\right) dt + j \int_0^T \sin\left((k-n)w_0 t\right) dt$$

$k=n$

0

$k=n \Rightarrow 0$

$k \neq n \Rightarrow 0$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

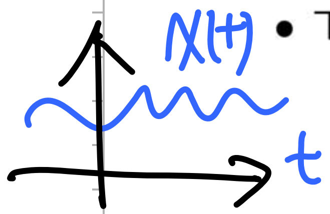
$$\Rightarrow \boxed{\int_0^T x(t) e^{-jnw_0 t} dt} = a_n T \quad \Rightarrow \quad a_n = \frac{1}{T} \int_0^T x(t) e^{-jnw_0 t} dt$$

$$\Rightarrow \boxed{a_k = \frac{1}{T} \int_0^T x(t) e^{-jkw_0 t} dt}$$

- Furthermore,

$$\int_T e^{j(k-n)w_0 t} dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases} \quad \Rightarrow \boxed{a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt}$$

In Summary:



- The synthesis equation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

- $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ CT Fouries series pair

- $\{a_k\}$: the Fourier series coefficients
  or the spectral coefficients of $x(t)$

- $a_0 = \frac{1}{T}\int_T x(t)dt$, the dc or constant component of $x(t)$

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Fourier Series of Real Periodic Signals:

- If $x(t)$ is real, then $x^*(t) = x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$(a+b)^* = (a^* + b^*)$

$(a \times b)^* = (a^* \times b^*)$

$$\Rightarrow x(t) = x(t)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

FS,

$$= \sum_{m=+\infty}^{-\infty} a_{-m}{}^* e^{jm\omega_0 t} \qquad m = -k$$

$$= \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t} \qquad k = m$$

$$\Rightarrow \boxed{a_{-k}^* = a_k} \quad \text{or,} \quad \boxed{a_k^* = a_{-k}}$$

Alternative Forms of the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$1 \rightarrow \infty$ $\quad -1 \rightarrow -\infty$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jkw_0t} + a_{-k} e^{-jkw_0t}\right]$$

$a_{-k} = a_k^*$

$$= a_0 + \sum_{k=1}^{\infty}\left[a_k e^{jkw_0t} + a_k^* e^{-jkw_0t}\right]$$

$a_k \quad e^* \quad a_k^* \quad e^*$

$$a_k e^{jkw_0t} + a_k^* e^{-jkw_0t} = (R+jI)(C+jS)+(R-jI)(C-jS)$$

$$= (RC-IS)+j\cancel{(RS+IC)}+(RC-IS)-j\cancel{(RS+IC)}$$

$$= 2(RC-IS)$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathcal{R}e\left\{a_k e^{jkw_0t}\right\}$$

Alternative Forms of the Fourier Series:

- If $a_k = A_k\, e^{j\theta_k}$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathcal{R}e\left\{A_k\, e^{j\theta_k}\, e^{jkw_0t}\right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathcal{R}e\left\{A_k\, e^{j(kw_0t+\theta_k)}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(kw_0t+\theta_k)$$

$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$

$\boxed{(a+jb)(c+jd) = (ac-bd) + j(ad+bc)}$

$\boxed{C(a+b) = C(a)C(b) - S(a)S(b)}$

- If $a_k = B_k + j\, C_k$

$$\Rightarrow x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathcal{R}e\left\{(B_k + j\,C_k)\, e^{jkw_0t}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty}\left[B_k\cos(kw_0t) - C_k\sin(kw_0t)\right]$$

- **Example 3.4:**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

e^{j5t}

$$e^{j\theta} = \cos(\theta) + j\sin(\theta); \quad \cos(\theta) = \frac{1}{2}(e^{j\theta}+e^{-j\theta}); \quad \sin(\theta) = \frac{1}{2j}(e^{j\theta}-e^{-j\theta})$$

$$x(t) = 1 + \sin w_0 t + 2\cos w_0 t + \cos\left(2w_0 t + \frac{\pi}{4}\right) \rightarrow a_k = \frac{1}{T}\int \ldots$$

$$\Rightarrow \; x(t) = 1 + \frac{1}{2j}\left[e^{jw_0 t} - e^{-jw_0 t}\right] + \left[e^{jw_0 t} + e^{-jw_0 t}\right] + \frac{1}{2}\left[e^{j(2w_0 t+\pi/4)} + e^{-j(2w_0 t+\pi/4)}\right]$$

$$\Rightarrow \; x(t) = \underbrace{1}_{a_0} + \underbrace{\left(1+\frac{1}{2j}\right)}_{a_1} e^{jw_0 t} + \underbrace{\left(1-\frac{1}{2j}\right)}_{a_{-1}} e^{-jw_0 t} + \underbrace{\left(\frac{1}{2}e^{j(\pi/4)}\right)}_{a_2} e^{j2w_0 t} + \underbrace{\left(\frac{1}{2}e^{-j(\pi/4)}\right)}_{a_{-2}} e^{-j2w_0 t}$$

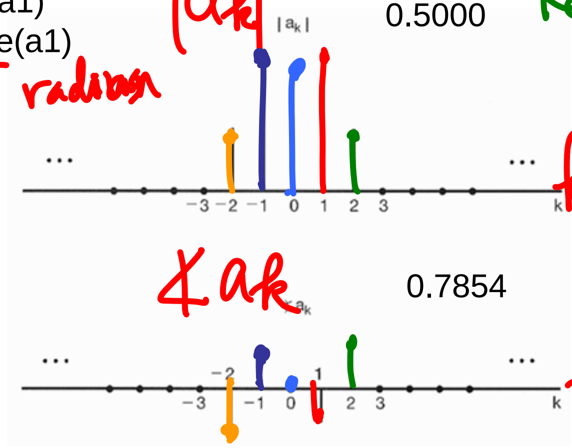
- **Example 3.4:**

$$\Rightarrow \begin{cases} a_0 &= 1, \\ a_1 &= \left(1+\frac{1}{2j}\right) = 1 - \frac{1}{2}j, \\ a_{-1} &= \left(1-\frac{1}{2j}\right) = 1 + \frac{1}{2}j, \\ a_2 &= \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j), \\ a_{-2} &= \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j), \\ a_k &= 0, \qquad |k| > 2. \end{cases}$$

```
>> a1 = 1-0.5j
>> abs(a1)      1.1180
>> angle(a1)
```

radian

$|a_k|$ $Re(a_k)$ $\measuredangle a_k$ $Im(a_k)$



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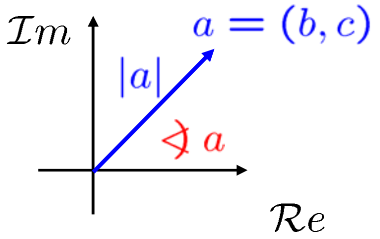
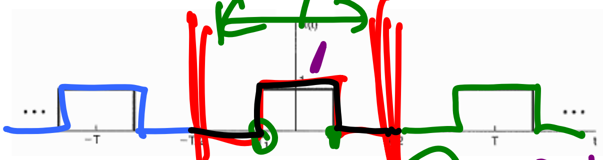


$$a = |a| e^{j\measuredangle a}$$

$$a = |a| \left[\cos(\measuredangle a) + j\sin(\measuredangle a)\right]$$

$$a = b + jc = \sqrt{b^2+c^2}\left[\frac{b}{\sqrt{b^2+c^2}} + j\frac{c}{\sqrt{b^2+c^2}}\right]$$

- **Example 3.5:** $a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$k = 0$ $\quad a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} = \frac{2\cdot 1}{4} = \frac{1}{2}$

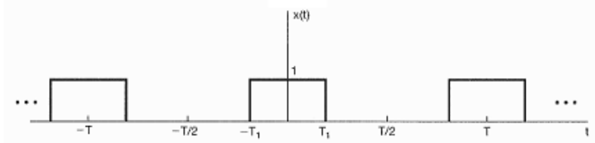
$k \neq 0$ $\quad a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0t}dt = \frac{1}{T}\frac{1}{(-jkw_0)}e^{-jkw_0t}\Big|_{-T_1}^{T_1}$

$$= \frac{2j}{jkw_0T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]/2j$$

$w_0 = \frac{2\pi}{T}$

$$= \frac{2\sin(kw_0T_1)}{kw_0T} = \frac{\sin(kw_0T_1)}{k\pi} = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, \quad k \neq 0$$

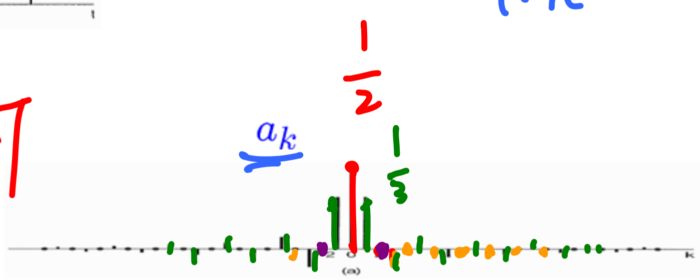
- **Example 3.5:** $T = 4T_1$



$a_0 = \frac{2\cdot 1}{4} = \frac{1}{2}$

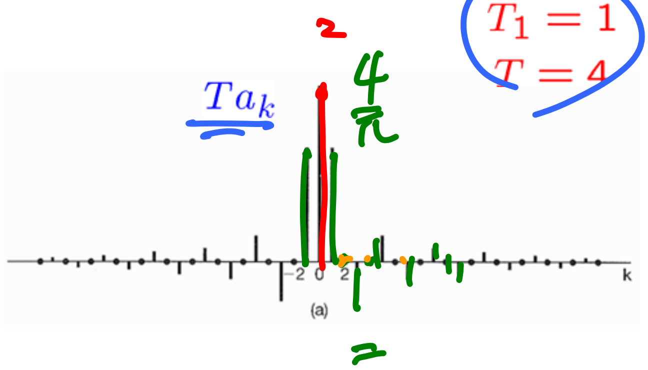
$a_1 = \frac{\sin(1\cdot\frac{\pi}{2})}{1\cdot\pi} = \frac{1}{\pi}$

$$a_k = \frac{\sin(k2\pi\frac{T_1}{T})}{k\pi} = \frac{\sin(k\frac{\pi}{2})}{k\pi}$$



$$T\,a_k = T\frac{\sin(k2\pi\frac{T_1}{T})}{k\pi} = T\frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$T_1 = 1$
$T = 4$



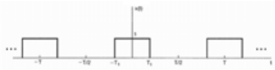
- **Example 3.5:** $a_k = \dfrac{\sin(k 2\pi \frac{T_1}{T})}{k\pi}$

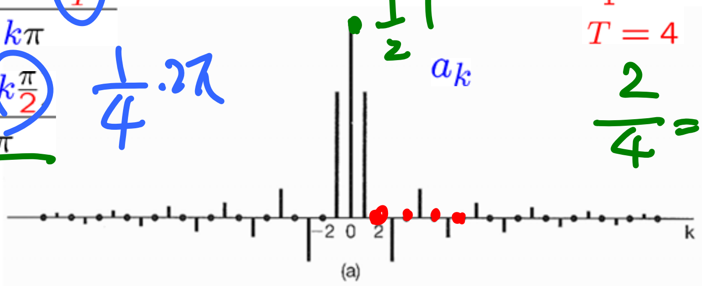
$a_0 = \dfrac{2T_1}{T} = \dfrac{1}{2}$

$T_1 = 1$
$T = 4$

$T = 4T_1$

$a_k = \dfrac{\sin(k\frac{\pi}{2})}{k\pi}$ $\quad \frac{1}{4}\cdot 2\pi$ $\quad \frac{2}{4} = \frac{1}{2}$



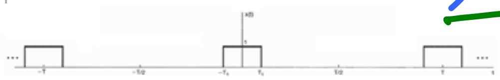


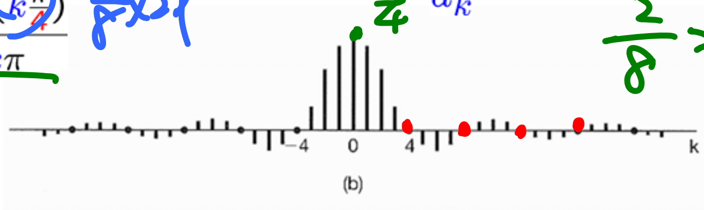
(a)

$k\frac{\pi}{4}$

$T = 8T_1$

$a_k = \dfrac{\sin(k\frac{\pi}{4})}{k\pi}$ $\quad \frac{1}{8}\times 2\pi$ $\quad \frac{2}{8} = \frac{1}{4}$

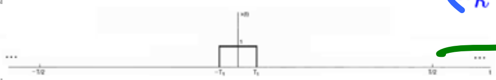


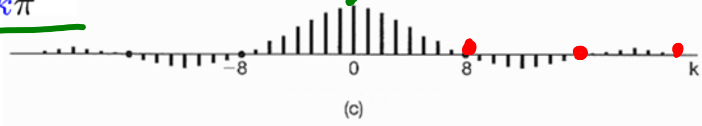


(b)

$T = 16T_1$

$a_k = \dfrac{\sin(k\frac{\pi}{8})}{k\pi}$ $\quad \frac{1}{16}\times 2\pi$ $\quad \frac{2}{16} = \frac{1}{8}$



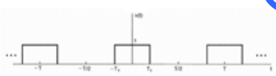


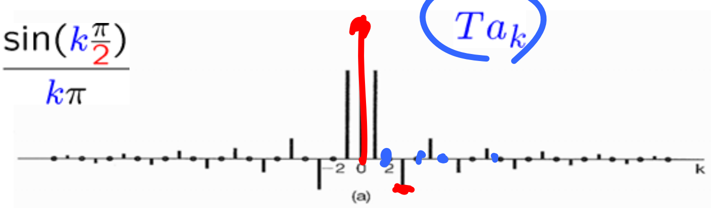
(c)

- **Example 3.5:** $T\, a_k = T\, \dfrac{\sin(k 2\pi \frac{T_1}{T})}{k\pi}$

$T = 4T_1$

$T a_k = T\, \dfrac{\sin(k\frac{\pi}{2})}{k\pi}$

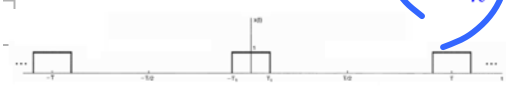


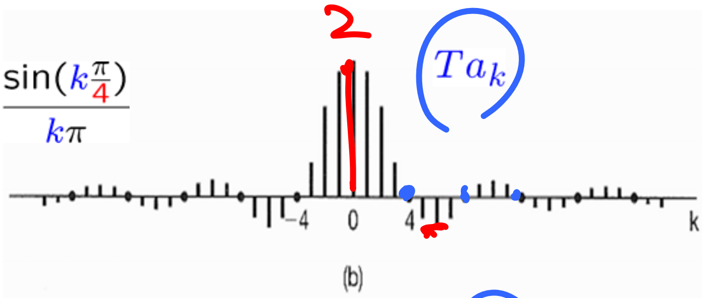


(a)

$T = 8T_1$

$T a_k = T\, \dfrac{\sin(k\frac{\pi}{4})}{k\pi}$

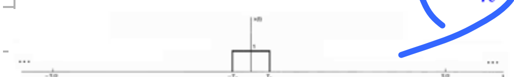


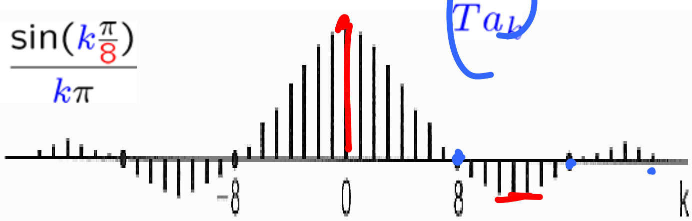


(b)

$T = 16T_1$

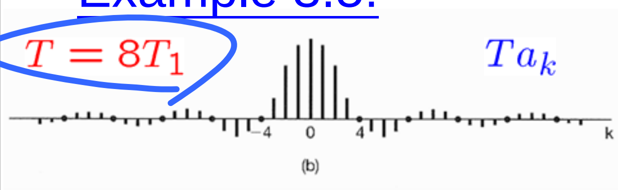
$T a_k = T\, \dfrac{\sin(k\frac{\pi}{8})}{k\pi}$

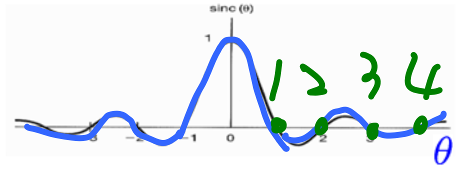




Example 3.5:

$T = 8T_1$





$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

$$\frac{\sin(x)}{x}$$

$$Ta_k = T\,\frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$= \frac{1}{4}\,T\,\frac{\sin(\pi\frac{k}{4})}{\pi\frac{k}{4}}$$

$$= \frac{1}{4}\,T\,\mathrm{sinc}(\frac{k}{4})$$

$$\frac{k}{4} = \theta = 1, 2, 3, 4$$

$$w_0 = \frac{2\pi}{T} \qquad Ta_k = \frac{2\sin(wT_1)}{w}$$

$$w = kw_0 \qquad wT_1 = k\left(\frac{2\pi}{T}\right)\cdot T_1$$

$$= \frac{2k\pi}{A}$$

Example 3.5:

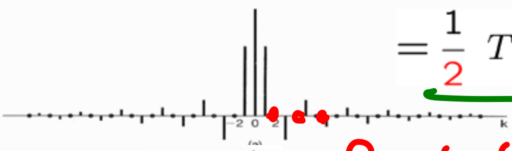
$$T\ a_k = T\,\frac{\sin(k2\pi\frac{T_1}{T})}{k\pi}$$

$\boxed{T = 4T_1}$

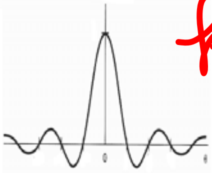
$$Ta_k = T\,\frac{\sin(k\frac{\pi}{2})}{k\pi}$$

$$= \frac{1}{2}\,T\,\mathrm{sinc}(\frac{k}{2})$$

$$\theta = \frac{k}{2}$$



$k = \pm 2, 4, 6\ \cdot$

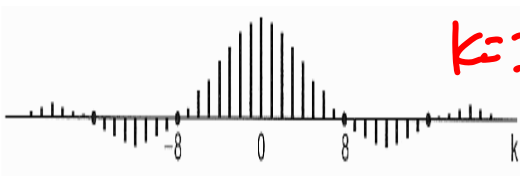


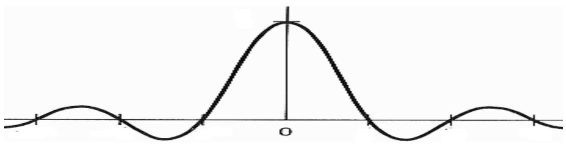
$$Ta_k = T\,\frac{\sin(k\frac{\pi}{8})}{k\pi}$$

$\boxed{T = 16T_1}$

$$= \frac{1}{8}\,T\,\mathrm{sinc}(\frac{k}{8})$$

$k = \pm 8, \pm 16$

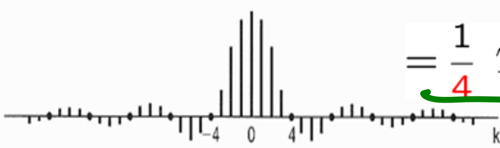




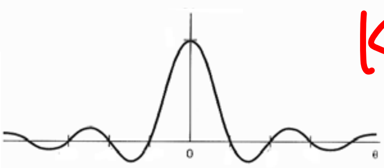
$\boxed{T = 8T_1}$

$$Ta_k = T\,\frac{\sin(k\frac{\pi}{4})}{k\pi}$$

$$= \frac{1}{4}\,T\,\mathrm{sinc}(\frac{k}{4})$$



$k = \pm 4, 8, \cdots$

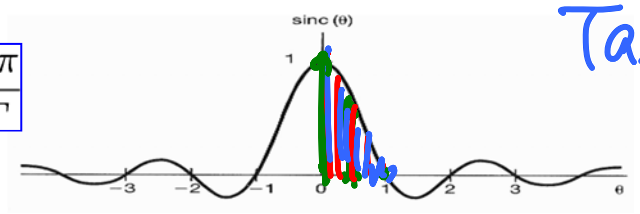


- Example 3.5:

$$Ta_k = T\,\frac{2\sin(kw_0T_1)}{kw_0T}$$

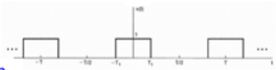
$$\boxed{w_0 = \frac{2\pi}{T}}$$

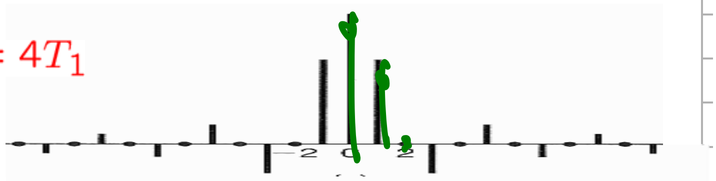
Tak



$$= T_1\,\frac{2\sin(kw_0T_1)}{kw_0T_1}$$

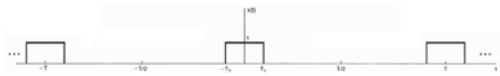
$T = 4T_1$

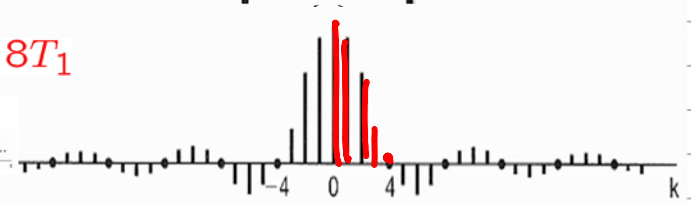




$$= \frac{1}{2}\;T\;\text{sinc}\left(\frac{k}{2}\right)$$

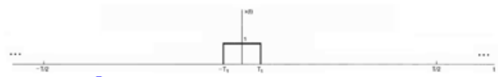
$T = 8T_1$

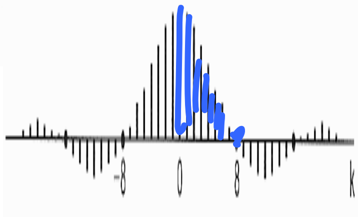




$$= \frac{1}{4}\;T\;\text{sinc}\left(\frac{k}{4}\right)$$

$T = 16T_1$





$$= \frac{1}{8}\;T\;\text{sinc}\left(\frac{k}{8}\right)$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Fourier maintained that
  "any" periodic signal could be represented
  by a Fourier series
- The truth is that
  Fourier series can be used to represent
  an extremely large class of periodic signals
- The question is that
  when a periodic signal x(t) does in fact have a
  Fourier series representation?

$x(t)$

$$x_{FS}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk(2\pi/T)t}$$

- One class of periodic signals:
  - Which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \quad \Rightarrow \quad a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt < \infty$$

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0 t}$$

$$e_N(t) = x(t) - x_N(t) \qquad e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = 0$$

$$E_N(t) = \int_T |e_N(t)|^2 \, dt \qquad E(t) = \int_T |e(t)|^2 \, dt = 0$$

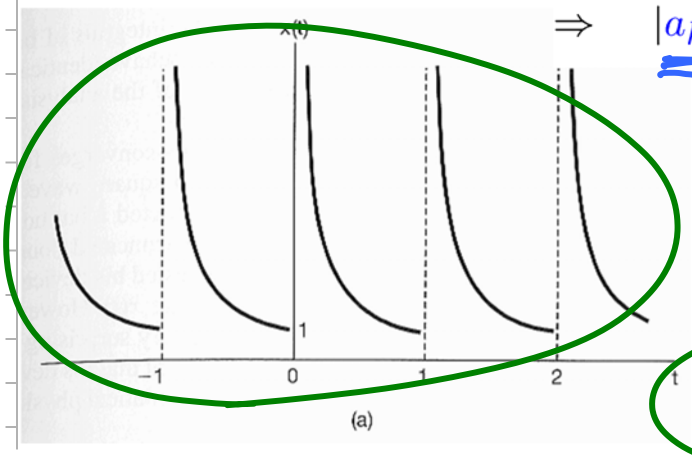
$$\to 0 \quad \text{as } N \to \infty \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}, \quad \forall t\ ???$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 1:
    - Over any period, x(t) must be absolutely integrable, i.e., $\int_T |x(t)|\, dt < \infty$

Johann Peter Gustav Lejeune Dirichlet
1805-1859
Born in Germany
Photo from wikipedia

$$\Rightarrow \quad |a_k| \leq \frac{1}{T}\int_T \left| x(t) e^{-jkw_0 t} \right| dt$$

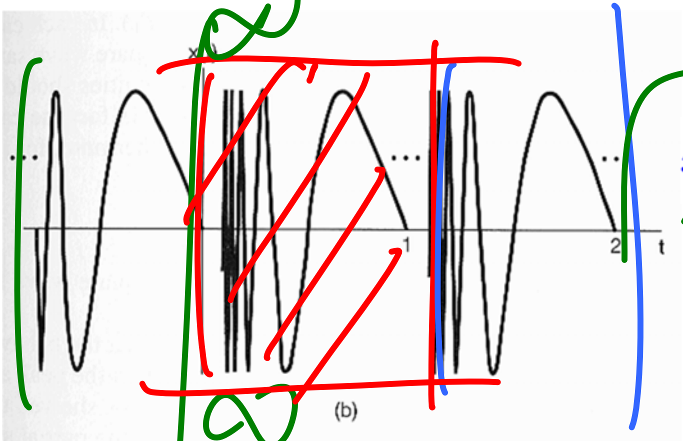
$$= \frac{1}{T}\int_T |x(t)|\, dt < \infty$$



(a)

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 2:
    - In any finite interval, x(t) is of bounded variation; i.e.,
    - There are no more than a finite number of maxima and minima during any single period of the signal

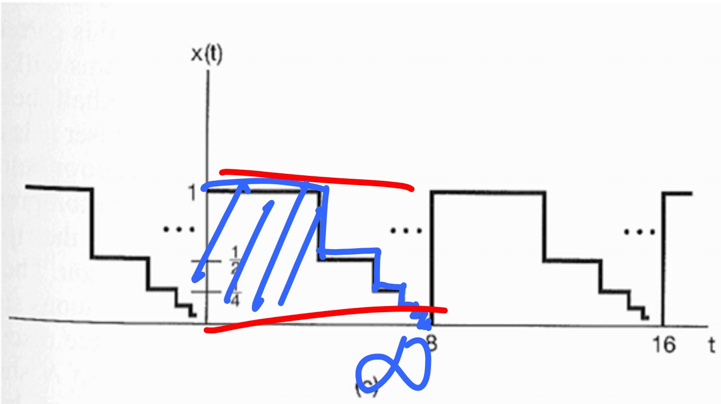


(b)

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$

$$\int_0^1 |x(t)|\, dt < 1$$

- The other class of periodic signals:
  - Which satisfy Dirichlet conditions:
  - Condition 3:
    - In any finite interval, x(t) has only finite number of discontinuities.
    - Furthermore, each of these discontinuities is finite

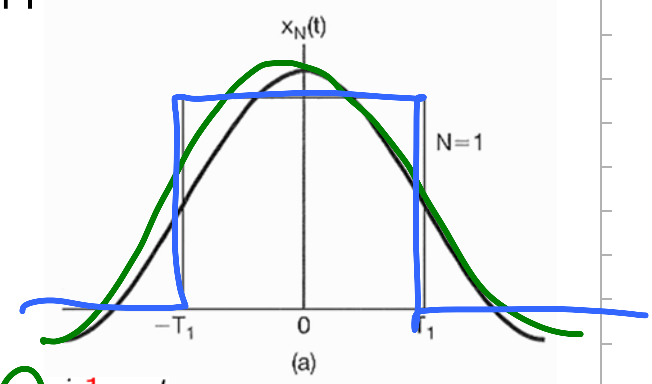


- How the Fourier series converges for a periodic signal with discontinuities

Albert Abraham Michelson
1852-1931
Polish-born German-American
Photo from wikipedia

  - In 1898,
    Albert Michelson (an American physicist)
    used his harmonic analyzer
    to compute
    the truncated Fourier series approximation
    for the square wave

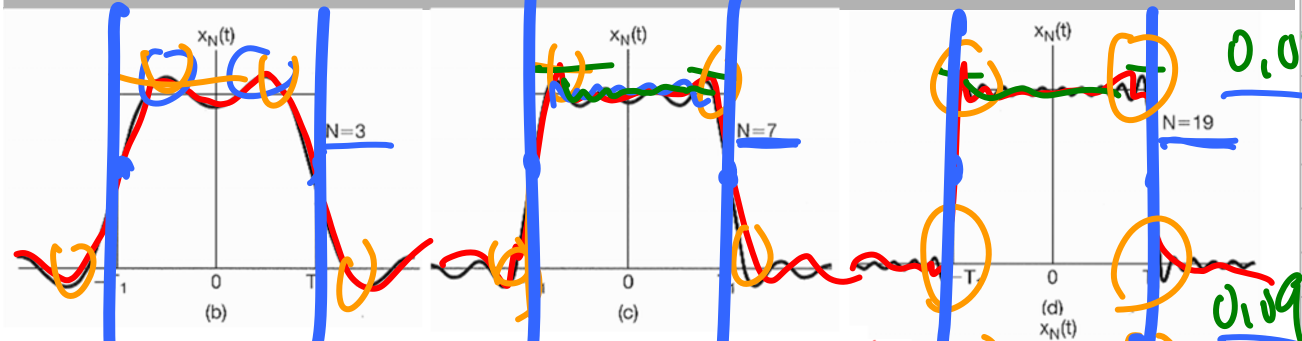
$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$



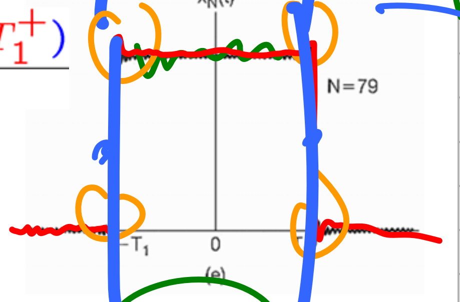
$$x_1(t) = a_{-1} e^{-j\cdot 1\cdot \omega_0 t} + a_0 + a_1 e^{j\cdot 1\cdot \omega_0 t}$$

Convergence of the Fourier Series

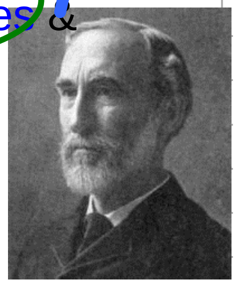
$x(t) = \sum \cdots$ $\forall t$?



$$x_N(T_1) = \frac{x(T_1^-) + x(T_1^+)}{2}$$



- Michelson wrote to Josiah Gibbs
- In 1899, Gibbs showed that
  - the partial sum near discontinuity exhibits ripples &
  - the peak amplitude remains constant with increasing N
- The Gibbs phenomenon



Josiah Willard Gibbs
1839-1903
Born in USA
Photo from wikipedia

3/15/12
1:35p
↓
2:26p

Outline

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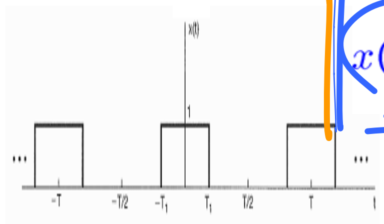
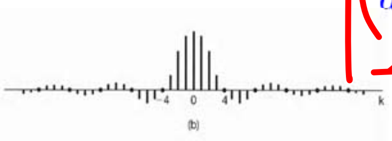
CT Fourier Series Representation:

- The synthesis equation:



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

- The analysis equation:



$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$$

- $x(t) \xleftrightarrow{\mathcal{FS}} a_k$: Fouries series pair

Section	Property
3.5.1	Linearity
3.5.2	Time Shifting
	Frequency Shifting
3.5.6	Conjugation
3.5.3	Time Reversal
3.5.4	Time Scaling
	Periodic Convolution
3.5.5	Multiplication
	Differentiation
	Integration
3.5.6	Conjugate Symmetry for Real Signals
3.5.6	Symmetry for Real and Even Signals
3.5.6	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.5.7	Parseval's Relation for Periodic Signals

- **Linearity:** $x(t+T)=x(t)$
  - $x(t), y(t)$: periodic signals with period T

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0t}$$

$$\Rightarrow \; z(t) = Ax(t)+By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k+Bb_k$$

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0t}$$

$$c_k = \frac{1}{T}\int_T z(t)e^{-jkw_0t}dt$$

$$= \frac{1}{T}\int_T \left(Ax(t)+By(t)\right)e^{-jkw_0t}dt$$

$$= \frac{1}{T}\int_T Ax(t)e^{-jkw_0t}dt + \frac{1}{T}\int_T By(t)e^{-jkw_0t}d\tau$$

$$Aa_k \qquad + Bb_k$$

Add

- **Time Shifting:**
  - $x(t)$: periodic signal with period T $\quad e^{jkw_0(-t_0)}$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow \; x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = \boxed{e^{-jkw_0t_0}}a_k = e^{-jk\left(\frac{2\pi}{T}\right)t_0}a_k$$

b/c $\quad b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jkw_0t}dt \qquad t-t_0=\tau,\quad t=\tau+t_0,\quad dt=d\tau$

$$= \frac{1}{T}\int_T x(\tau)e^{-jkw_0(\tau+t_0)}d\tau$$

$$= e^{-jkw_0t_0}\boxed{\frac{1}{T}\int_T x(\tau)e^{-jkw_0\tau}d\tau} = a_k$$

Time Reversal:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow \boxed{x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}}$$

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}} \to -t$$

$$x(-t) = \sum_{k=-\infty}^{+\infty} a_k e^{-jk\left(\frac{2\pi}{T}\right)t}$$

m

$$= \sum_{m=-\infty}^{+\infty} a_{-m} e^{jm\left(\frac{2\pi}{T}\right)t}$$

$$-k = m$$

- If $x(t)$ is even, i.e., $x(-t) = x(t)$

  $\Rightarrow$ a_k is even, i.e., $a_{-k} = a_k$

- If $x(t)$ is odd, i.e., $x(-t) = -x(t)$

  $\Rightarrow$ a_k is odd, i.e., $a_{-k} = -a_k$

Time Scaling:

- $x(t)$: periodic signals with period T
  and fundamental frequency w_0

$x(2t)$

- $x(\alpha t)$: periodic signals with period $\frac{T}{\alpha}$
  and fundamental frequency αw_0

$\frac{T}{2}$

$2w_0$

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t}}$$

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\,w_0\,(\alpha\, t)} = \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\,\alpha\left(\frac{2\pi}{T}\right)t}$$

$$= \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\,(\alpha\, w_0)\,t} = \sum_{k=-\infty}^{+\infty} a_k\, e^{j\,k\left(\frac{2\pi}{\frac{T}{\alpha}}\right)t}$$

- **Multiplication:**

$$z(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jkw_0 t}$$

- $x(t), y(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t) = \sum_{l=-\infty}^{+\infty} a_l e^{jlw_0 t}$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y(t) = \sum_{m=-\infty}^{+\infty} b_m e^{jmw_0 t}$$

$\Rightarrow x(t)y(t)$: also periodic with T

$$z(t) = x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = \boxed{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$

$$(a+b+c)(d+e+f) = ad+ae+af + bd+be+bf + cd+ce+cf$$

$$= \sum_l a_l e^{jlw_0 t} \sum_m b_m e^{jmw_0 t}$$

$$= \sum_l \left(\sum_m a_l b_m \right) e^{j(\underbrace{l+m}_{k})w_0 t}$$

$k = l+m$

$l = k-m$

$$= \sum_k \boxed{\left(\sum_m a_{k-m} b_m \right)} e^{jkw_0 t} = c_k$$

Add

- **Differentiation:**

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

- $x(t)$: periodic signals with period T

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\frac{d}{dt}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0 a_k$$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\sum_{k=-\infty}^{+\infty} a_k \frac{d}{dt} e^{jkw_0 t}$$

$$jkw_0 e^{jkw_0 t}$$

Integration:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

- $x(t)$: periodic signals with period T

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{FS}}{\longleftrightarrow} \frac{1}{jkw_0} a_k$$

only if $a_0 = 0$,
it is finite velaued
and periodic

$$\int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0\tau} d\tau$$

$$\frac{1}{jkw_0} e^{jkw_0\tau}\Big|_{-\infty}^{t}$$

Conjugation & Conjugate Symmetry:

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t)^* \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{-k}^*$$

$$\left(x(t)\right)^* = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* \left(e^{jkw_0 t}\right)^*$$

$$= \sum_{k=-\infty}^{+\infty} a_k^* e^{j(-k)w_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jkw_0 t}$$

$-k \rightarrow m$

$m \rightarrow k$

Conjugation & Conjugate Symmetry:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x(t)^* \overset{\mathcal{FS}}{\longleftrightarrow} {a_{-k}}^*$$

- $x(t) = x(t)^* \Rightarrow a_{-k} = a_k^*$

  $x(t)$ is real $\Rightarrow \{a_k\}$ are conjugate symmetric

- $x(t) = x(t)^*$ & $x(-t) = x(t) \Rightarrow a_{-k} = a_k^*$ & $a_{-k} = a_k$

  $\Rightarrow a_k = a_k^*$

  $x(t)$ is real & even $\Rightarrow \{a_k\}$ are real & even

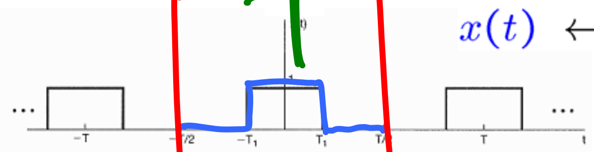
- $x(t)$ is real & odd $\Rightarrow \{a_k\}$ are purely imaginary & odd

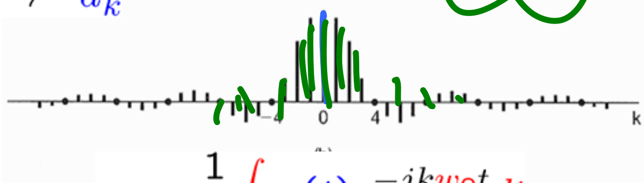
  $\Rightarrow a_k^* = -a_k$

Parseval's relation for CT periodic signals:

- As shown in Problem 3.46:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$





$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

$$\frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{+\infty} \left|a_k\right|^2$$

- Parseval's relation states that
  the total average power in a periodic signal
  equals
  the sum of the average powers
  in all of its harmonic components

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ } Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$a_k = a^*_{-k}$; $\Re e\{a_k\} = \Re e\{a_{-k}\}$; $\Im m\{a_k\} = -\Im m\{a_{-k}\}$; $\lvert a_k\rvert = \lvert a_{-k}\rvert$; $\measuredangle a_k = -\measuredangle a_{-k}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]; $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{a_k\}$; $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

$$\int uv' = (uv) - \int u'v$$

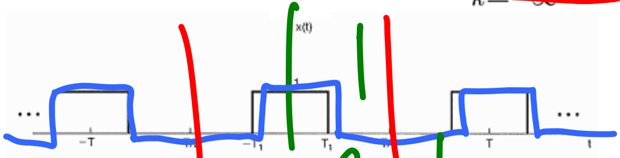
3/15/12
3:14 pm

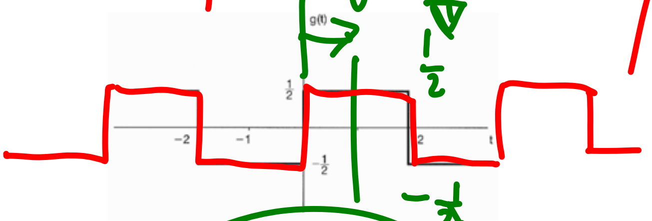
Properties of CT Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt$$

K=0

- Example 3.6: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$





$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$\begin{cases} a_0 = \frac{2T_1}{T} \\ a_k = \frac{\sin(k(2\pi/T)T_1)}{k\pi}, & k \neq 0 \end{cases}$$

$$\boxed{g(t) = x(t-1) - 1/2}$$

with $T = 4, T_1 = 1$

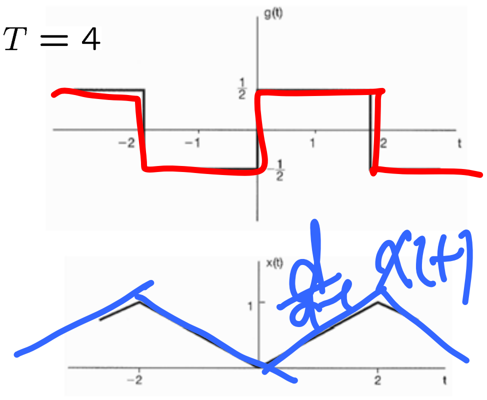
$$x(t-1) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k = a_k e^{-jk\pi/2}$$

$$g(t) = x(t-1) - 1/2 \stackrel{\mathcal{FS}}{\longleftrightarrow} \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0 \\ a_0 - 1/2, & \text{for } k = 0 \end{cases}$$

$$g(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0 \\ 0, & \text{for } k = 0 \end{cases}$$

- ## Example 3.7:

$T = 4$



$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} d_k$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} e_k$$

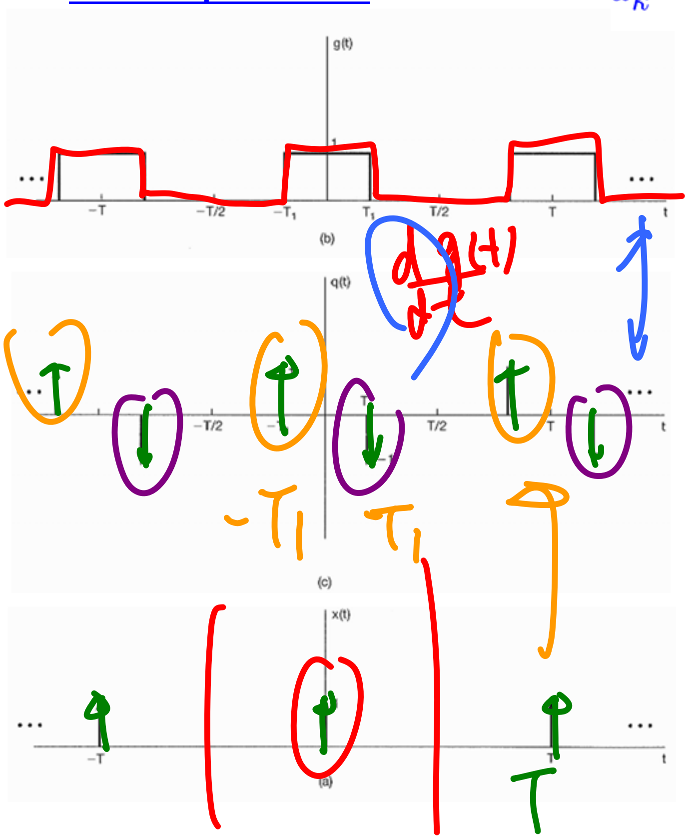
$$\frac{d}{dt}x(t) \overset{\mathcal{FS}}{\longleftrightarrow} jkw_0\, e_k$$

$$g(t) = \frac{d}{dt}x(t) \iff d_k = jk(\pi/2)e_k$$

$$e_k = \begin{cases} \frac{2}{jk\pi} d_k = \boxed{\frac{2\sin(\pi k/2)}{j(k\pi)^2} e^{-jk\pi/2}}, & \text{for } k \neq 0 \\ \frac{1}{2}, & \text{for } k = 0 \end{cases}$$

- ## Example 3.8:

$$a_k = \frac{1}{T}\int_T \cancel{x(t)}\, \cancel{e^{-jkw_0 t}}\, dt$$ $t = 0$ $= \frac{1}{T}$



$$g(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

$$q(t) = \frac{d}{dt}g(t) \iff b_k = jkw_0 c_k$$

$$q(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$q(t) = x(t+T_1) - x(t-T_1)$$

$$\iff b_k = e^{jkw_0T_1}a_k - e^{-jkw_0T_1}a_k$$

$\frac{1}{T}$ $\frac{1}{T}$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{T}$$

- Example 3.8:

$$b_k = e^{jkw_0T_1} a_k - e^{-jkw_0T_1} a_k$$

$$= \frac{1}{T}\left[e^{jkw_0T_1} - e^{-jkw_0T_1}\right]$$

$$= \frac{2j\sin(kw_0T_1)}{T}$$

$$b_k = jkw_0 c_k$$

$$\begin{cases} k \neq 0 & c_k = \dfrac{b_k}{jkw_0} = \dfrac{2j\sin(kw_0T_1)}{jkw_0T} = \dfrac{\sin(kw_0T_1)}{k\pi} \\ k = 0 & c_0 = \dfrac{2T_1}{T} \end{cases}$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Fourier Series Representation of Discrete-Time Periodic Signals
- Properties of Discrete-Time Fourier Series
- Fourier Series & LTI Systems
- Filtering & Examples of CT & DT Filters

- Harmonically related complex exponentials

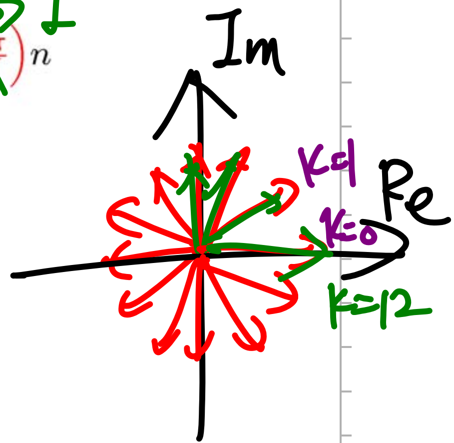
$n \in \mathbb{Z}$

$$\phi_k[n] = e^{jkw_0 n} = e^{jk\left(\frac{2\pi}{N}\right)n}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

$N=12$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n}\, \cancel{e^{jN\left(\frac{2\pi}{N}\right)n}} \; 1$$

$$\Rightarrow \; \phi_k[n] = \phi_{k+N}[n] = \cdots = \phi_{k+rN}[n]$$



- The Fourier Series Representation:

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$<N>$

- Procedure of Determining the Coefficients:

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)}$$

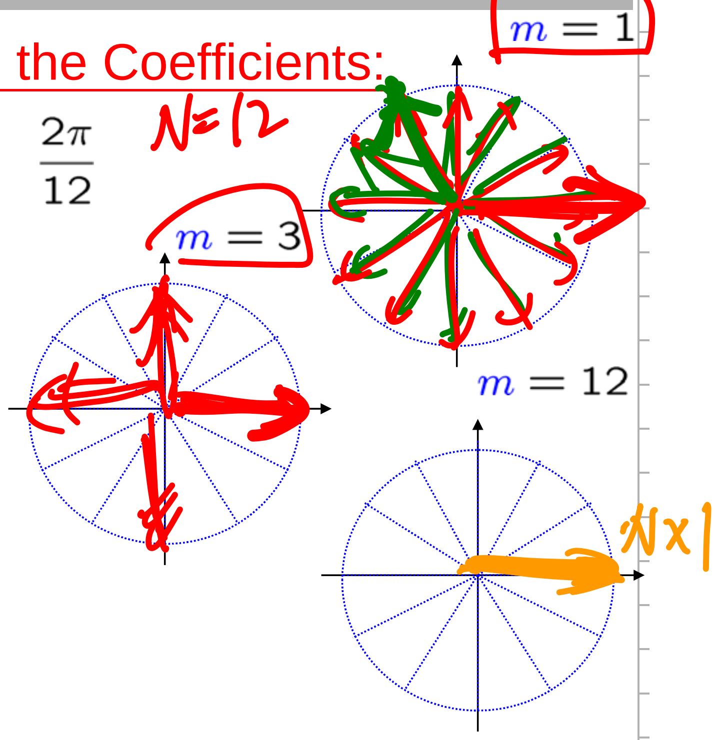
$$x[2] = \sum_{k=<N>} a_k e^{jk2\left(\frac{2\pi}{N}\right)}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(N-1)\left(\frac{2\pi}{N}\right)}$$

$$x[N] = \sum_{k=<N>} a_k e^{jk(N)\left(\frac{2\pi}{N}\right)}$$

$\frac{2\pi}{12}$ $N=12$



and
$$\sum_{n=<N>} e^{jm\left(\frac{2\pi}{N}\right)n} = \begin{cases} N, & m = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$$

- **Procedure of Determining the Coefficients:**

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \qquad \sum_{r=<N>} e^{-jr\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$\sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n} = \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

k=r

= 0

0 ≠ N ≠ 2N

k=r
k=r+N
k=r+2N

$$= a_r N$$

m
r
k

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jr\left(\frac{2\pi}{N}\right)n}$$

- **In Summary:**
  - The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

  - The analysis equation:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = a_{k+N}$$

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair
- $\{a_k\}$: the Fourier series coefficients

  or the spectral coefficients of $x[n]$

Example 3.11:

$e^{j1\omega_0 n}$ $\quad e^{j2\omega_0 n}$ $\quad e^{j3\omega_0 n}$

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

$\omega_0 = \frac{2\pi}{N}$

$$\Rightarrow x[n] = 1 + \frac{1}{2j}\left[e^{j\left(\frac{2\pi}{N}\right)n} - e^{-j\left(\frac{2\pi}{N}\right)n}\right] + \frac{3}{2}\left[e^{j\left(\frac{2\pi}{N}\right)n} + e^{-j\left(\frac{2\pi}{N}\right)n}\right] + \frac{1}{2}\left[e^{j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)} + e^{-j\left(\frac{4\pi}{N}n+\frac{\pi}{2}\right)}\right]$$

$$\Rightarrow x[n] = \underbrace{1}_{a_0} + \underbrace{\left(\frac{3}{2} + \frac{1}{2j}\right)}_{a_1} e^{j\left(\frac{2\pi}{N}\right)n} + \underbrace{\left(\frac{3}{2} - \frac{1}{2j}\right)}_{a_{-1}} e^{-j\left(\frac{2\pi}{N}\right)n} + \underbrace{\frac{1}{2}e^{j\left(\frac{\pi}{2}\right)}}_{a_2} e^{j2\left(\frac{2\pi}{N}\right)n} + \underbrace{\frac{1}{2}e^{-j\left(\frac{\pi}{2}\right)}}_{a_{-2}} e^{-j2\left(\frac{2\pi}{N}\right)n}$$

Example 3.11:

$$\Rightarrow \begin{cases} a_0 = 1 \quad 0 \\ a_1 = \left(\frac{3}{2} + \frac{1}{2j}\right) = \frac{3}{2} - \frac{1}{2}j \\ a_{-1} = \left(\frac{3}{2} - \frac{1}{2j}\right) = \frac{3}{2} + \frac{1}{2}j \\ a_2 = 0 \quad \frac{1}{2}j \\ a_{-2} = 0 \quad -\frac{1}{2}j \\ a_k = 0, \text{ others in } < N > \end{cases}$$

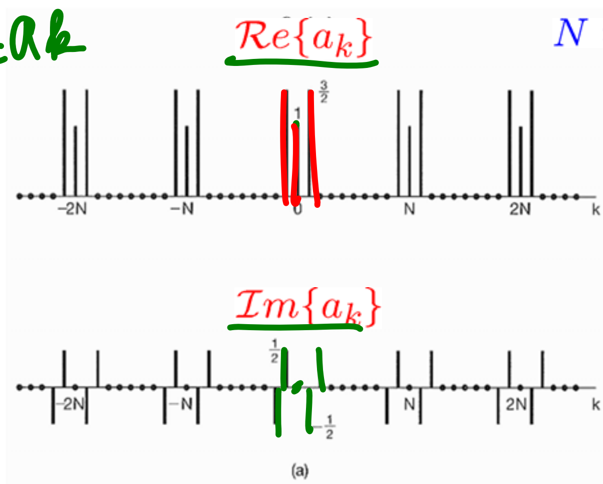
$$\boxed{a = |a| e^{j\measuredangle a}}$$

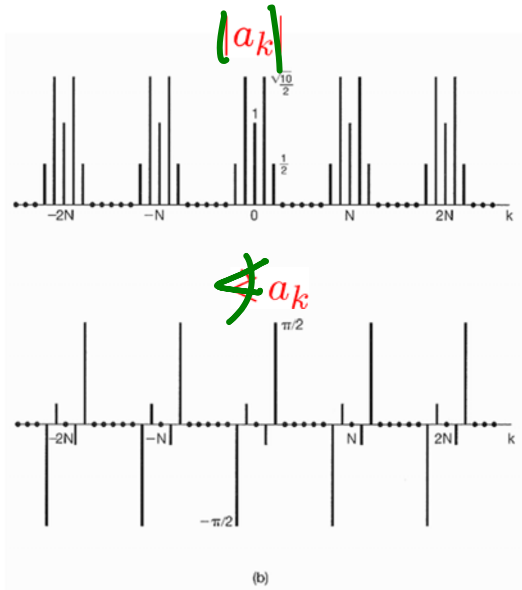
$$\boxed{a = |a| \left[\cos(\measuredangle a) + j\sin(\measuredangle a)\right]}$$

$$\boxed{a = b + jc = \underbrace{\sqrt{b^2+c^2}}_{|a|}\left[\frac{b}{\sqrt{b^2+c^2}} + j\frac{c}{\sqrt{b^2+c^2}}\right]}$$

$a_{k+N} = a_k$

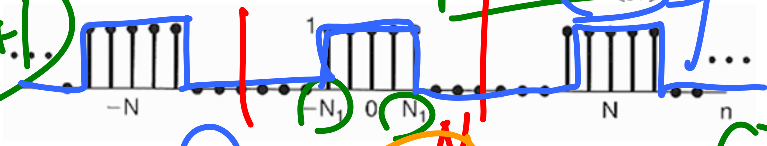
$N = 10$





- Example 3.12: $a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$

$2N_1+1$



$$x[n] = \begin{cases} 1, & -N_1 \le n \le N_1 \\ 0, & \text{others in } <N> \end{cases}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N} \sum_{n=-N_1}^{N_1} \left(e^{-jk\left(\frac{2\pi}{N}\right)}\right)^n$$

$$= \frac{1}{N}\left[\left(\cdot\right)^{-N_1} + \left(\cdot\right)^{-N_1+1} + \cdots + \left(\cdot\right)^{N_1}\right]$$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[\frac{1-\left(\cdot\right)^{(2N_1+1)}}{1-\left(\cdot\right)}\right] \qquad \left(\cdot\right) \neq 1$$

$2N_1+1$

$$= \frac{1}{N}\left(\cdot\right)^{-N_1}\left[1 + \left(\cdot\right)^1 + \cdots + \left(\cdot\right)^{2N_1}\right]$$

$2N_1+1$

- Let $m = n + N_1$ or $n = m - N_1$

$$a_k = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)(m-N_1)} = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \sum_{m=0}^{2N_1} e^{-jk\left(\frac{2\pi}{N}\right)m}$$

- Example 3.12:

- $k = 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{2N_1+1}{N}$$

- $k \neq 0, \pm N, \pm 2N, \ldots$

$$a_k = \frac{1}{N} e^{jk\left(\frac{2\pi}{N}\right)N_1} \left(\frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}}\right)$$

$(N_1 + \frac{1}{2})$

$$= \frac{1}{N} \frac{\cancel{e^{-jk\left(\frac{2\pi}{2N}\right)}}\left[e^{jk\left(\frac{2\pi}{2N}\right)(2N_1+1)} - e^{-jk\left(\frac{2\pi}{2N}\right)(2N_1+1)}\right]}{\cancel{e^{-jk\left(\frac{2\pi}{2N}\right)}}\left[e^{jk\left(\frac{2\pi}{2N}\right)} - e^{-jk\left(\frac{2\pi}{2N}\right)}\right]}$$

$2j$

$2j$

$$= \frac{1}{N} \frac{\sin\left[\left(\frac{2\pi}{N}\right) k \left(N_1 + \frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right) k\right]}$$

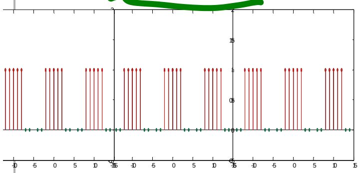
$$1 - e^{-j\theta}$$
$$= e^{-j\theta/2} e^{j\theta/2} - e^{-j\theta/2} e^{-j\theta/2}$$
$$= e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right)$$

$2j$

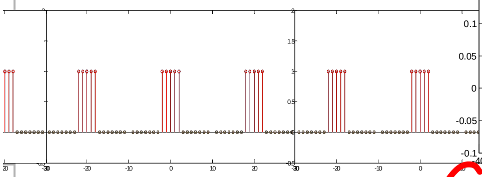
- Example 3.12: a_k

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]} = 5$$

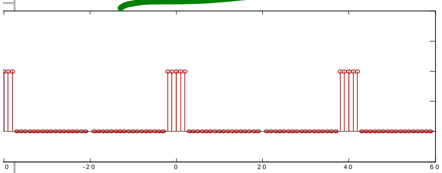
  - $2N_1 + 1 = 5$
  - $N = 10$

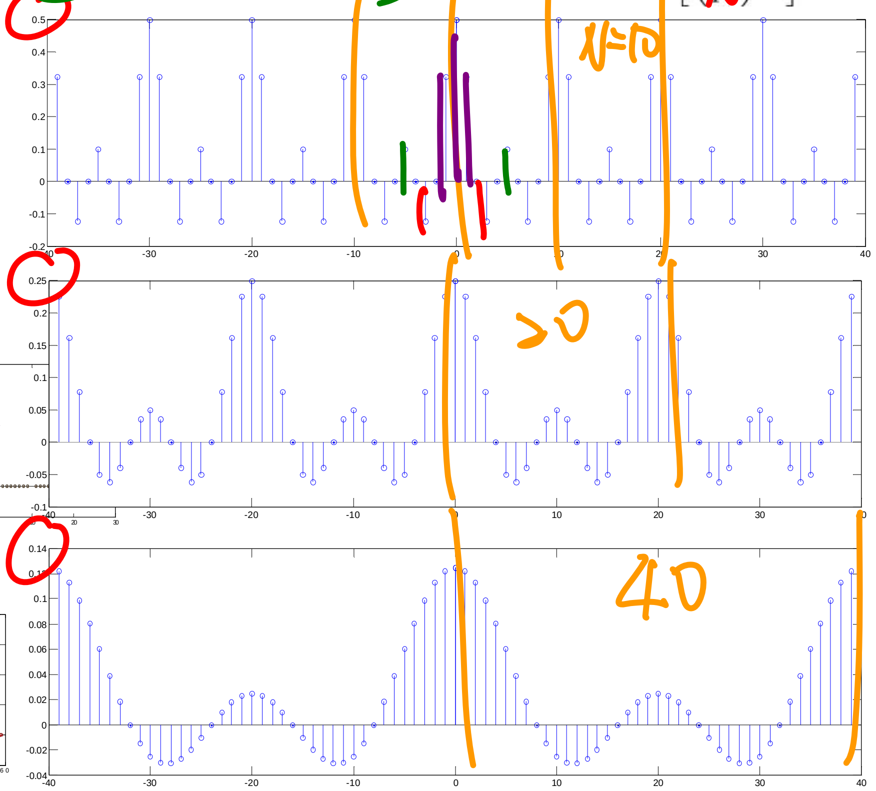


  - $N = 20$



  - $N = 40$

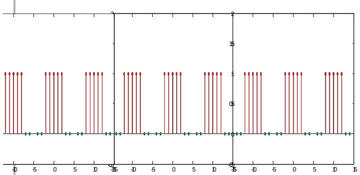




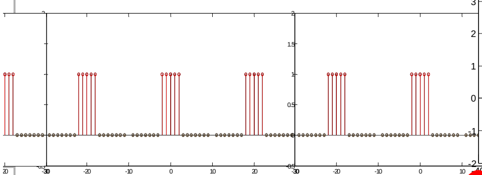
- Example 3.12: $N a_k$

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$

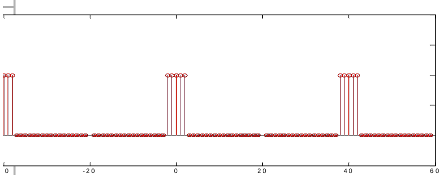
  - $2N_1 + 1 = 5$
  - $N = 10$

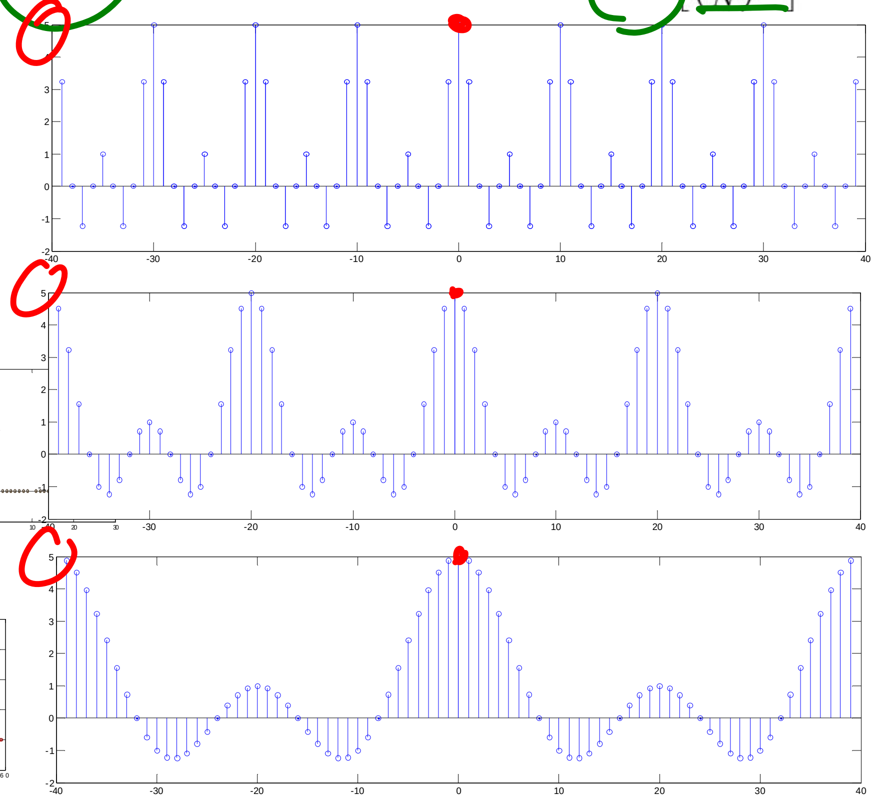


  - $N = 20$

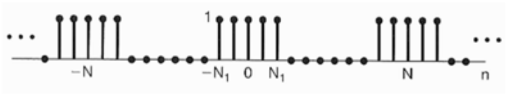


  - $N = 40$



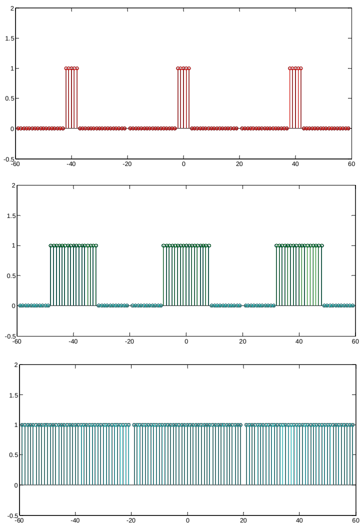


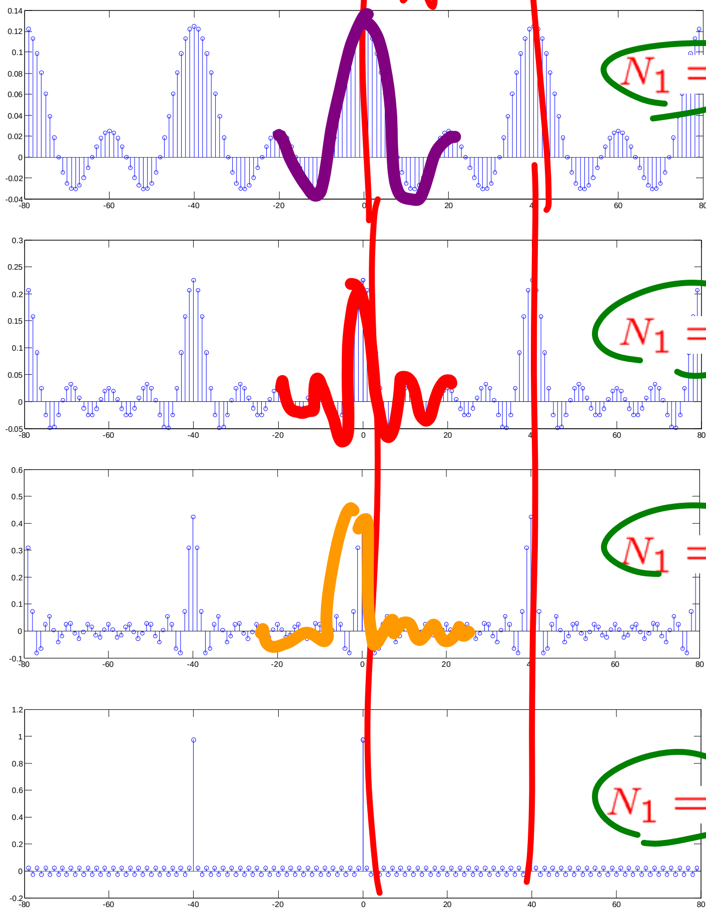
- Example 3.12:



- $N = 40$

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$





N=40

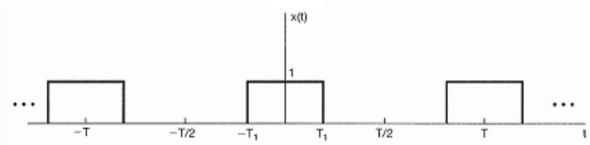
$N_1 = 2$ 5

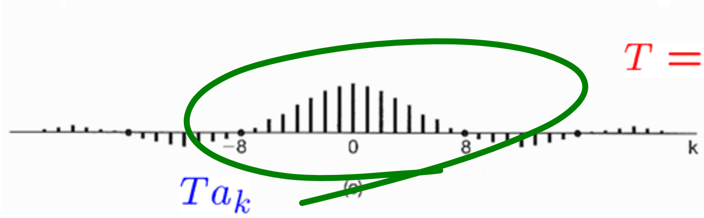
$N_1 = 4$ 9

$N_1 = 8$ 16

$N_1 = 19$ 38

- Examples 3.5 (CT) & 3.12 (DT):





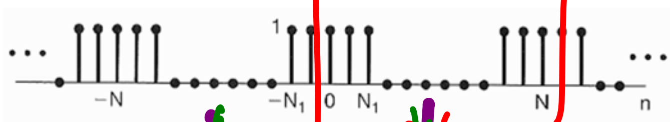
$$Ta_k = T\frac{\sin(k\frac{\pi}{8})}{k\pi}$$

$T = 16T_1$

Ta_k

e^{jkw_0t}

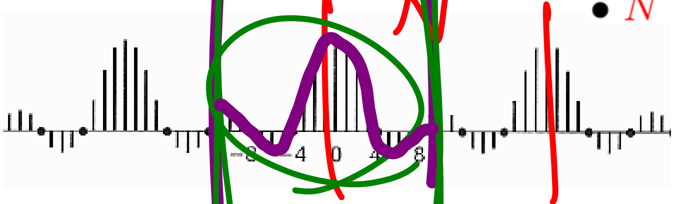
N



- $N = 20$

$$a_k = \frac{1}{N}\frac{\sin\left[\left(\frac{2\pi}{N}\right)k\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[\left(\frac{\pi}{N}\right)k\right]}$$

$$a_k = \frac{2N_1+1}{N}$$



N

e^{jkw_0n}

- Partial Sum:

$$x[n] = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

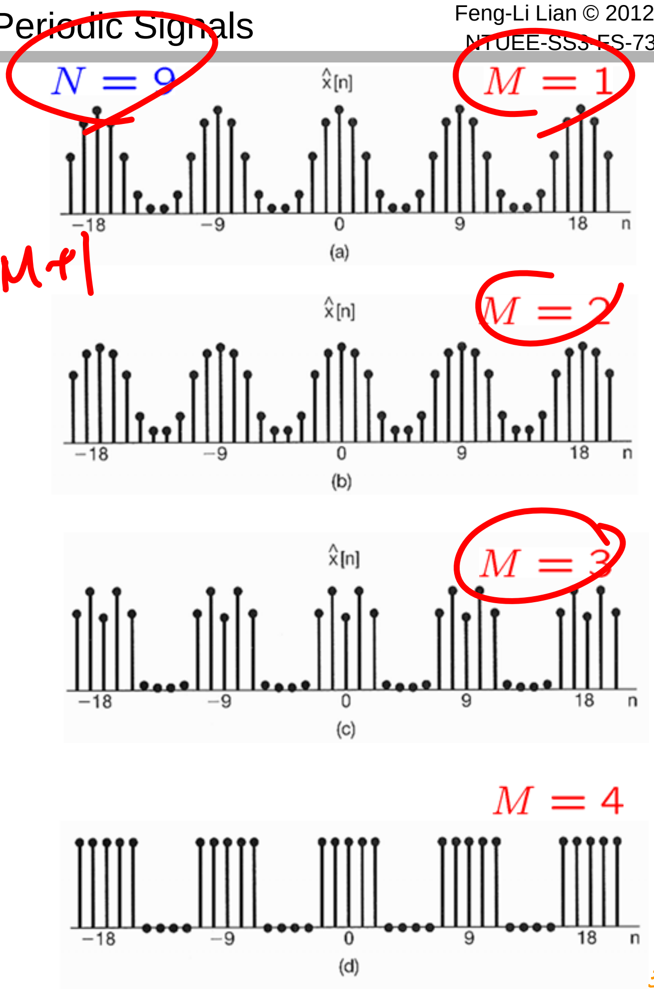
2M+1

- If N is odd

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

- If N is even

$$\hat{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$



3
5
7
9

3/19/12
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- A Historical Perspective
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Outline

Section	Property
	Linearity
	Time Shifting
	Frequency Shifting
	Conjugation
	Time Reversal
	Time Scaling
	Periodic Convolution
3.7.1	Multiplication
3.7.2	First Difference
	Running Sum
	Conjugate Symmetry for Real Signals
	Symmetry for Real and Even Signals
	Symmetry for Real and Odd Signals
	Even-Odd Decomposition for Real Signals
3.7.3	Parseval's Relation for Periodic Signals

Properties of DT Fourier Series

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

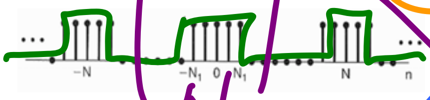
$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

In Summary:

- The synthesis equation:

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

N +



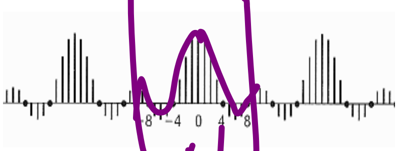
N

$x[n+N] = x[n]$

- The analysis equation:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

N N −



N

$a_k = a_{k+N}$

- $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$: DT Fouries series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n}$$

Linearity:

- $x[n], y[n]$: periodic signals with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$\Rightarrow z[n] = Ax[n] + By[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

Time Shifting:

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

Multiplication:

- $x[n], y[n]$: periodic signals with period N

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad x[n] = \sum_{l=<N>} a_l e^{jlw_0 n}$$

$$y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k \qquad y[n] = \sum_{m=<N>} b_m e^{jmw_0 n}$$

N

$\Rightarrow x[n]y[n]$: also periodic with N

$$c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} d_k = \sum_{l=<N>} a_l\, b_{k-l}$$

N

$\Rightarrow$ a periodic convolution

Add

First Difference:

$$x[n] = \sum_{l=<N>} a_k e^{jkw_0 n}$$

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Rightarrow x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0 n_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)n_0} a_k$$

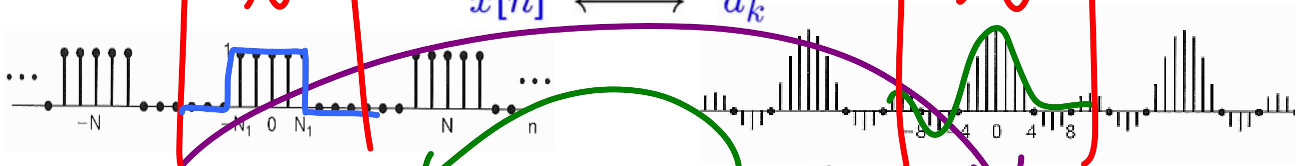
-no

$$\Rightarrow x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jkw_0} a_k = e^{-jk\left(\frac{2\pi}{N}\right)} a_k$$

$$\boxed{x[n] - x[n-1]} \overset{\mathcal{FS}}{\longleftrightarrow} \left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k$$

- Parseval's relation for DT periodic signals:
  - As shown in Problem 3.57:

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$



$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

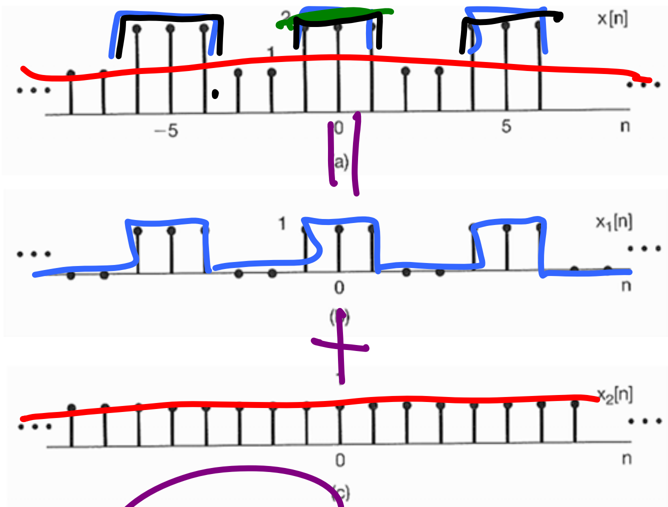
$$\frac{1}{N}\sum_{k=<N>} \left|x[n]\right|^2 = \sum_{k=<N>} \left|a_k\right|^2$$

  - Parseval's relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components (only N distinct harmonic components in DT)

- Example 3.13:

$$\boxed{a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}}$$

$N_1=1$
$N=5$
$N_1=2$
$N=5$



$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x_1[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$x_2[n] \overset{\mathcal{FS}}{\longleftrightarrow} c_k$$

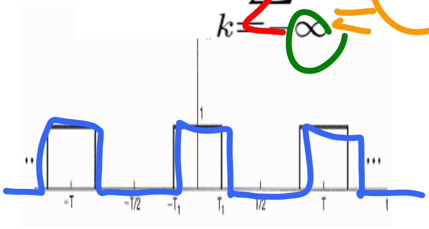
$$x[n] = x_1[n] + x_2[n]$$

$$\Rightarrow b_k = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

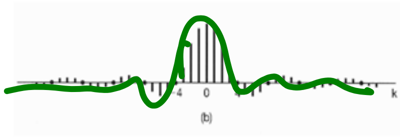
$$\Rightarrow c_k = \begin{cases} 0, & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ 1, & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases} \quad \Rightarrow \quad a_k = b_k + c_k$$

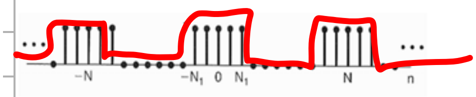
CT & DT Fourier Series Representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

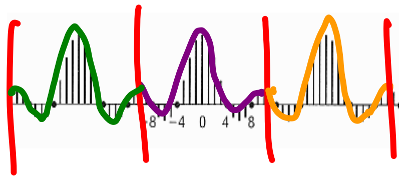


$$x(t) \xleftrightarrow{\mathcal{CTFS}} a_k$$





$$x[n] \xleftrightarrow{\mathcal{DTFS}} a_k$$



$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

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The Response of an LTI System:

$k\omega_0$

On pages 12-14

$k\omega_0$

$in \rightarrow$ LTI $\rightarrow out$

$\delta(t)$ $h(t)$

$$\begin{cases} \text{CT:} & e^{st} \longrightarrow H(s)e^{st} \\ \text{DT:} & z^n \longrightarrow H(z)z^n \end{cases}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt \quad \Rightarrow \text{the impulse response}$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \quad \Rightarrow \text{the system functions}$$

$w = k\omega_0$

- If $s = jw$ or $z = e^{jw}$:

w

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$w = k\omega_0$ $\Rightarrow$ the frequency response

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

In Summary:

$$\boxed{a = |a|e^{j\measuredangle a}}$$

$$\boxed{H = |H|e^{j\measuredangle H}}$$

$in \rightarrow$ LTI H(*s/z/w*) $\rightarrow out$

($s_i = jw_i$ or $z_i = e^{jw_i}$)

$$\begin{cases} \text{CT:} & e^{s_i t} \longrightarrow H(s_i)e^{s_i t} \\ \text{DT:} & z_i^n \longrightarrow H(z_i)z_i^n \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n} \longrightarrow y[n] = \sum_{k=<N>} a_k H(e^{jk(\frac{2\pi}{N})}) e^{jk(\frac{2\pi}{N})n}$$

Examples 3.16 & 3.17

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- Filtering:

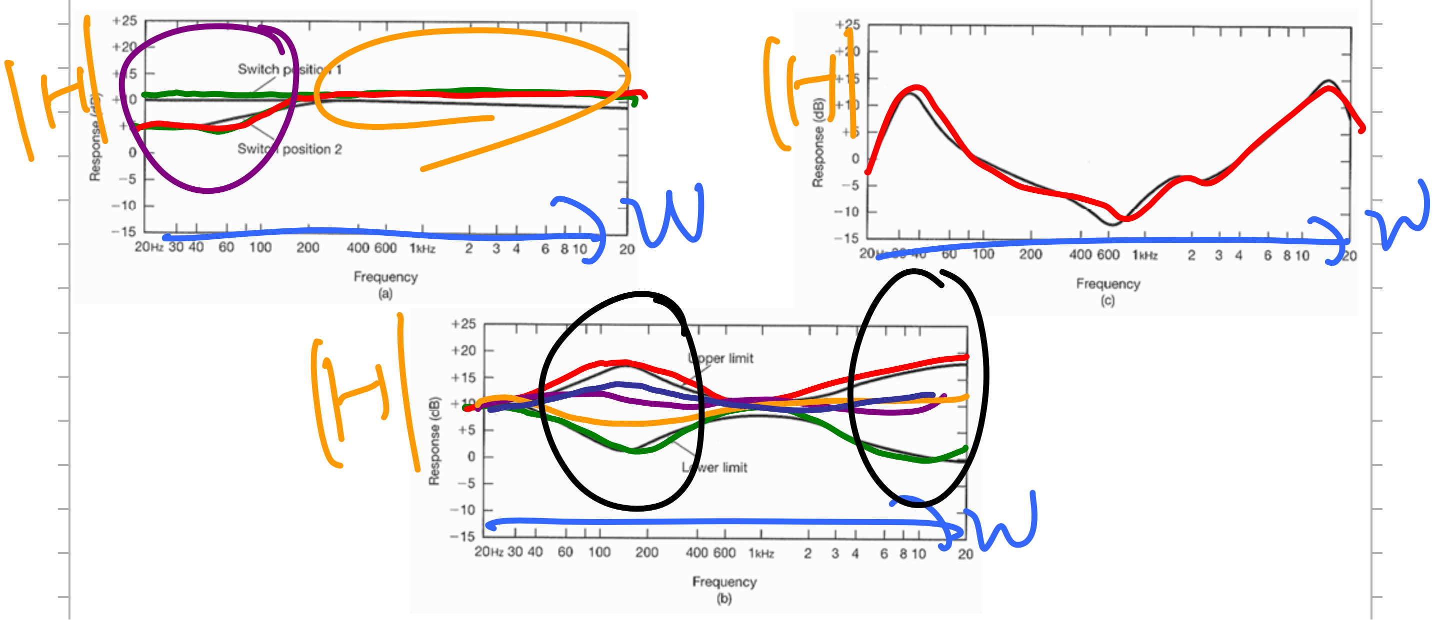
$in \longrightarrow$ filter $\longrightarrow out$

  - Change the relative amplitudes of the frequency components in a signal,
    – Frequency-shaping filters
  - OR, significantly attenuate or eliminate some frequency components entirely
    – Frequency-selective filters

- **Frequency-Shaping Filters:**
  - Audio System:

Microphone or Tape → Bass → Treble → Equalizer → Speaker

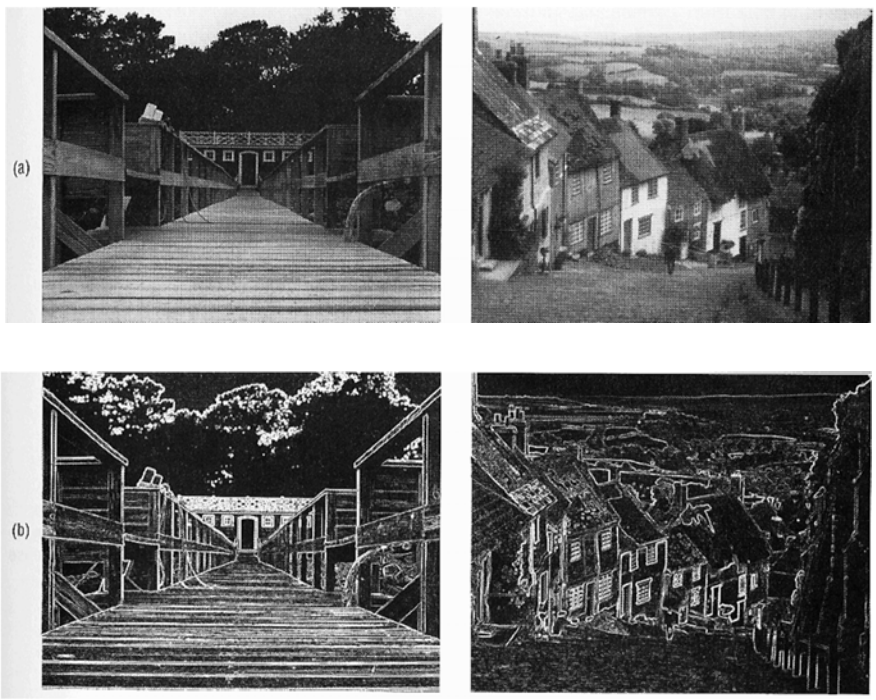
H



- **Frequency-Shaping Filters:**
  - Differentiating filter on enhancing edges:

$x(t)$ → Differentiating Filter → $y(t) = \frac{d}{dt}x(t)$

$H(jw) = jw$



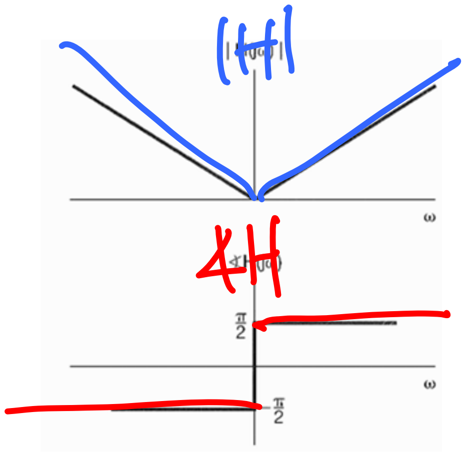
Frequency-Shaping Filters:

- Differentiating filter:

$x(t) \rightarrow$ Differentiating Filter $\rightarrow y(t) = \frac{d}{dt}x(t)$

$H(jw) = jw$

$Y(jw) = jw X(jw)$



Frequency-Shaping Filters:

$$1 \pm e^{-j\theta} = e^{-j\theta/2}\left(e^{j\theta/2} \pm e^{-j\theta/2}\right)$$

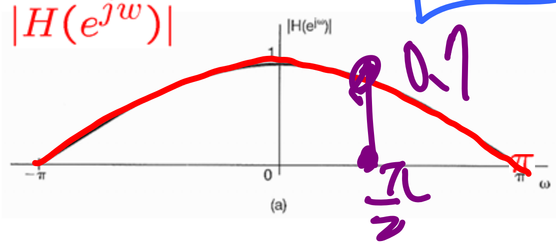
- A simple DT filter: Two-point average

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right) = \frac{1}{2}\left(1 + e^{-jw}\right) x[n] = H(e^{jw})\, x[n]$$

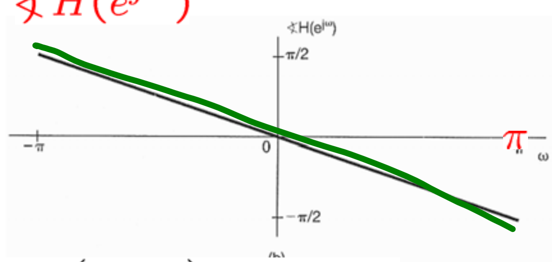
$$\Rightarrow \; H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = \frac{1}{2} e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} + e^{-j\left(\frac{w}{2}\right)}\right] \frac{1}{2}$$

$$= e^{-j\left(\frac{w}{2}\right)} \boxed{\cos\left(\frac{w}{2}\right)}$$

$|H(e^{jw})|$



$\sphericalangle H(e^{jw})$

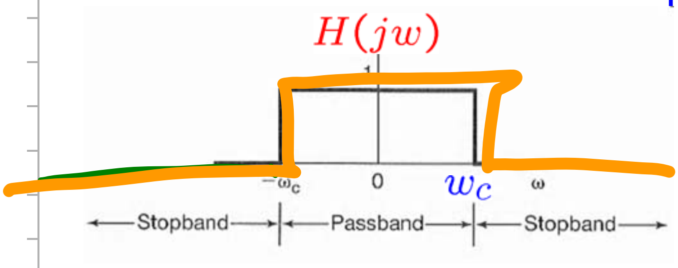


if $x[n] = K e^{j\left(\frac{\pi}{2}\right)\cdot n}$ $\quad W = \frac{\pi}{2}$ then $y[n] = H\left(e^{j\left(\frac{\pi}{2}\right)}\right) K e^{j\left(\frac{\pi}{2}\right)\cdot n}$

0.7

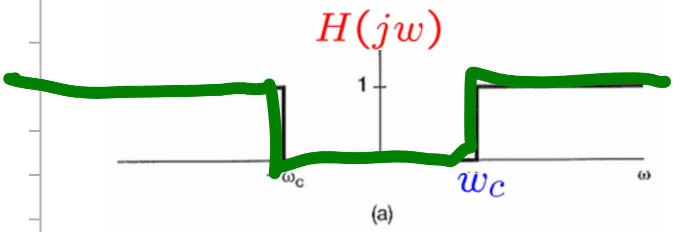
Frequency-Selective Filters:

- Select some bands of frequencies and reject others



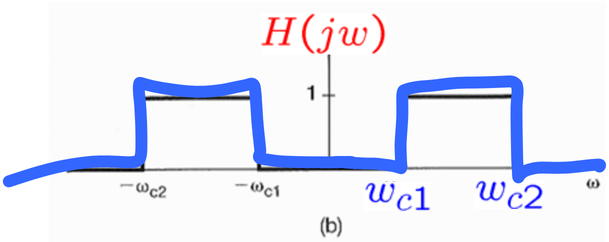
CT ideal lowpass filter

$$H(jw) = \begin{cases} 1, & |w| \leq w_c \\ 0, & |w| > w_c \end{cases}$$



CT ideal highpass filter

$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \geq w_c \end{cases}$$

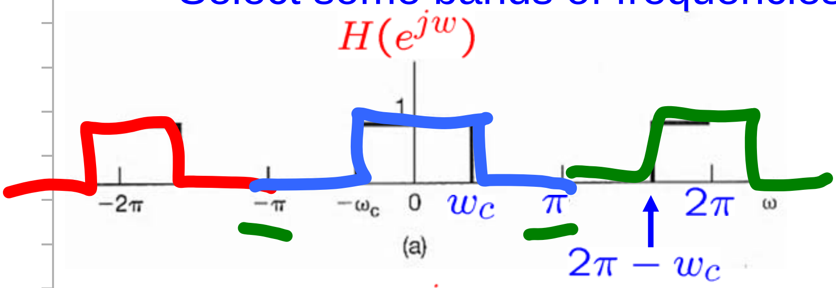


CT ideal bandpass filter

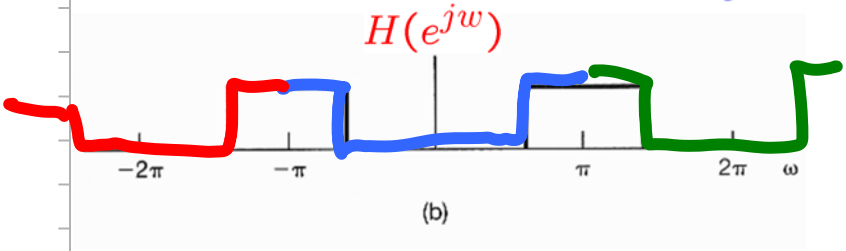
$$H(jw) = \begin{cases} 1, & w_{c1} \leq |w| \leq w_{c2} \\ 0, & \text{otherwise} \end{cases}$$

Frequency-Selective Filters:

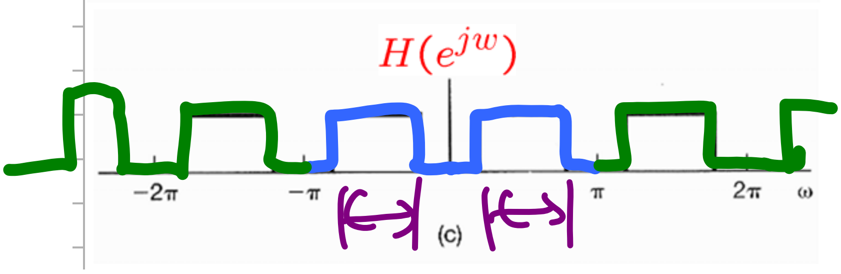
- Select some bands of frequencies and reject others



DT ideal lowpass filter



DT ideal highpass filter



DT ideal bandpass filter

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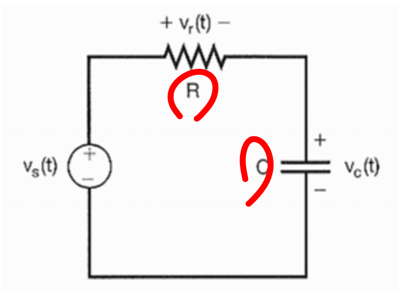
- A Simple RC Lowpass Filter:

Input signal:

$v_s(t) = e^{jwt}$

$\delta(t)$

$u(t)$



Output signal:

$v_c(t) = H(jw)e^{jwt}$

$h(t)$

$s(t)$

$$\Rightarrow \boxed{RC\,\frac{d}{dt}v_c(t) + v_c(t) = v_s(t)}$$

$$\Rightarrow RC\,\frac{d}{dt}\left[H(jw)e^{jwt}\right] + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow RCjwH(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$$

$$\Rightarrow H(jw)\cancel{e^{jwt}} = \frac{1}{1 + RCjw}\cancel{e^{jwt}}$$

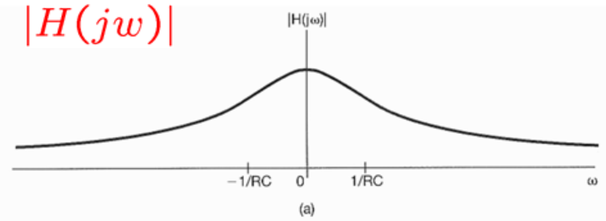
- A Simple RC Lowpass Filter: $H(jw) = \int_{-\infty}^{+\infty} h(t) e^{-jwt} dt$

$$\Rightarrow H(jw) = \frac{1}{1 + RCjw} \qquad \Rightarrow h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\Rightarrow s(t) = \left[1 - e^{-t/RC}\right] u(t)$$

$$\boxed{H = |H| e^{j\measuredangle H}}$$

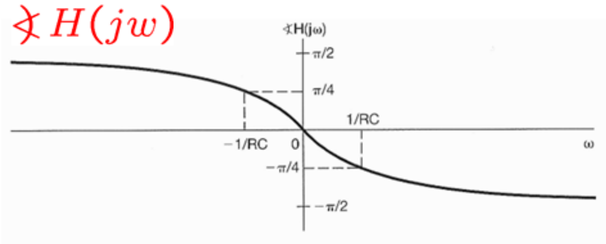
$|H(jw)|$



(a)

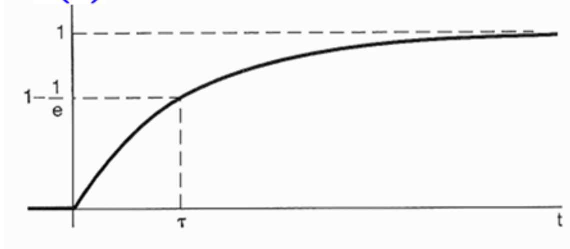
$h(t)$ impulse response

$\measuredangle H(jw)$



(b)

$s(t)$ step response



(b)

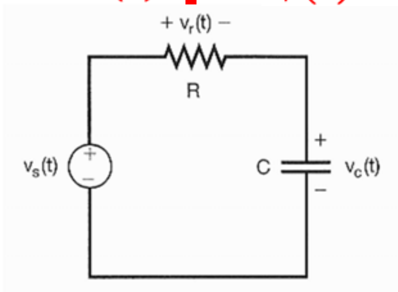
- A Simple RC Highpass Filter:

$h(t)$
$s(t)$

Output signal:
$v_r(t) = G(jw) e^{jwt}$

Input signal:
$v_s(t) = e^{jwt}$

$\delta(t)$
$u(t)$



$$\Rightarrow RC \frac{d}{dt} v_r(t) + v_r(t) = RC \frac{d}{dt} v_s(t)$$

$$\Rightarrow RC \frac{d}{dt} \left[G(jw) e^{jwt}\right] + G(jw) e^{jwt} = RC \frac{d}{dt} e^{jwt}$$

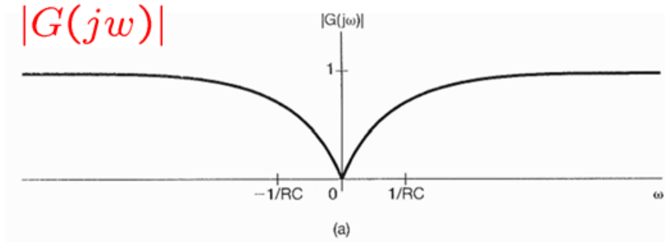
$$\Rightarrow RC\, jw\, G(jw) e^{jwt} + G(jw) e^{jwt} = RC\, jw\, e^{jwt}$$

$$\Rightarrow \underline{G(jw) e^{jwt}} = \frac{jw\, RC}{\underline{1 + jw\, RC}} e^{jwt}$$

A Simple RC Highpass Filter:

$$\Rightarrow \quad G(jw) = \frac{jw\ RC}{1 + jw\ RC}$$

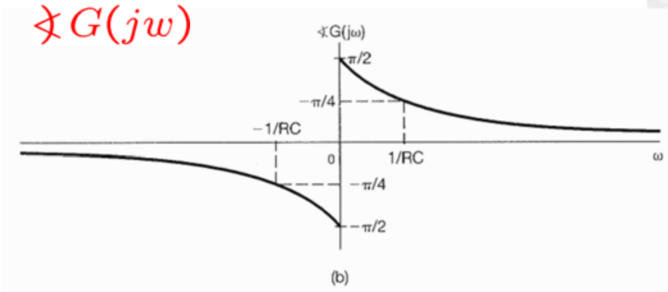
$|G(jw)|$



$$v_r(t) = v_s(t) - v_c(t)$$

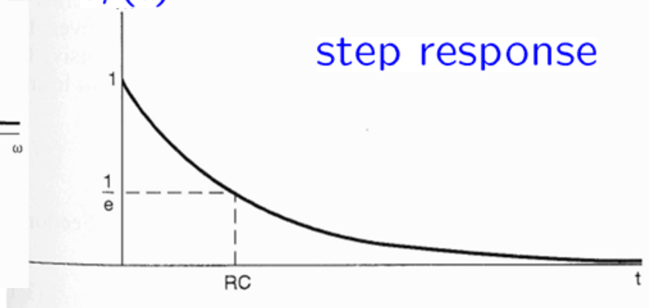
$$\Rightarrow \quad v_r(t) = e^{-t/RC}\ u(t)$$

$\measuredangle G(jw)$



$v_r(t)$

step response



First-Order Recursive DT Filters:

$$y[n] - a y[n-1] = x[n]$$

- If $x[n] = e^{jwn}$, then $y[n] = H(e^{jw}) e^{jwn}$

  where $H(e^{jw})$: the frequency response

$$\Rightarrow \quad H(e^{jw})\ e^{jwn} - a\ H(e^{jw})\ e^{jw(n-1)} = e^{jwn}$$

$$\Rightarrow \quad \left[1 - a\, e^{-jw}\right] H(e^{jw})\ e^{jwn} = e^{jwn}$$

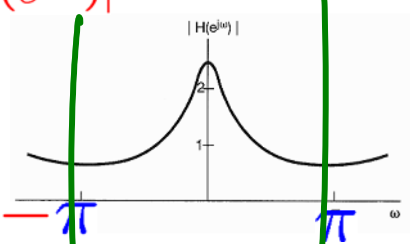
$$\Rightarrow \quad H(e^{jw}) = \boxed{\frac{1}{1 - a\ e^{-jw}}}$$

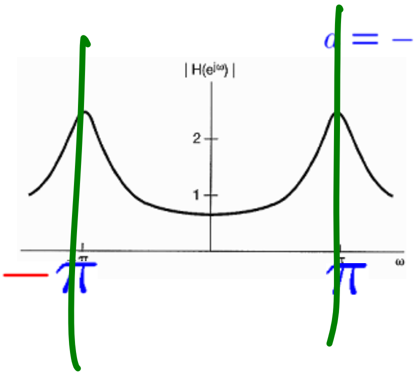
▪ First-Order Recursive DT Filters:

$$H(e^{jw}) = \frac{1}{1 - a\, e^{-jw}} \qquad y[n] = ay[n-1] + x[n]$$

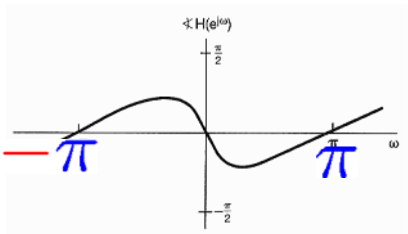
lowpass filter: $0 < a < 1$ highpass filter: $-1 < a < 0$

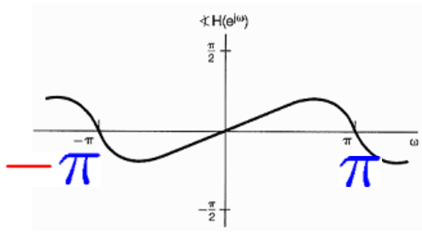
$|H(e^{jw})|$ $a = 0.6$ $a = -0.6$





$\sphericalangle H(e^{jw})$

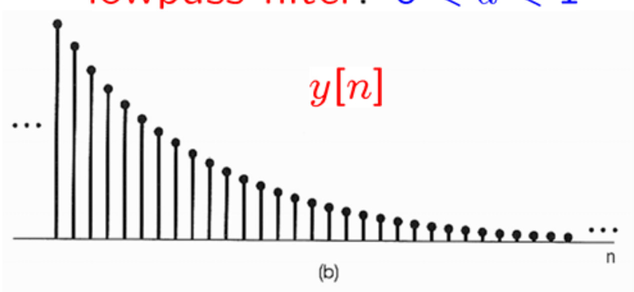




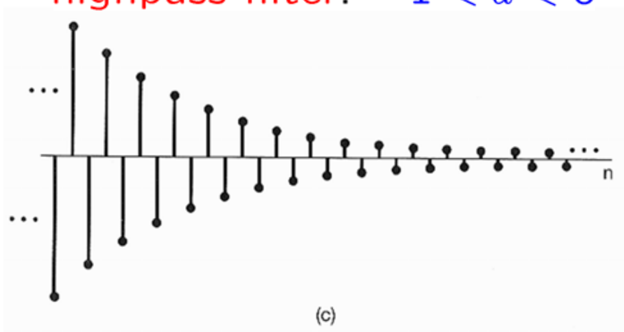
▪ First-Order Recursive DT Filters:

$$y[n] = ay[n-1] + x[n]$$

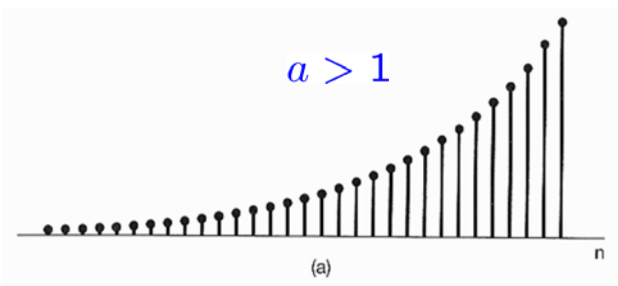
lowpass filter: $0 < a < 1$ highpass filter: $-1 < a < 0$



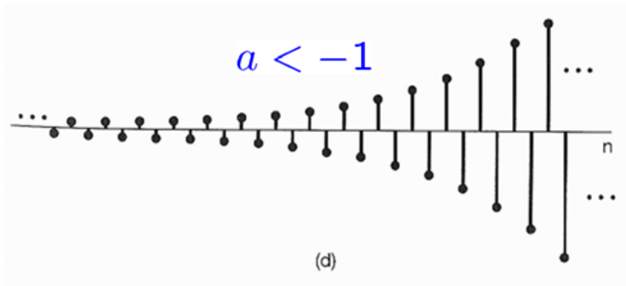
(b)



(c)



(a)



(d)

Nonrecursive DT Filters:

- An FIR nonrecursive difference equation:

$$y[n] = \sum_{k=-N}^{M} b_k\, x[n-k]$$

N+M+1

$$= b_{-N}\, x[n+N] + b_{-N+1}\, x[n+N-1] + \cdots$$

$+b_{-1}\, x[n+1]$

$$+b_0\, x[n] + b_1\, x[n-1] + \cdots + b_M\, x[n-M]$$

$$b_k = \frac{1}{N+M+1}$$

$$b_k = \frac{\sin w_c \cancel{n} k}{\pi \cancel{n} k}$$

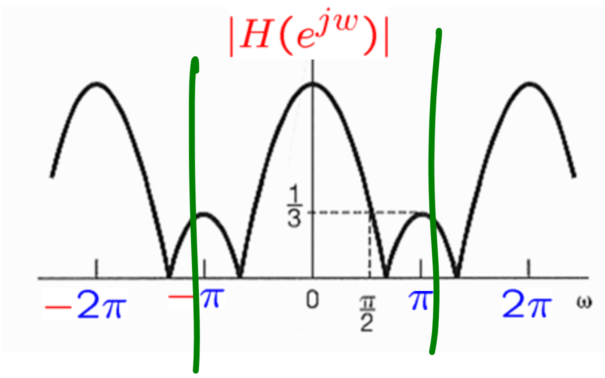
Nonrecursive DT Filters:

- Three-point moving average (lowpass) filter:

$$y[n] = \frac{1}{3}\big(x[n+1] + x[n] + x[n-1]\big)$$

$$\Rightarrow\ h[n] = \frac{1}{3}\big(\delta[n+1] + \delta[n] + \delta[n-1]\big)$$

$$\Rightarrow\ H(e^{jw}) = \frac{1}{3}\big(e^{jw} + 1 + e^{-jw}\big) = \frac{1}{3}\big(1 + 2\cos w\big)$$



$e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

ss3-67

- Nonrecursive DT Filters:
  - N+M+1 moving average (lowpass) filter:

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-jwk}$$

+M +M+1 +1

$( ) + ( ) \cdots - ( )$

$( )^{0} + ( )^{-1} \cdots + ( )^{-N}$

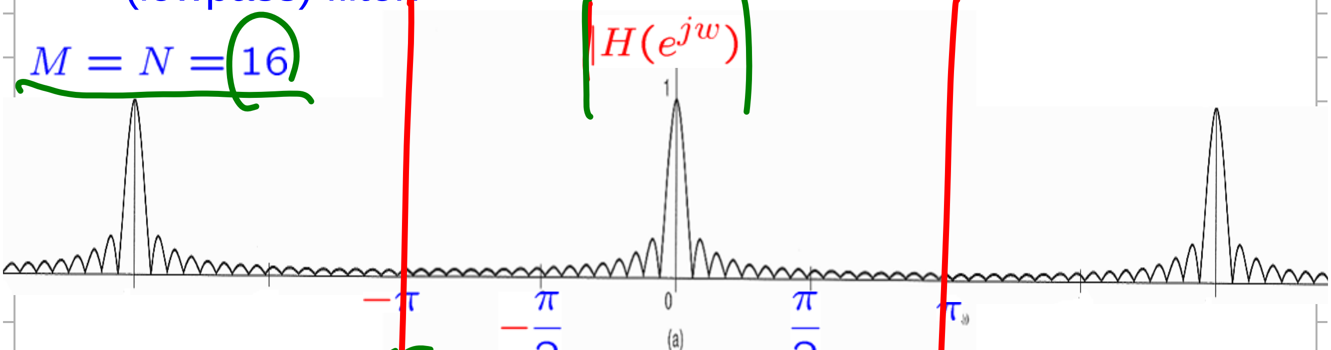
$$\Rightarrow H(e^{jw}) = \frac{1}{N+M+1} e^{jw\left(\frac{N-M}{2}\right)} \frac{\sin\left((M+N+1)\frac{w}{2}\right)}{\sin\left(\frac{w}{2}\right)}$$

$$\frac{1-e^{-ja}}{1-e^{-jb}} = \frac{e^{-ja/2}\left(e^{ja/2} - e^{-ja/2}\right)}{e^{-jb/2}\left(e^{jb/2} - e^{-jb/2}\right)}$$

- Nonrecursive DT Filters:
  - N+M+1 moving average (lowpass) filter:

33

$M = N = 16$



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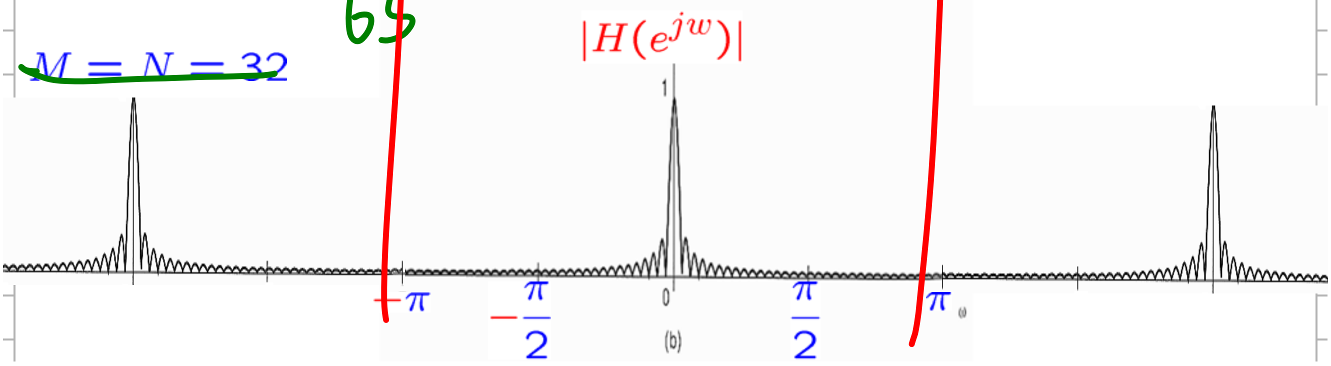
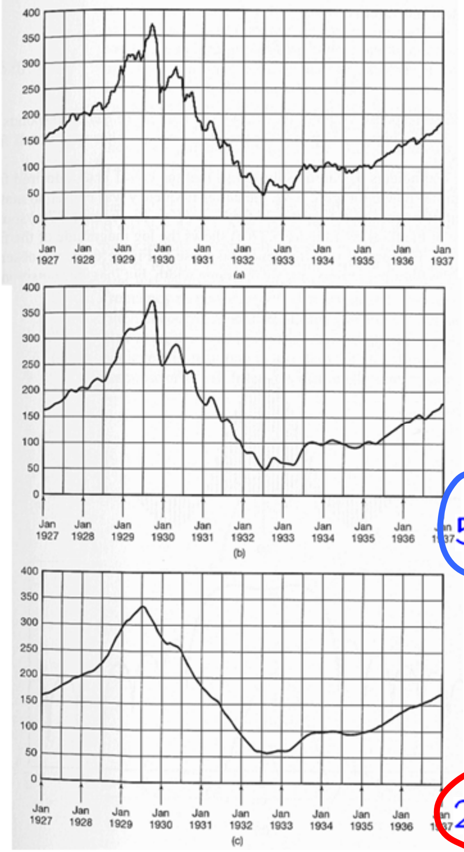
$M = N = 32$



Lowpass Filtering on Dow Jones Weekly Stock Market Index:



http://big5.jrj.com.cn/

51-day moving average

201-day moving average

Nonrecursive DT Filters:

$$\boxed{1 \pm e^{-j\theta} = e^{-j\theta/2}\left(e^{j\theta/2} \pm e^{-j\theta/2}\right)}$$

- Highpass filters:

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

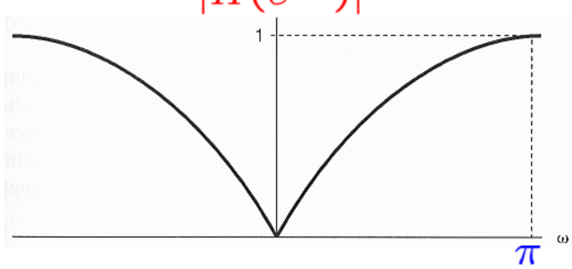
$$\Rightarrow h[n] = \frac{1}{2}\left\{\delta[n] - \delta[n-1]\right\}$$

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 - e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} - e^{-j\left(\frac{w}{2}\right)}\right]$$

2j

$$= j\, e^{-j\left(\frac{w}{2}\right)} \sin\left(\frac{w}{2}\right)$$

$|H(e^{jw})|$



- On page 235, Eq. 3.139

$$1 \pm e^{-j\theta} = e^{-j\theta/2}\left(e^{j\theta/2} \pm e^{-j\theta/2}\right)$$

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} + e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= e^{-j\left(\frac{w}{2}\right)} \cos\left(\frac{w}{2}\right)$$

- On page 249, Eq. 3.164

$$\Rightarrow H(e^{jw}) = \frac{1}{2}\left[1 - e^{-jw}\right] = \frac{1}{2}e^{-j\left(\frac{w}{2}\right)}\left[e^{j\left(\frac{w}{2}\right)} - e^{-j\left(\frac{w}{2}\right)}\right]$$

$$= j\, e^{-j\left(\frac{w}{2}\right)} \sin\left(\frac{w}{2}\right)$$

- A Historical Perspective
- The Response of LTI Systems to Complex Exponentials
- FS Representation of CT Periodic Signals
- Convergence of the FS
- Properties of CT FS
  - Linearity, Time Shifting, Frequency Shifting, Conjugation
  - Time Reversal, Time Scaling, Periodic Convolution, Multiplication
  - Differentiation, Integration, Conjugate Symmetry for Real Signals
  - Symmetry for Real and Even Signals, Symmetry for Real and Odd Signals
  - Even-Odd Decomposition for Real Signals, Parseval's Relation for Periodic Signals
- FS Representation of DT Periodic Signals
- Properties of DT FS
  - Multiplication, First Difference, Running Sum
- FS & LTI Systems
- Filtering
  - Frequency-shaping filters & Frequency-selective filters
- Examples of CT & DT Filters

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS (Chap 3) CT DT

Aperiodic

FT CT (Chap 4) DT (Chap 5)

Unbounded/Non-convergent

LT CT (Chap 9)

zT DT (Chap 10)

Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)