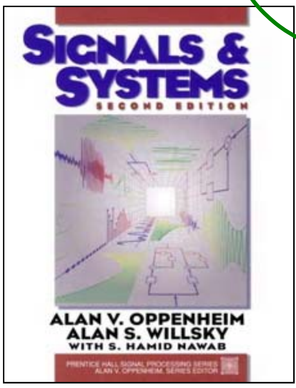
Spring 2012

信號與系統
Signals and Systems

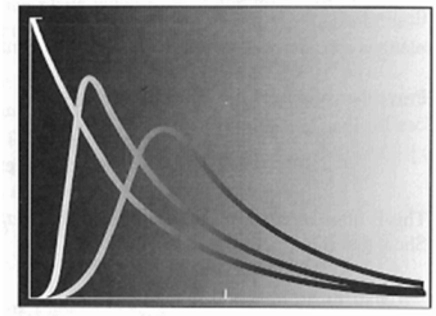
Chapter SS-2
Linear Time-Invariant Systems



Feng-Li Lian

NTU-EE

Feb12 – Jun12



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

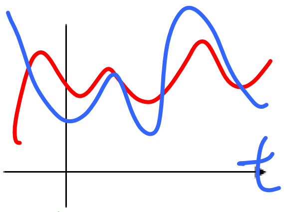
Outline

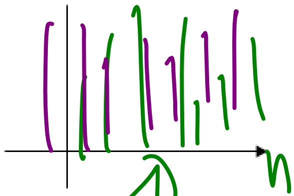
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$ $x(t) \rightarrow$ h(t) $\rightarrow y(t)$

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$ $\quad x(t) \rightarrow$ h(t) $\rightarrow y(t)$

Signals





ce^{at}

$A\cos(\omega t+\phi)$

$\delta(t)$ $\delta[n]$

$u(t)$ $u[n]$

Systems

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$

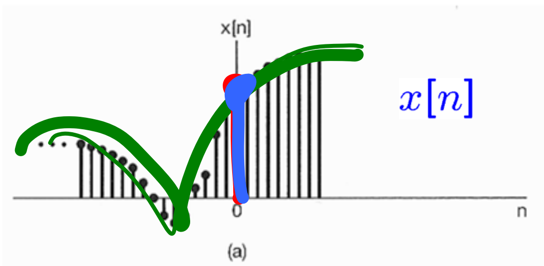
$$\Rightarrow y[n] + ay[n-1] = bx[n]$$

- ✓ memory $\Rightarrow$ $\forall t$ $\forall n$
- ✓ inverse $\forall$ $x(t)$, $x[n]$
- ✓ causal
- ✓ stable
- ✓ TI
- ✓ Linear

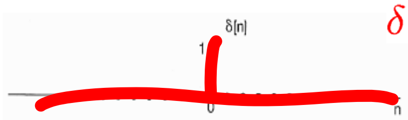
$x(t)$, $x[n]$ $\rightarrow$ $y(t)$, $y[n]$

- Sample by Unit Impulse
- For x[n]

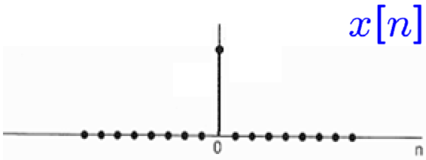
$$x[n]\delta[n] = x[0]\delta[n]$$



$x[n]$



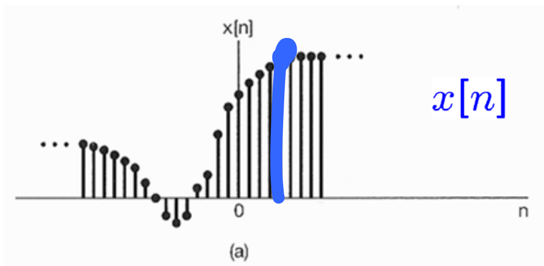
$\delta[n]$



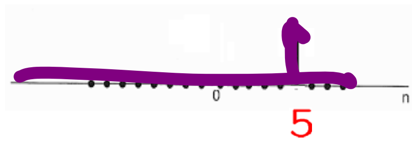
$x[n]\delta[n]$

- More generally,

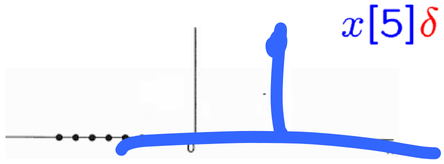
$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



$x[n]$

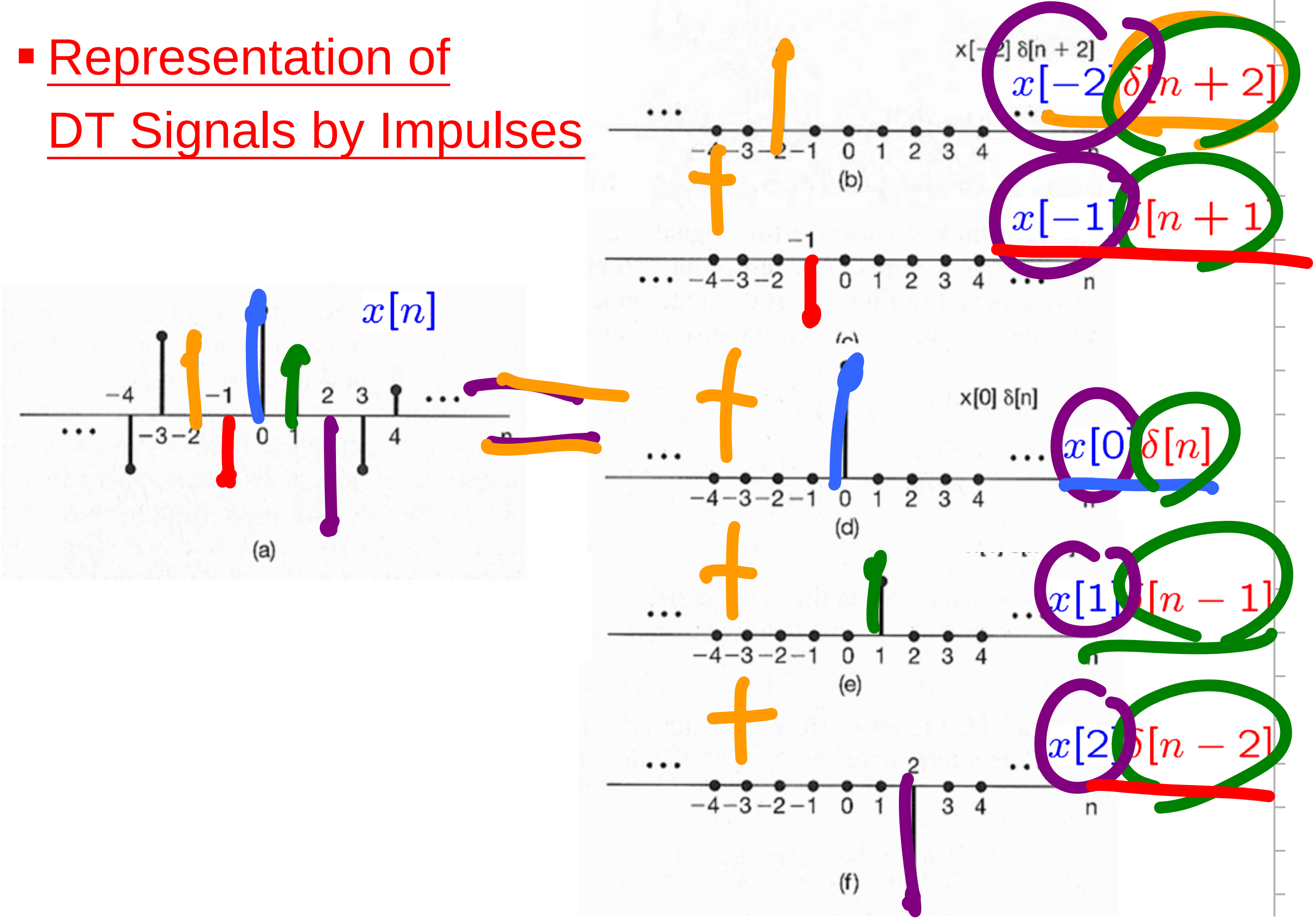


$\delta[n-5]$



$x[5]\delta[n-5]$

- Representation of DT Signals by Impulses



- Representation of DT Signals by Impulses:
  - More generally,

$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2]$$
$$+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$
$$+ x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$
$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$\sum x[k]\delta[n-k]$

- The sifting property of the DT unit impulse
- x[n] = a superposition of scaled versions of shifted unit impulses δ[n-k]

▪ DT Unit Impulse Response & Convolution Sum:

input $\longrightarrow$ Linear System $\longrightarrow$ output

TI

$a\,\delta[n] \longrightarrow$ Linear System $\longrightarrow a\,h_0[n]$ =

+ +

$b\,\delta[n-1] \longrightarrow$ Linear System $\longrightarrow b\,h_1[n]$

+ +

$c\,\delta[n-2] \longrightarrow$ Linear System $\longrightarrow b\,h_2[n]$

+ +

⋮

+

$\delta[n-k] \longrightarrow$ Linear System $\longrightarrow h_k[n]$

$h_0[n-2] = h_2[n]$

▪ DT Unit Impulse Response & Convolution Sum:

$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

$x[0] \cdot \delta[n] \longrightarrow$ Linear System $\longrightarrow h_0[n] \cdot x[0]$

+ +

$x[1] \cdot \delta[n-1] \longrightarrow$ Linear System $\longrightarrow h_1[n] \cdot x[1]$

+ +

$x[2] \cdot \delta[n-2] \longrightarrow$ Linear System $\longrightarrow h_2[n] \cdot x[2]$

+ +

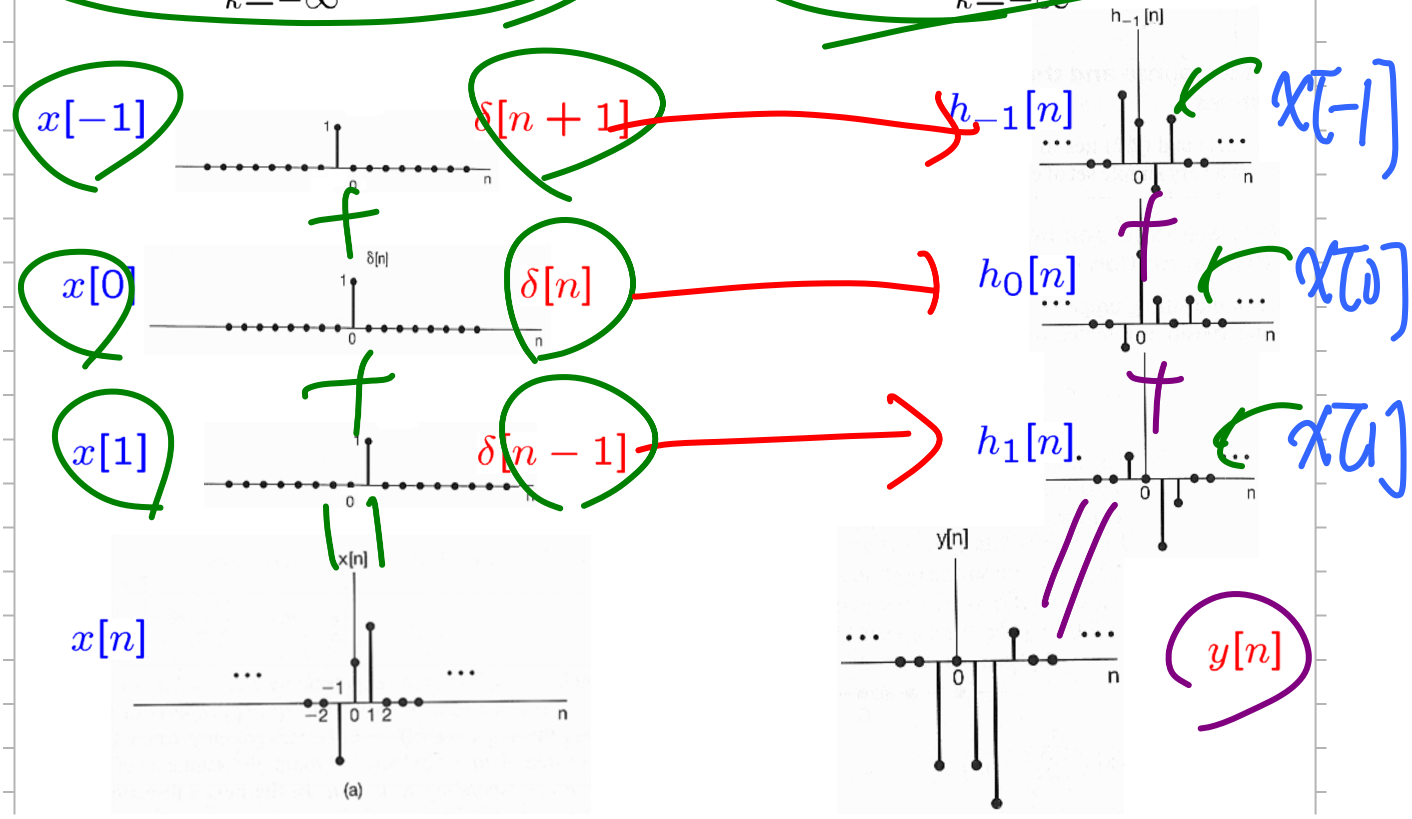
⋮

$x[k] \cdot \delta[n-k] \longrightarrow$ Linear System $\longrightarrow h_k[n] \cdot x[k]$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h_k[n]$$

DT LTI Systems: Convolution Sum

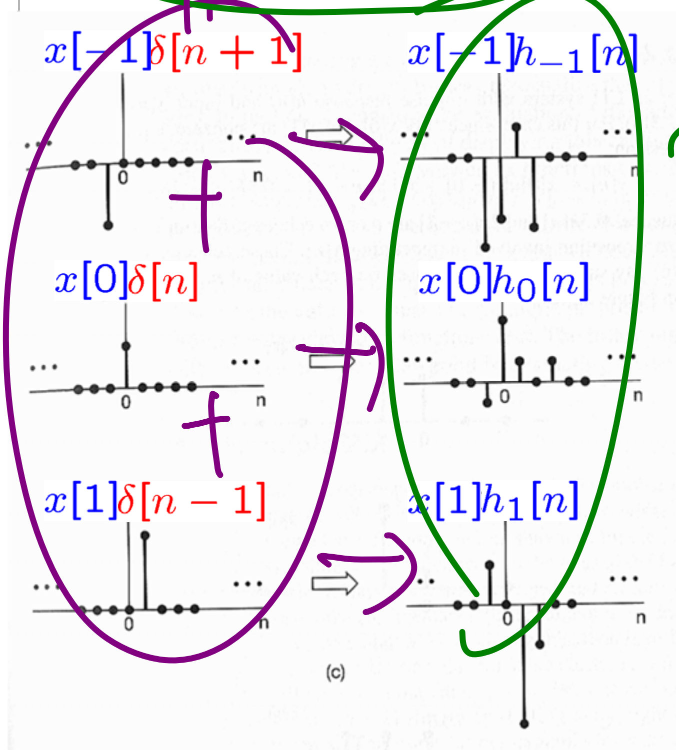
$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

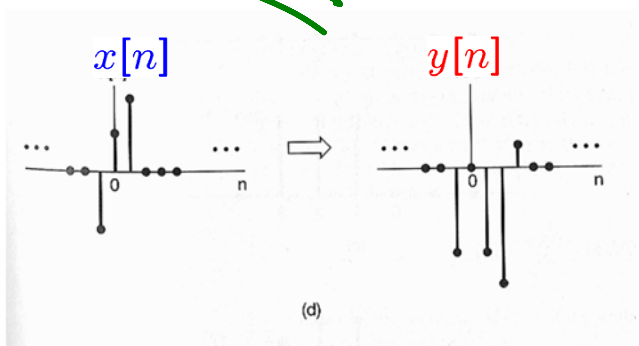


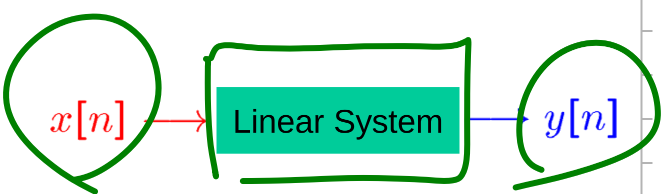
DT LTI Systems: Convolution Sum

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

h[n-k]







- If the linear system (L) is also time-invariant (TI)
  - Then,

$$h_k[n] = h_0[n-k] = h[n-k]$$

$\delta[n] \rightarrow h[n]$

- Hence, for an LTI system,

$h_k[n]$ $h[n] = h_0[n]$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

$m = n-k$

  - Known as the convolution of x[n] & h[n]
  - Referred as the convolution sum or superposition sum

- Symbolically, $y[n] = x[n] * h[n] = h[n] * x[n]$

$*$

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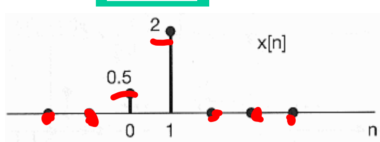
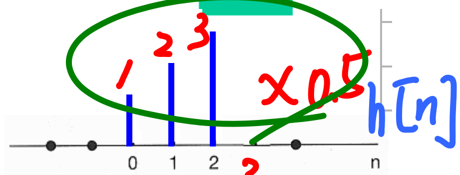
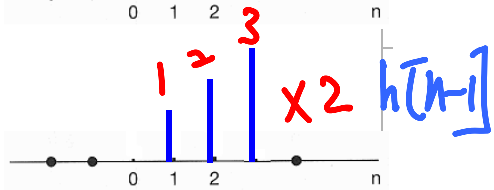
- Example 2.1: $x[n] \longrightarrow$ LTI $\longrightarrow y[n]$

$\delta[n] \longrightarrow$ LTI $\longrightarrow h[n]$



x[n]

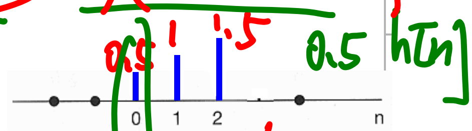




$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

0.5 2

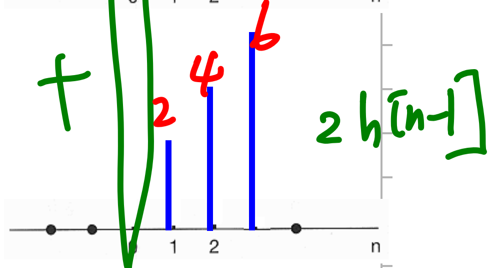
$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

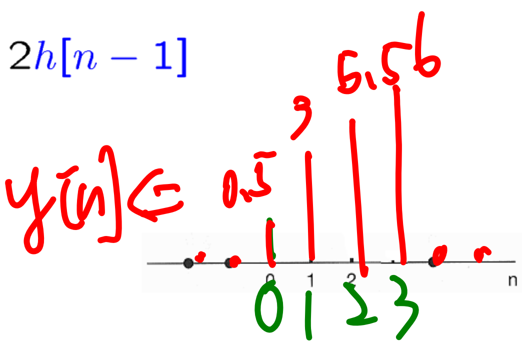


$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

$$= 0.5h[n] + 2h[n-1]$$

+

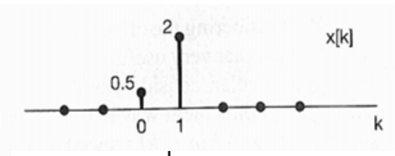


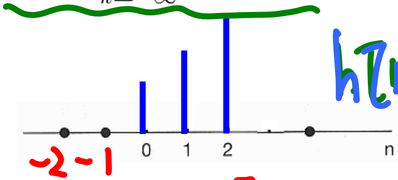


Example 2.2:

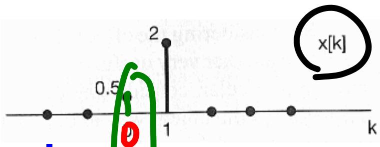
$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

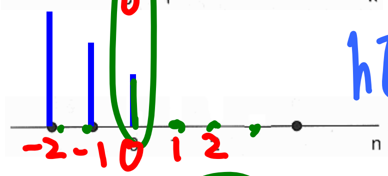
$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$





h[n]



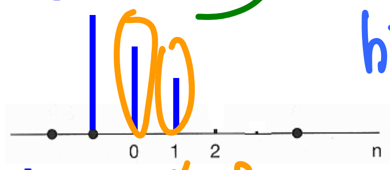


h[-k]

n=0

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k] = h[-k]$$

$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

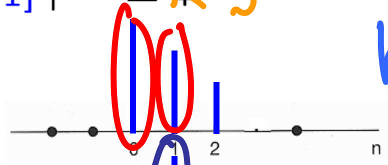


h[1-k]

n=1

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k]$$

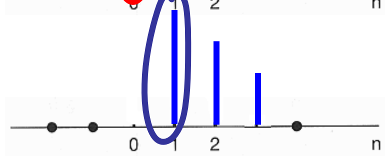
$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = \cancel{4}\ 3$$



h[2-k]

n=2

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 5.5$$



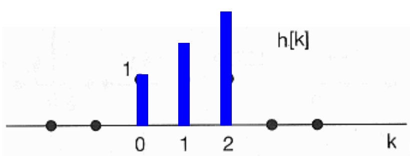
h[3-k]

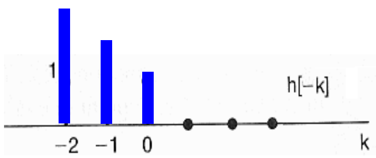
n=3

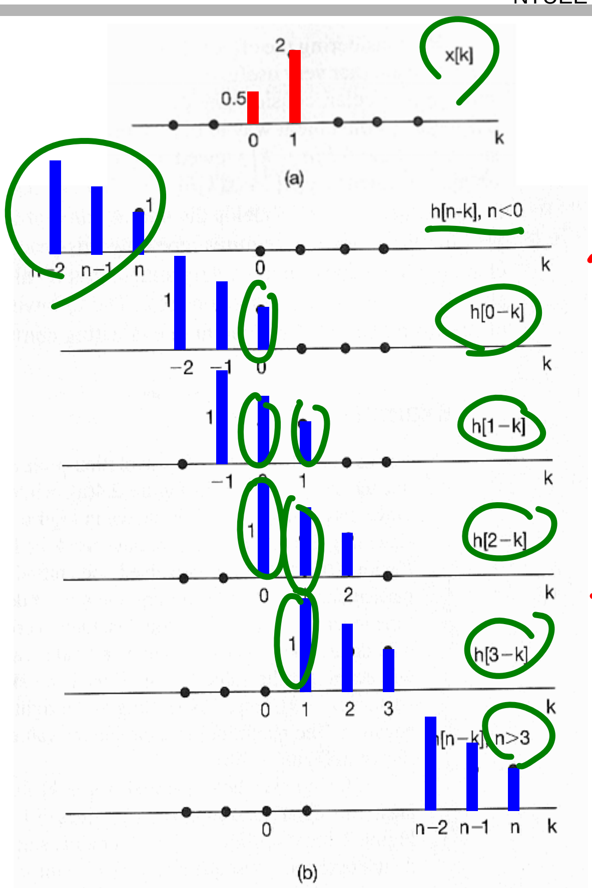
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = \cancel{4.0}\ 6.0$$

Example 2.2:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

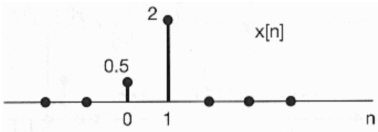


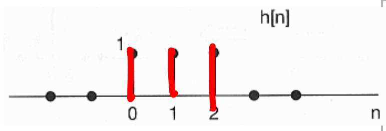




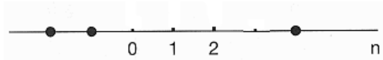
Example 2.1:

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$





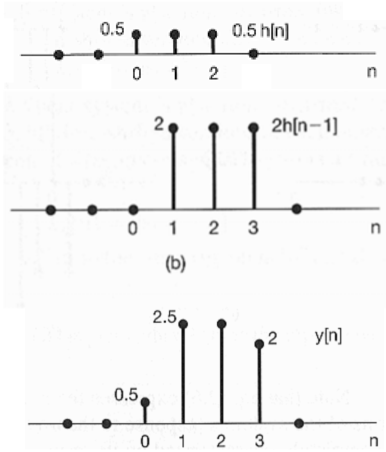
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

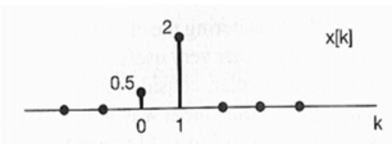
$$= 0.5h[n] + 2h[n-1]$$

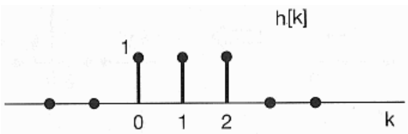


Example 2.2:

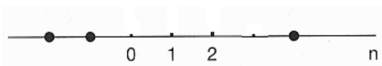
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



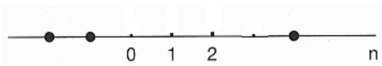


$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$



$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$



$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 2.5$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

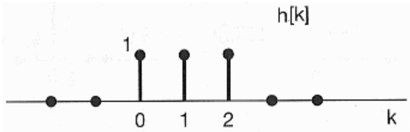
$$y[n] = 0 \text{ for } n < 0$$

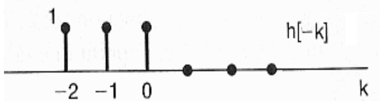
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

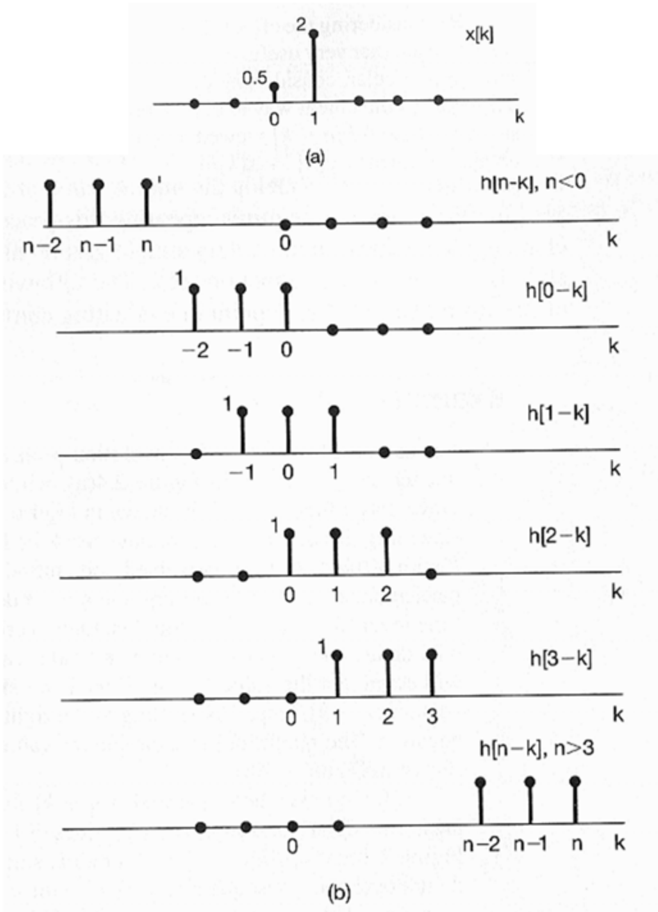
$$y[n] = 0 \text{ for } n > 3$$

- Example 2.2:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

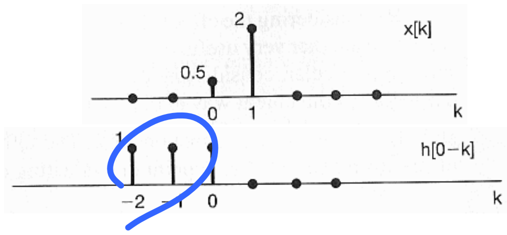


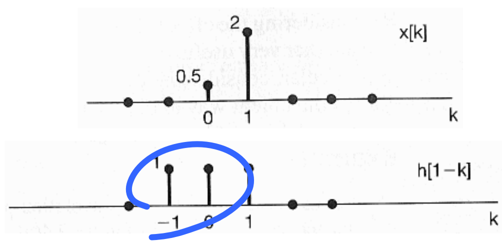


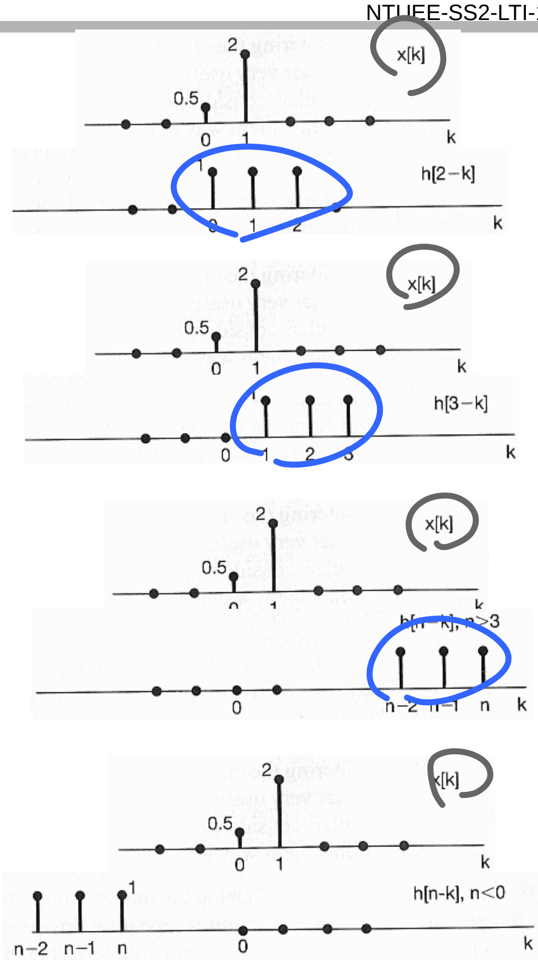


- Example 2.2:

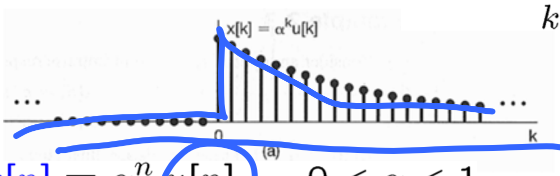
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$





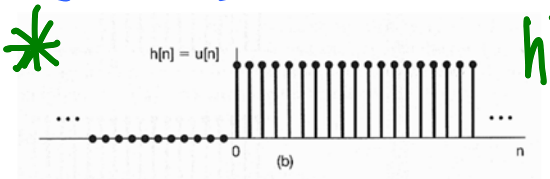


- **Example 2.3:** $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

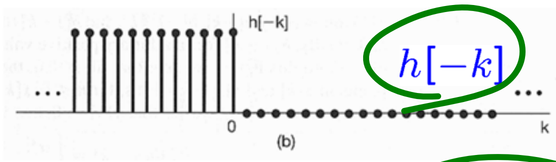


$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$

$*$ $h[n]$

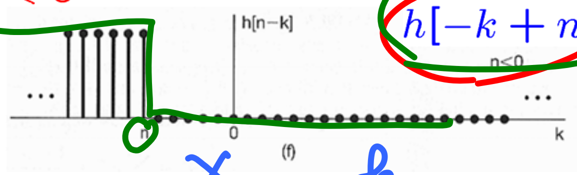


$h[n] = u[n]$



$h[-k]$

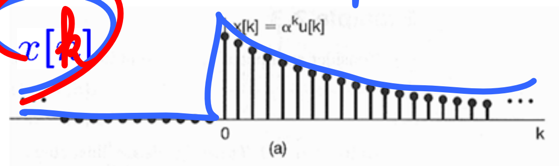
$n < 0$



$h[-k+n]$

$\times$ k

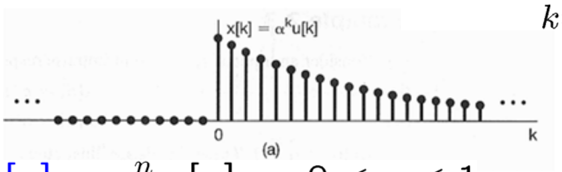
$x[k]$



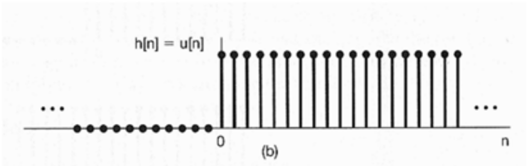
for $n < 0$, $\quad x[k]\ h[n-k] = 0$

$\Rightarrow \ y[n] = 0$

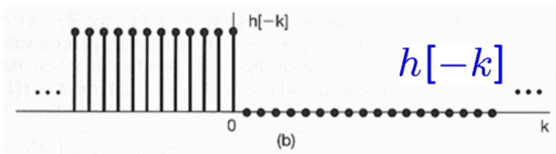
- **Example 2.3:** $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$

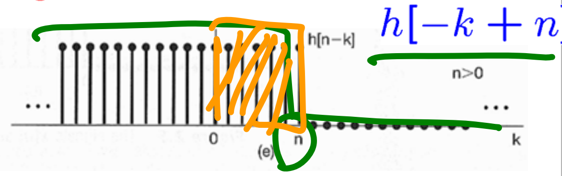


$h[n] = u[n]$



$h[-k]$

$n > 0$

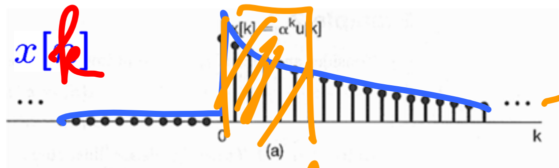


$h[-k+n]$

for $n \geq 0$,

$$x[k]\ h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$x[k]$



k

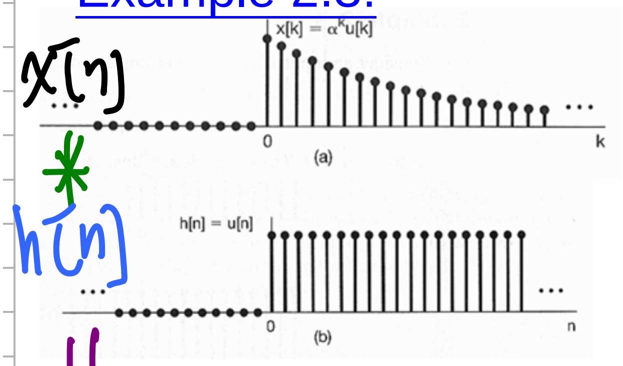
$0 \quad n$

$$\Rightarrow \ y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Example 2.3:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$x[n]$

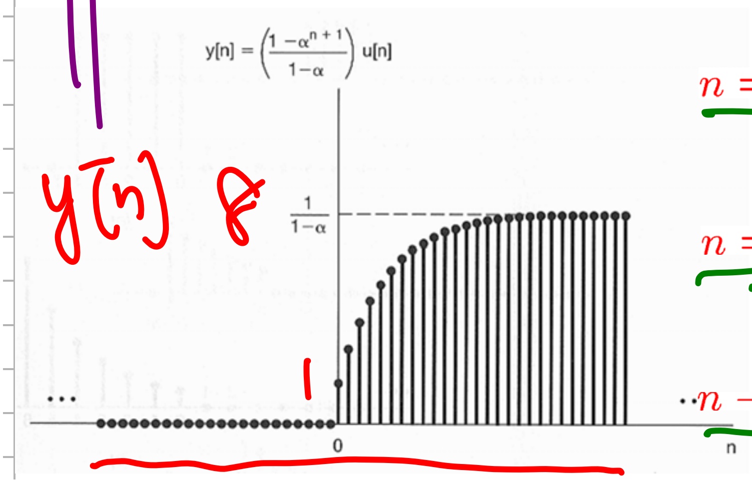


$*$

$h[n]$

$\|$

$y[n]$



for all n, $y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$

$\alpha = \frac{7}{8}$

$n = 0$ $\quad y[0] = \frac{1-\frac{7}{8}}{1-\frac{7}{8}} = 1$

$n = 1$ $\quad y[1] = \frac{1-(\frac{7}{8})^2}{1-\frac{7}{8}} = \frac{15}{8}$

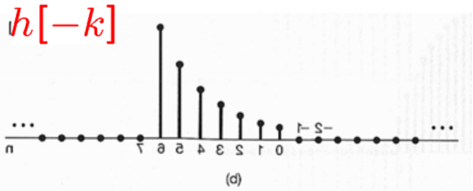
$n \to \infty$ $\quad y[n] = \frac{1-0}{1-\frac{7}{8}} = 8$

Example 2.4:

$x[n] = 1, \ 0 \le n \le 4$ 5

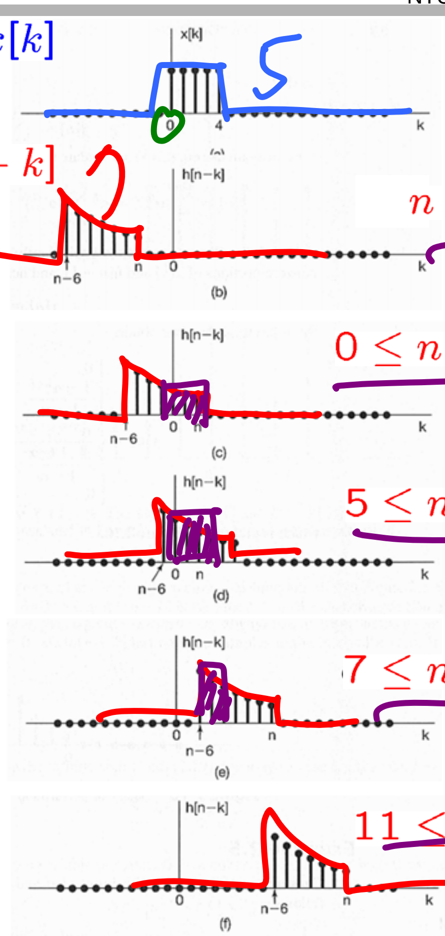
$h[n] = \alpha^n, \ 0 \le n \le 6$ 7

$h[-k]$



$x[k]$ 5

$h[n-k]$



$n \le -1$

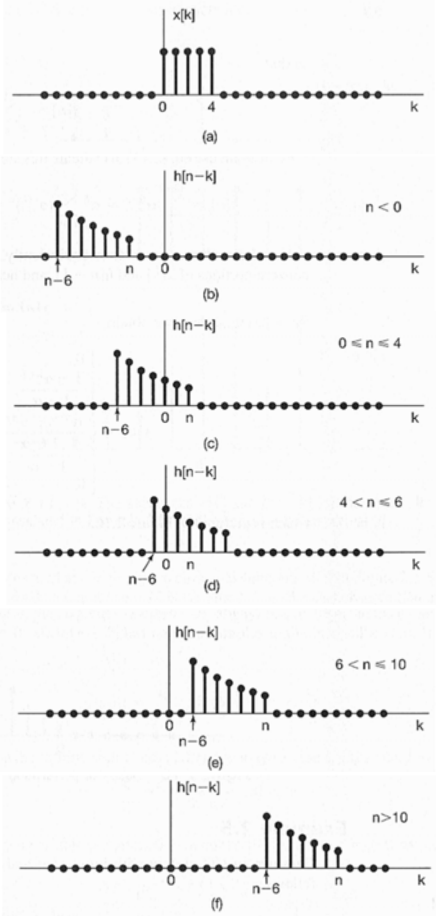
$0 \le n \le 4$

$5 \le n \le 6$

$7 \le n \le 10$

$11 \le n$

DT LTI Systems: Convolution Sum



for $n < 0$, $x[k]\,h[n-k] = 0 \;\;\Rightarrow\;\; y[n] = 0$

for $0 \le n \le 4$, $x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $4 < n \le 6$, $x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

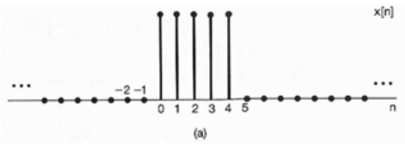
for $6 < n \le 10$, $x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

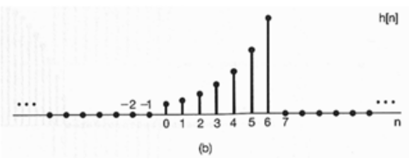
for $n > 10$, $y[n] = 0$

DT LTI Systems: Convolution Sum

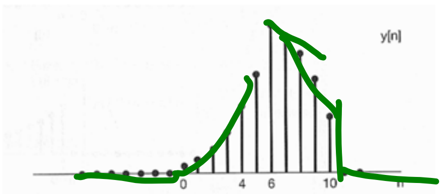
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$



$x[n] = 1,\; 0 \le n \le 4$



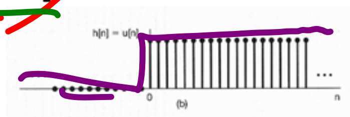
$h[n] = \alpha^n,\; 0 \le n \le 6$

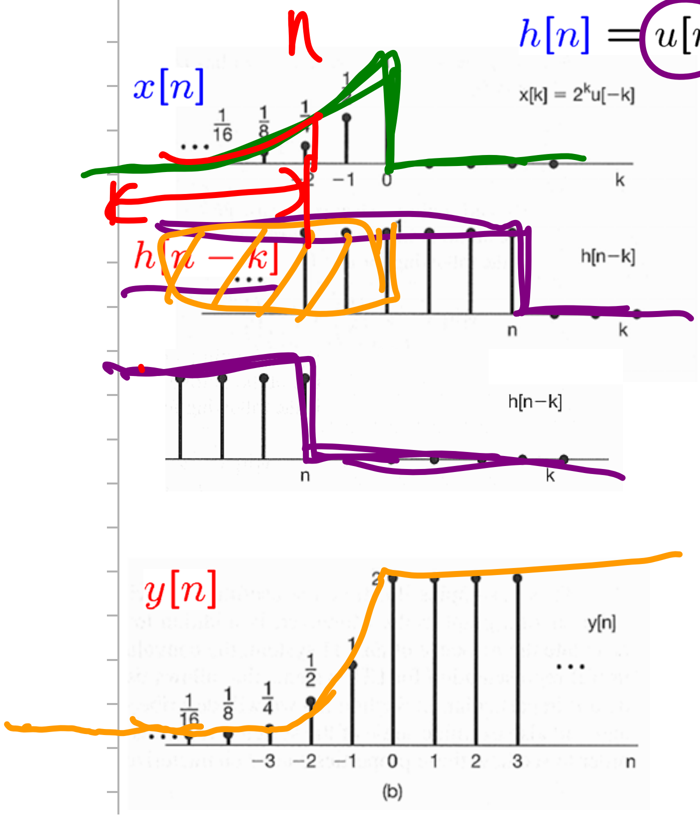


$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 4 < n \le 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 6 < n \le 10 \\ 0, & 10 < n \end{cases}$$

- Example 2.5: $x[n] = 2^n u[-n]$ $\quad h[n] = u[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$





for $n \geq 0$, $\quad y[n] = \sum_{k=-\infty}^{0} x[k]\, h[n-k] = \sum_{k=-\infty}^{0} 2^k$

$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

for $n < 0$, $\quad y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} 2^k$

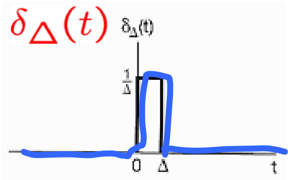
$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

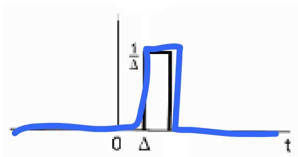
Outline

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $\quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

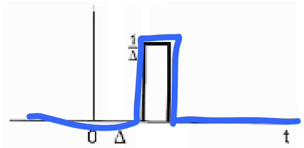
Representation of CT Signals by Impulses:



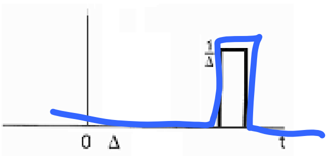
$$\delta_\triangle(t) = \begin{cases} \frac{1}{\triangle}, & 0 \le t < \triangle \\ 0, & \text{otherwise} \end{cases}$$



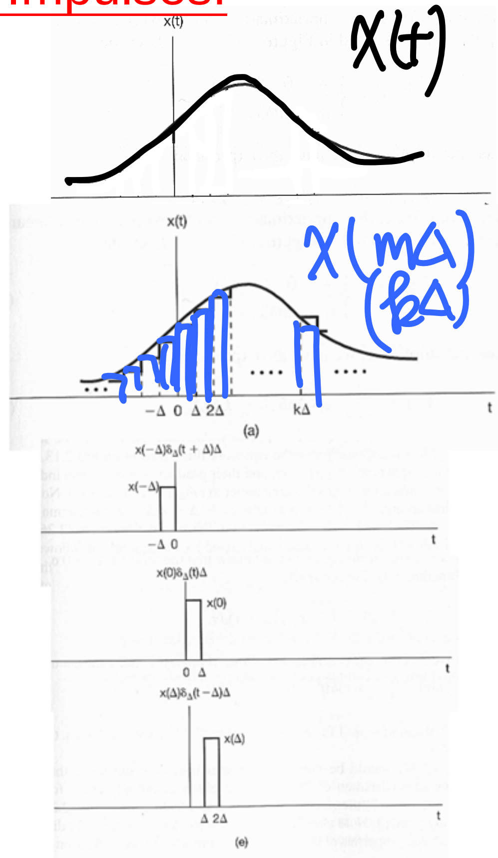
$\delta_\triangle(t - \triangle)$



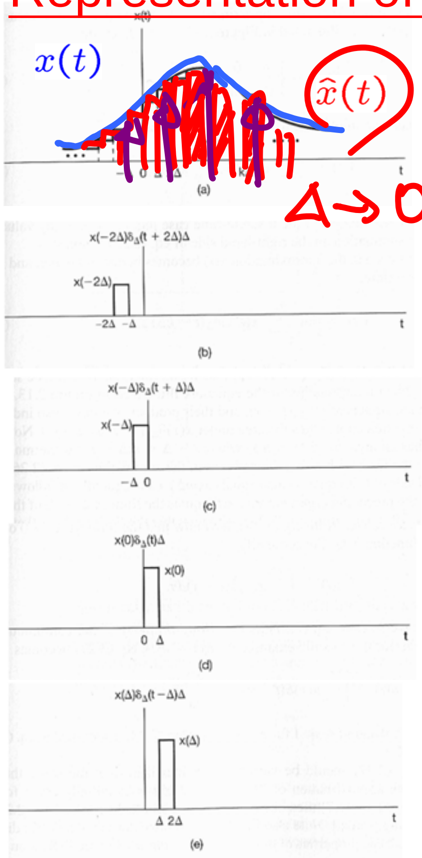
$\delta_\triangle(t - 2\triangle)$



$\delta_\triangle(t - k\triangle)$



Representation of CT Signals by Impulses:



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\triangle)\delta_\triangle(t - k\triangle)\triangle$$

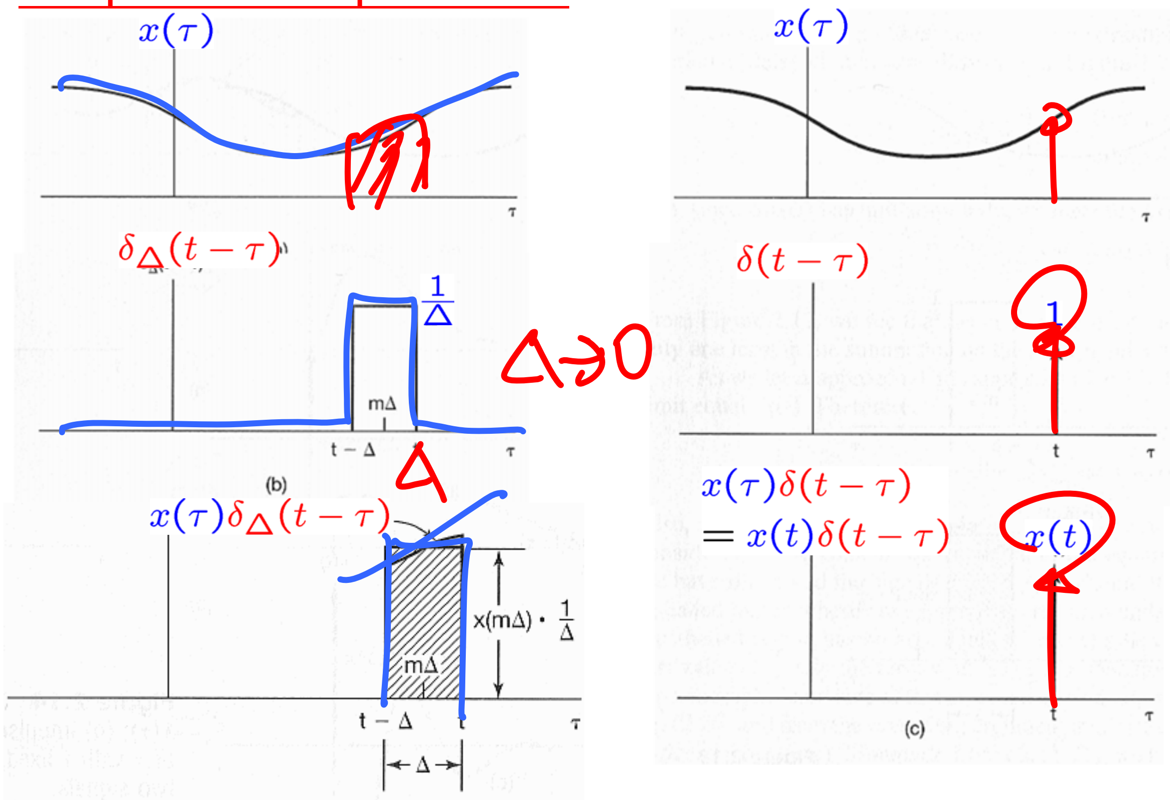
$$x(t) = \lim_{\triangle \to 0} \sum_{k=-\infty}^{+\infty} x(k\triangle)\delta_\triangle(t - k\triangle)\triangle$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)\,d\tau$$

the sifting property of CT impulse

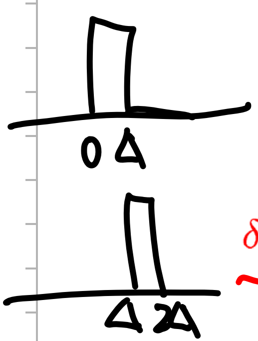
$x(t)$ = an integral of weighted, shifted impulses

- Graphical interpretation:



- CT Impulse Response & Convolution Integral:

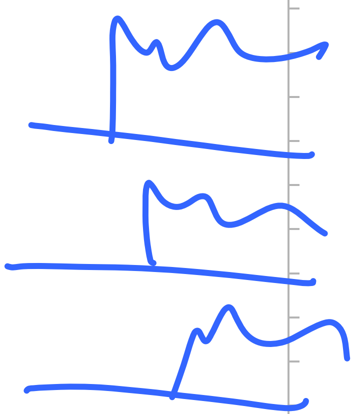
input $\longrightarrow$ Linear System $\longrightarrow$ output



$\delta_\Delta(t)$ $\longrightarrow$ Linear System $\longrightarrow$ $\hat{h}_{0\Delta}(t)$

$\delta_\Delta(t-1\Delta)$ $\longrightarrow$ Linear System $\longrightarrow$ $\hat{h}_{1\Delta}(t)$

$\delta_\Delta(t-2\Delta)$ $\longrightarrow$ Linear System $\longrightarrow$ $\hat{h}_{2\Delta}(t)$



⋮

$\delta_\Delta(t-k\Delta)$ $\longrightarrow$ Linear System $\longrightarrow$ $\hat{h}_{k\Delta}(t)$

CT Impulse Response & Convolution Integral:

input $\longrightarrow$ Linear System $\longrightarrow$ output

$x(0\Delta)$ $\quad \delta_\Delta(t) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{0\Delta}(t)$ $\quad x(0\Delta)\,\Delta$

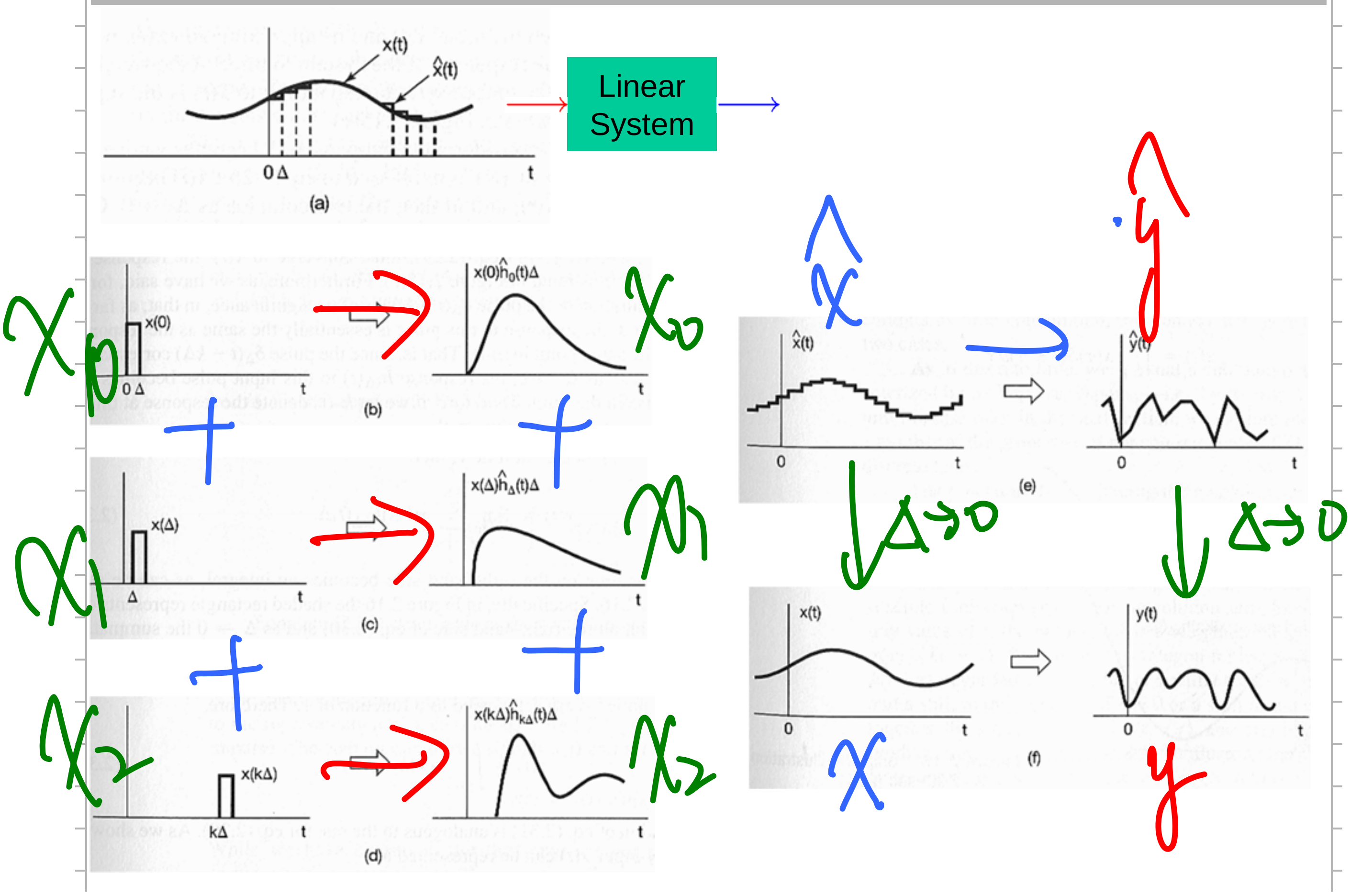
$x(1\Delta)$ $\quad \delta_\Delta(t-1\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{1\Delta}(t)$ $\quad x(1\Delta)\,\Delta$

$x(2\Delta)$ $\quad \delta_\Delta(t-2\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{2\Delta}(t)$ $\quad x(2\Delta)\,\Delta$

$\vdots$

$x(k\Delta)$ $\quad \delta_\Delta(t-k\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{k\Delta}(t)$ $\quad x(k\Delta)\,\Delta$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \implies \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

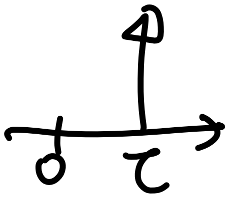


- CT Unit Impulse Response & Convolution Integral:

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\ \hat{h}_{k\Delta}(t)\ \Delta$$

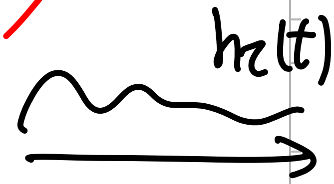
$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\ \hat{h}_{k\Delta}(t)\ \Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\ \ h_\tau(t)\ \ d\tau$$



$\delta(t-\tau) \longrightarrow$ Linear System $\longrightarrow h_\tau(t)$

$x(t) \longrightarrow$ Linear System $\longrightarrow y(t)$



$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \quad \Longrightarrow \quad y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

- If the linear system (L) is also time-invariant (TI)
  - Then,

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$h_\tau(t) = h_0(t-\tau) = h(t-\tau)$$

- Hence, for an LTI system,

σ
||

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

  - Known as the convolution of x(t) & h(t)
  - Referred as the convolution integral or the superposition integral

- Symbolically, $y(t) = x(t) * h(t) \qquad = h(t) * x(t)$

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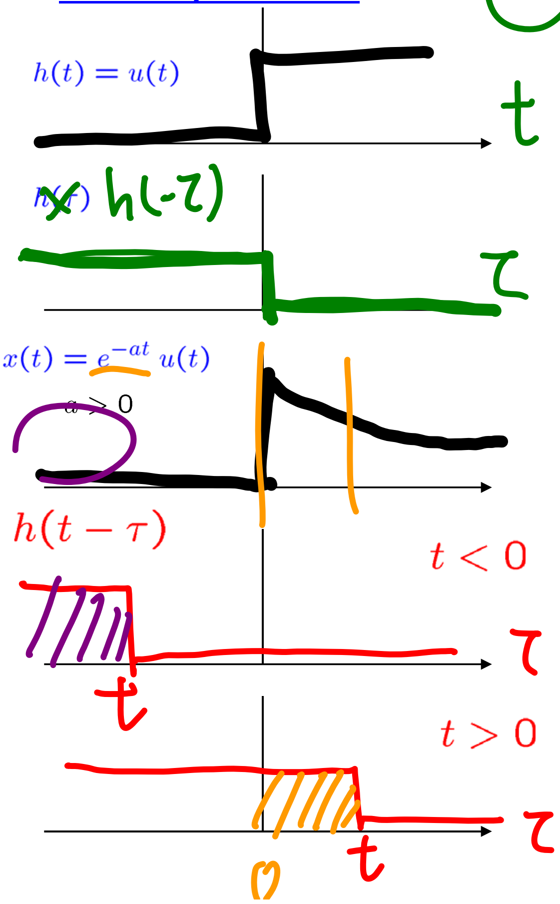
- **Example 2.6:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

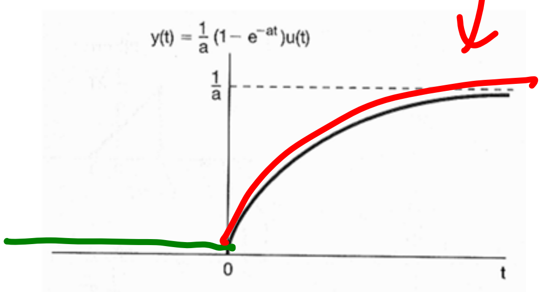


for $t < 0$, $x(\tau)\,h(t-\tau) = 0$

$$\Rightarrow\ y(t) = \int_{-\infty}^{t} 0\, d\tau = 0$$

for $t \geq 0$, $x(\tau)\,h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

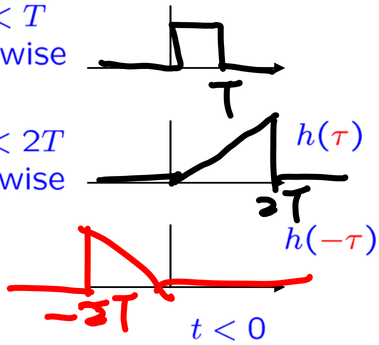
$$\Rightarrow\ y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a}(1 - e^{-at})$$

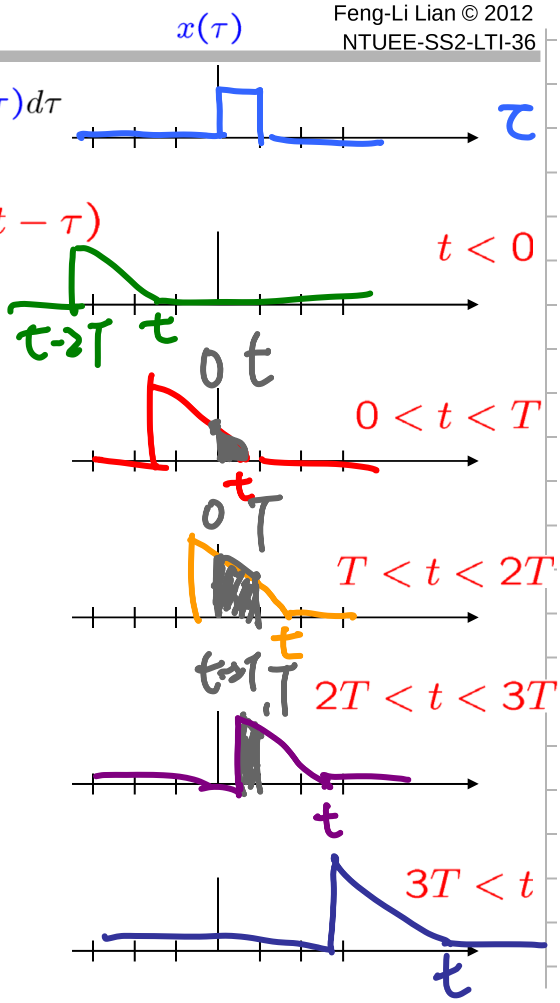


- **Example 2.7:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

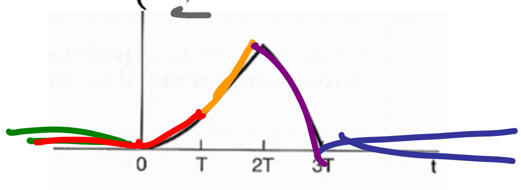
$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$

$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$





$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

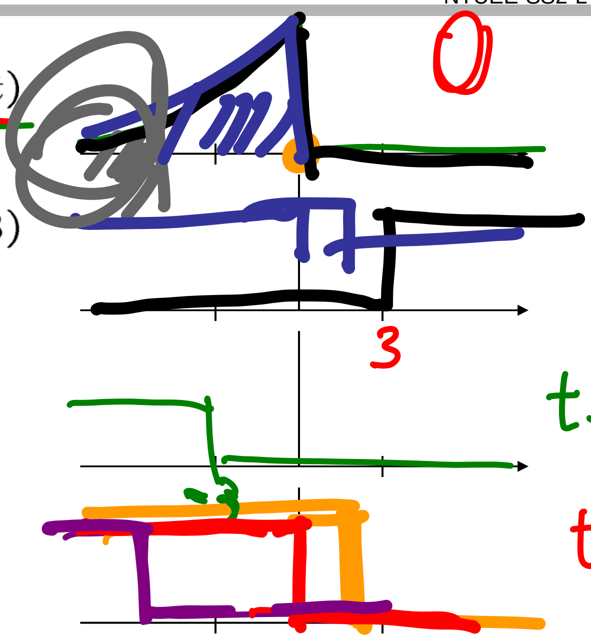


- Example 2.8: $x(t) = e^{2t}u(-t)$

$h(t) = u(t-3)$

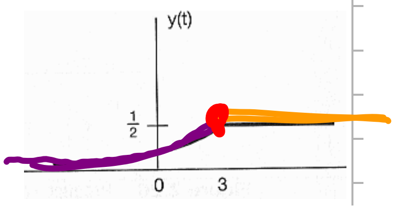
$h(-\tau)$

$h(t-\tau)$



for $t-3 \le 0$, $y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau \quad = \frac{1}{2} e^{2(t-3)}$

for $t-3 \ge 0$, $y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau \quad = \frac{1}{2}$



- Signal and System..

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] \quad = x(t) * h(t)$$

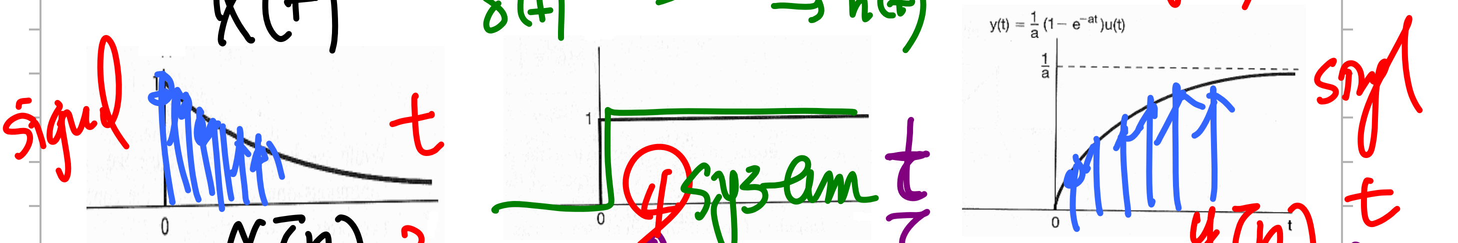
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \quad = x[n] * h[n]$$

$x(t) \rightarrow$ h(t) $\rightarrow y(t)$

δ(t) → h(t)

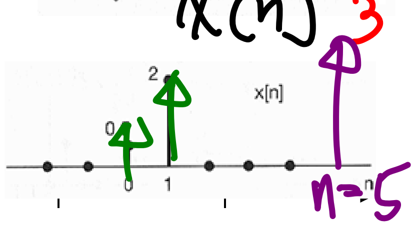
X(t)

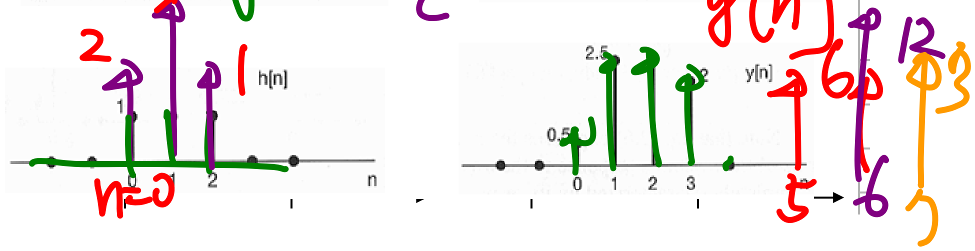
y(t)



X[n]

y[n]





- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

- Convolution Sum & Integral of LTI Systems:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[k] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

Properties of LTI Systems

$$y[n] = x[k] * h[n]$$
$$y(t) = x(t) * h(t)$$

1. Commutative property
2. Distributive property
3. Associative property
4. With or without memory
5. Invertibility
6. Causality
7. Stability
8. Unit step response

$$a \times b = b \times a$$
$$a + b = b + a$$
$$a \times (b + c) = a \times b + a \times c$$
$$a \times (b \times c) = (a \times b) \times c$$
$$= \cdots = a \times b \times c$$

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$

$x(t) \rightarrow$ h(t) $\rightarrow y(t)$

$$\forall x[n] \rightarrow \forall y[n] \Rightarrow h[n] = ?$$

$$\boxed{a \times b = b \times a}$$

Commutative Property:

$$n - k = r$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=+\infty}^{-\infty} x[n-r]h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r]x[n-r] = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$t - \tau = \sigma$$
$$-d\tau = d\sigma$$

$$= \int_{+\infty}^{-\infty} x(t-\sigma)h(\sigma)(-d\sigma) = \int_{-\infty}^{+\infty} x(t-\sigma)h(\sigma)d\sigma$$

$$= \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma = h(t) * x(t)$$

$a \times (b+c) = a \times b + a \times c$

▪ Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

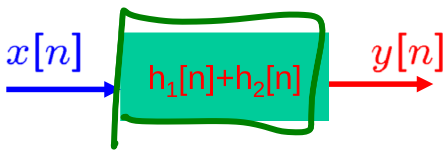
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

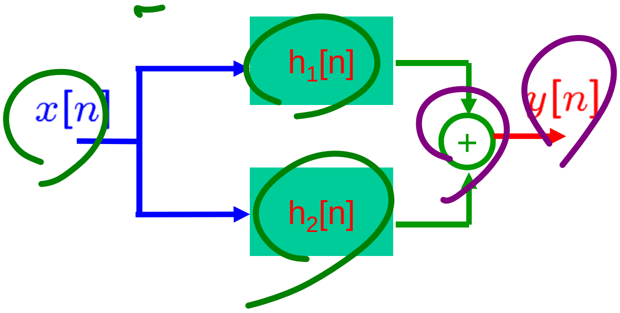
$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left(h_1[n-k] + h_2[n-k]\right) d\tau = \sum_k \left(x[k]\,h_1[n-k] + x[k]\,h_2[n-k]\right)$$

$$= \sum_k x[k]\,h_1[n-k] + \sum_k x[k]\,h_2[n-k] = x[n]*h_1[n] + x[n]*h_2[n]$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$





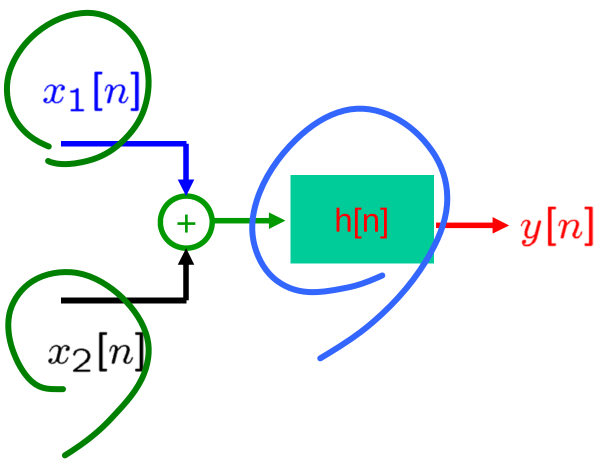
▪ Distributive Property:

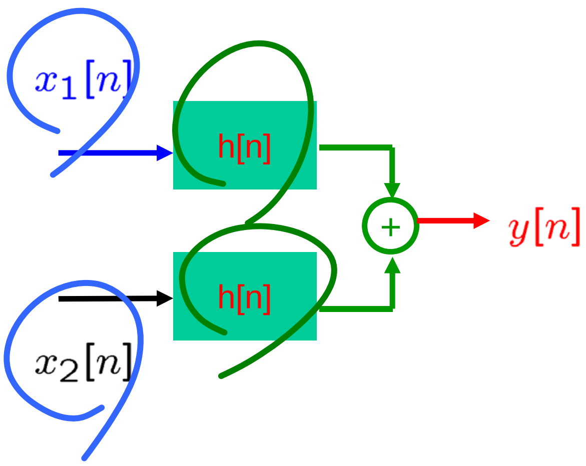
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\left(x_1[n] + x_2[n]\right) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$\left(x_1(t) + x_2(t)\right) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$



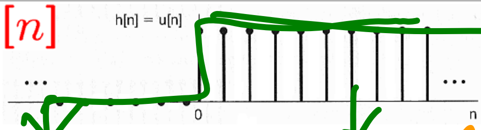


- Example 2.10

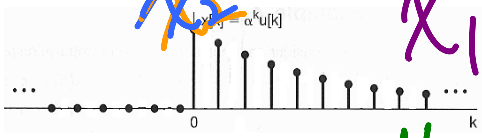
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$

x_1 x_2

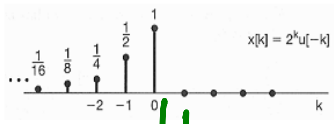
$h[n]$

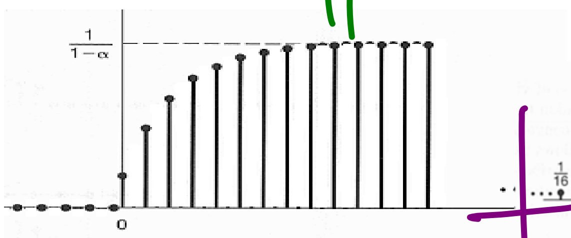


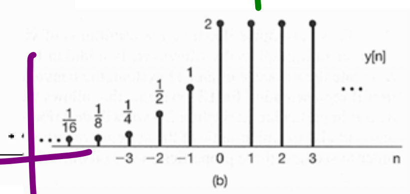
$* \quad x_1$

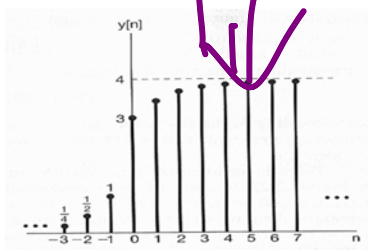


$* \quad x_2$









$$y[n] = x[n] * h[n]$$

$$= \left(x_1[n] + x_2[n]\right) * h[n]$$

$$= x_1[n] * h[n] + x_2[n] * h[n]$$

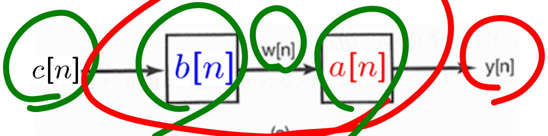
- Associative Property:

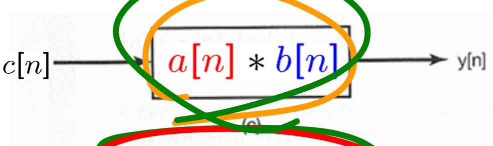
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

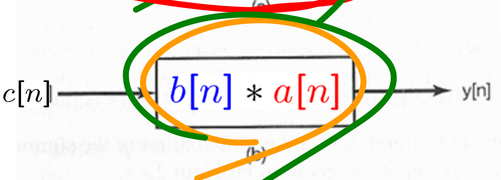
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

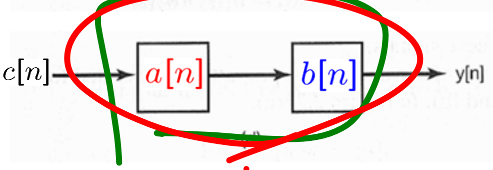
$$y[n] = a[n] * \left(\underbrace{b[n] * c[n]}_{w[n]}\right) = \left(a[n] * b[n]\right) * c[n] = a * b *$$

$$a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$$









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$$y[n] = a[n] * \left(b[n] * c[n] \right) = \left(a[n] * b[n] \right) * c[n]$$

$$y[n] = a[n] * \left(c[n] * b[n] \right)$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

$$= a[n] * \left(\sum_{k=-\infty}^{+\infty} c[k] \, b[n-k] \right)$$

$$= \sum_{m=-\infty}^{+\infty} a[m] \left(\sum_{k=-\infty}^{+\infty} c[k] \, b[\boxed{n-m}-k] \right)$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a[m] \, c[k] \, b[n-m-k]$$

$$= \sum_{k=-\infty}^{+\infty} c[k] \sum_{m=-\infty}^{+\infty} a[m] \, b[\boxed{n-k}-m]$$

$$= c[n] * \left(\sum_{m=-\infty}^{+\infty} a[m] \, b[n-m] \right) \qquad = c[n] * \left(a[n] * b[n] \right)$$

- Systems with or without memory
  - Memoryless systems
    - Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t) \qquad \text{(resistor)}$$

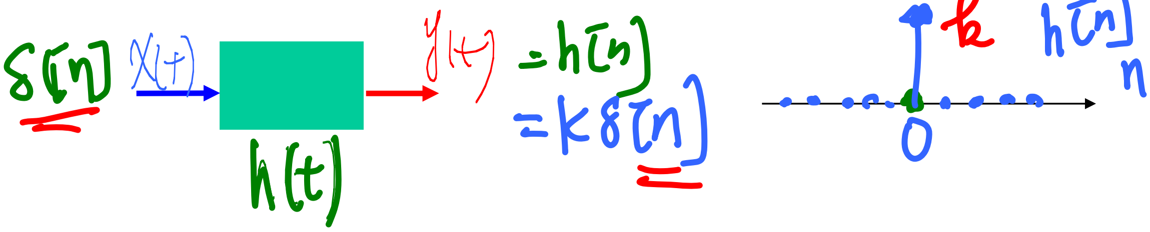
  - Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k] \quad \text{(accumulator)} \qquad y(t) = \frac{1}{C} \int_{-\infty}^{t} x(\tau) d\tau$$

$$y[n] = x[n-1] \qquad \text{(delay)}$$

$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

- **Memoryless:**
  - A DT LTI system is memoryless if $\boxed{h[n] = 0 \text{ for } n \neq 0}$



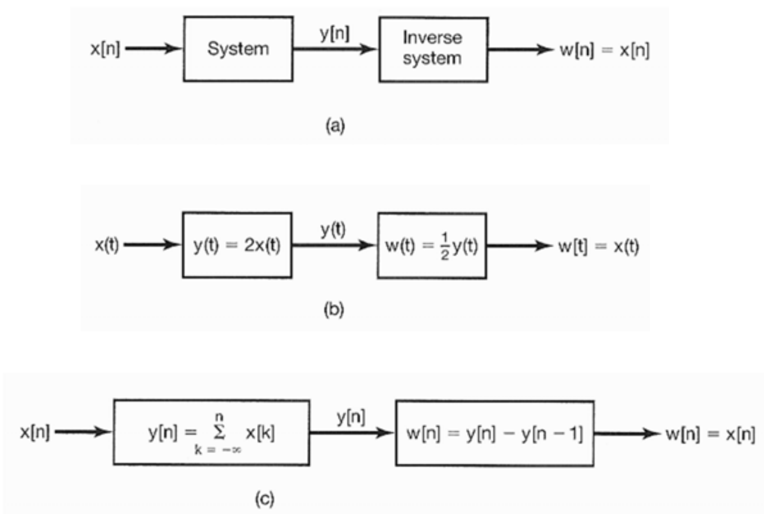
  - The impulse response: $h[n] = K\delta[n], \quad K = h[0]$
  - The convolution sum: $y[n] = x[n] * h[n]$

$= \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$, $k=n$, $\boxed{K\delta[n-k]}$

$= K\, x[n] = Kx[n]$, $k=n$

  - Similarly, for CT LTI system: $y(t) = x(t) * h(t) = Kx(t)$

- **Invertibility & Inverse Systems**
  - Invertible systems
    - Distinct inputs lead to distinct outputs



$y(t) = x(t)^2$ is not invertible

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- Invertibility:

$$x(t) \to \boxed{\text{h1(t)}} \to y(t) \to \boxed{\text{h2(t)}} \to w(t)$$

$$y(t) = x(t) * h_1(t) \qquad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t) = x(t)$$

$$x(t) \to \boxed{\text{Identity System } \delta(t)} \to x(t)$$

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(t)\delta(t-\tau)d\tau$$

$\tau = t$, $x(t)$

$$\int_{-\infty}^{+\infty} h_2(\tau)h_1(t-\tau)d\tau \qquad \Rightarrow h_2(t) * h_1(t) = \delta(t)$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- Example 2.11: Pure time shift

$$x(t) \to \boxed{\text{h1(t)}} \to y(t) \to \boxed{\text{h2(t)}} \to w(t)$$

- $y(t) = x(t - t_0)$
  - delay if $t_0 > 0$
  - advance if $t_0 < 0$

$$\Rightarrow h_1(t) = \delta(t - t_0) \quad \Rightarrow x(t) * \delta(t - t_0) = x(t - t_0)$$

- $w(t) = x(t) = y(t + t_0)$

$$\Rightarrow h_2(t) = \delta(t + t_0) \quad \Rightarrow y(t) * \delta(t + t_0) = y(t + t_0)$$

$$\Rightarrow h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

$$x[n]*h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- **Example 2.12**

$$x[n] \rightarrow \boxed{h1[n]} \rightarrow y[n] \rightarrow \boxed{h2[n]} \rightarrow w[n]$$

$$h_1[n] = u[n]$$

n

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\,u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

$\Rightarrow$ a running-sum operation .

- Its inverse is a first difference operation:

$$w[n] = y[n] - y[n-1] \Rightarrow h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

- **Causality:**

$$x[n]*h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t)*h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- The output of a causal system depends only on
  the present and past values of the input to the system
- Specifically, y[n] must not depend on x[k] for k > n

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

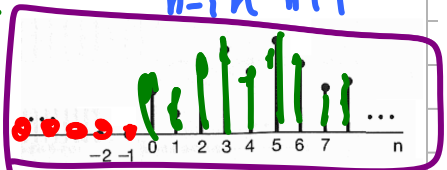
k>n

$h[n-k] = 0,\quad$ for $k > n$

$h[m] = 0,\quad$ for $m < 0$

h[n]=0 for n<0

n-1 n n+1



- It implies that the system is initially rest
- A CT LTI system is causal if

$h(t) = 0,\quad$ for $t < 0$

Convolution Sum & Integral

Causal

LTI

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \; h[n-k]$$

C.LTI

$$= \sum_{k=-\infty}^{n} x[k] \; h[n-k]$$

$$= \sum_{k=0}^{\infty} h[k] \; x[n-k]$$

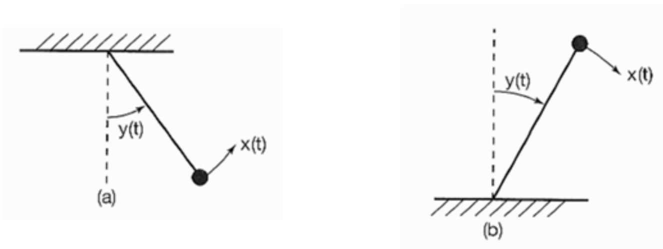
$$y(t) = \int_{-\infty}^{\infty} x(\tau) \; h(t-\tau) \; d\tau$$

$$= \int_{-\infty}^{t} x(\tau) \; h(t-\tau) \; d\tau$$

$$= \int_{0}^{\infty} h(\tau) \; x(t-\tau) \; d\tau$$

Stability

- Stable systems
  - Small inputs lead to responses that do not diverge
  - Every bounded input excites a bounded output
    - Bounded-input bounded-output stable (BIBO stable)
    - For all $|x(t)| < a$, then $|y(t)| < b$, for all t



- Balance in a bank account? $y[n] = 1.01 y[n-1] + x[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- **Stability:**
  - A system is stable
    if every bounded input produces a bounded output

x_1, x_2, x_3

$$x[n] \rightarrow \boxed{\text{Stable LTI}} \rightarrow y[n]$$

$h[n]$

y_1, y_2, y_3

$$\big|x[n]\big| < B \quad \text{for all } n \qquad \big|y[n]\big| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|$$

$$\Rightarrow \big|y[n]\big| \leq \sum_{k=-\infty}^{+\infty} \big|h[k]\big| \big|x[n-k]\big| \rightarrow B$$

$$\Rightarrow \big|y[n]\big| \leq B \left(\sum_{k=-\infty}^{+\infty} \big|h[k]\big| \right) < M$$

if $\boxed{\sum_{k=-\infty}^{+\infty} \big|h[k]\big| < \infty}$ then, $y[n]$ is bounded

absolutely summable

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

- **Stability:**
  - For CT LTI stable system:

$$x(t) \rightarrow \boxed{\text{Stable LTI}} \rightarrow y(t)$$

$$\big|x(t)\big| < B \quad \text{for all } t \qquad \big|y(t)\big| = \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right|$$

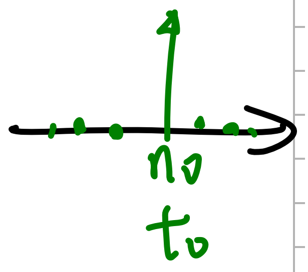
$$\Rightarrow \big|y(t)\big| \leq \int_{-\infty}^{+\infty} \big|h(\tau)\big| \big|x(t-\tau)\big| d\tau$$

$$\Rightarrow \big|y(t)\big| \leq B \left(\int_{-\infty}^{+\infty} \big|h[(\tau)\big| d\tau \right) < M$$

if $\boxed{\int_{-\infty}^{+\infty} \big|h(\tau)\big| d\tau < \infty}$ then, $y(t)$ is bounded

absolutely integrable

Example 2.13: Pure time shift



- $y[n] = x[n - n_0]$ & $h[n] = \delta[n - n_0]$
- $y(t) = x(t - t_0)$ & $h(t) = \delta(t - t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left| h[n] \right| = \sum_{n=-\infty}^{+\infty} \left| \delta[n - n_0] \right| = 1$$ absolutely summable

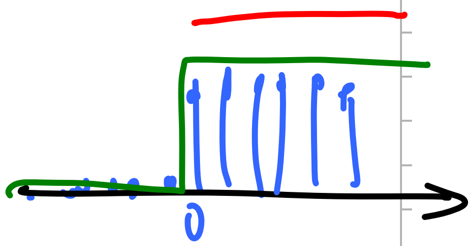
n=n0

$$\Rightarrow \int_{-\infty}^{+\infty} \left| h(\tau) \right| = \int_{-\infty}^{+\infty} \left| \delta(\tau - t_0) \right| d\tau = 1$$ absolutely integrable

t=t0

$\Rightarrow$ A (CT or DT) pure time shift is stable

Example 2.13: Accumulator



- $y[n] = \sum_{k=-\infty}^{n} x[k]$ & $h[n] = u[n]$
- $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ & $h(t) = u(t)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left| h[n] \right| = \sum_{n=0}^{+\infty} \left| u[n] \right| = \infty$$ NOT absolutely summable

$$\Rightarrow \int_{-\infty}^{+\infty} \left| h(\tau) \right| = \int_{0}^{\infty} \left| u(\tau) \right| d\tau = \infty$$ NOT absolutely integrable

$\Rightarrow$ A accumulator or integrator is NOT stable

$h[n] = \delta[n] * h[n]$

- **Unit Step Response:**
  - For an LTI system, its impulse response is:

$\delta[n] \rightarrow$ DT LTI $\rightarrow h[n]$ $\qquad$ $\delta(t) \rightarrow$ CT LTI $\rightarrow h(t)$

  - Its unit step response is:

$u[n] \rightarrow$ DT LTI $\rightarrow s[n]$ $\qquad$ $u(t) \rightarrow$ CT LTI $\rightarrow s(t)$

0 1 2

$$\Rightarrow s[n] = u[n] * h[n] \qquad = 0 \text{ or } 1$$

$$= \sum_{k=-\infty}^{+\infty} u[n-k]h[k]$$

$$= \boxed{\sum_{k=-\infty}^{n} h[\cancel{k}]} \qquad k$$

$$\Rightarrow h[n] = s[n] - s[n-1]$$

$$\Rightarrow s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{t} h(\tau)d\tau$$

$$\Rightarrow h(t) = \frac{ds(t)}{dt}$$

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- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $\quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $\quad y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $\quad x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $\quad x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $\quad a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $\quad h[n] = 0$ for $n \neq 0$
  5. Invertibility $\quad h_2(t) * h_1(t) = \delta(t)$
  6. Causality $\quad h(t) = 0,$ for $t < 0$
  7. Stability $\quad$ if $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $\quad h[n] = \left(\frac{1}{2}\right)^n u[n]$

Singularity Functions

奇異

- CT unit impulse function is one of singularity functions

$$\delta(t) = u_0(t) \qquad u(t) = u_{-1}(t)$$

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = u(t)$$

$$\frac{d}{dt}\delta(t) = u_1(t) \qquad \int_{-\infty}^{t} u(\tau)d\tau = u_{-2}(t)$$

$$\frac{d^2}{dt^2}\delta(t) = u_2(t) \qquad \int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} u(\sigma)d\sigma\right)d\tau = u_{-3}(t)$$

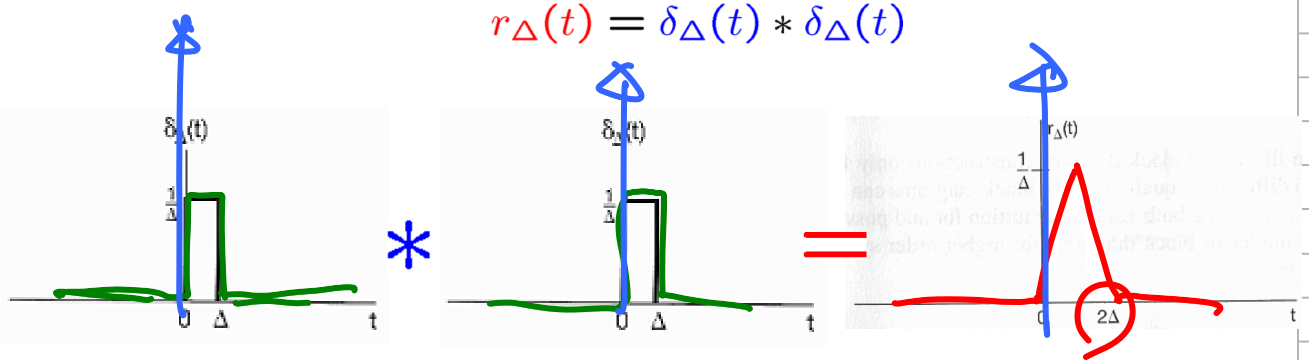
$$\frac{d^k}{dt^k}\delta(t) = u_k(t) \qquad \int_{-\infty}^{t}\cdots\left(\int_{-\infty}^{\tau} u(\sigma)d\sigma\right)\cdots d\tau = u_{-k}(t)$$

Singularity Functions

$$\boxed{x(t) = x(t) * \delta(t)}$$

$$\boxed{\delta(t) = \delta(t) * \delta(t)}$$

$$r_\Delta(t) = \delta_\Delta(t) * \delta_\Delta(t)$$



$\Delta \to 0$

$$\lim_{\Delta\to 0} \delta_\Delta(t) = \delta(t) \qquad \Rightarrow \lim_{\Delta\to 0} r_\Delta(t) = \delta(t)$$

- Example 2.16 $\frac{d}{dt}y(t) + 2y(t) = x(t)$ $\delta(t)$

with initial-rest condition

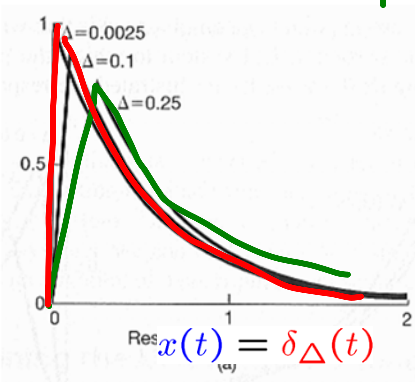
$x(t) = \delta_\triangle(t)$

$x(t) = r_\triangle(t)$

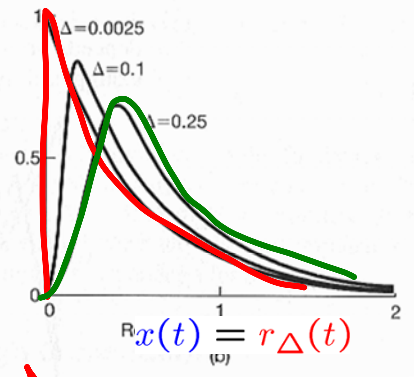
$x(t) = \delta_\triangle(t) * r_\triangle(t)$

$x(t) = r_\triangle(t) * r_\triangle(t)$

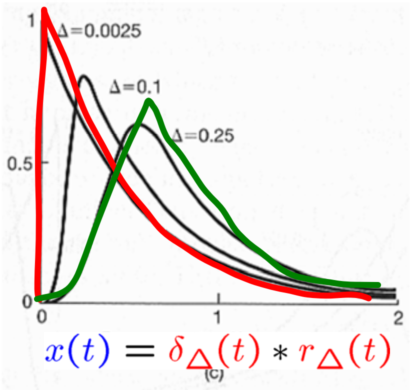
$x(t) = \delta(t)$



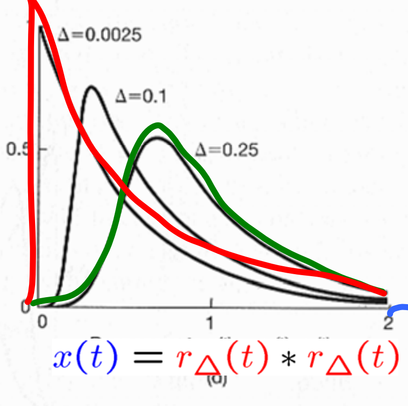
$x(t) = \delta_\triangle(t)$



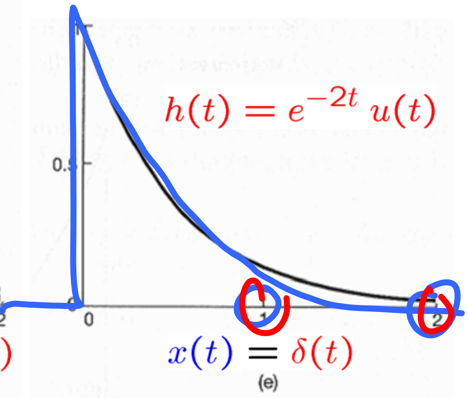
$x(t) = r_\triangle(t)$



$x(t) = \delta_\triangle(t) * r_\triangle(t)$



$x(t) = r_\triangle(t) * r_\triangle(t)$



$h(t) = e^{-2t}\,u(t)$

$x(t) = \delta(t)$

- Example 2.16 $\frac{d}{dt}y(t) + 20y(t) = x(t)$

with initial-rest condition

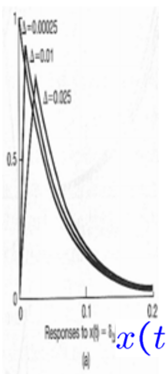
$x(t) = \delta_\triangle(t)$

$x(t) = r_\triangle(t)$

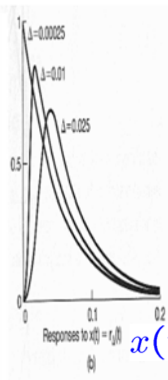
$x(t) = \delta_\triangle(t) * r_\triangle(t)$

$x(t) = r_\triangle(t) * r_\triangle(t)$

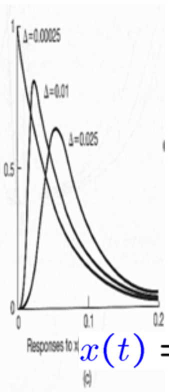
$x(t) = \delta(t)$



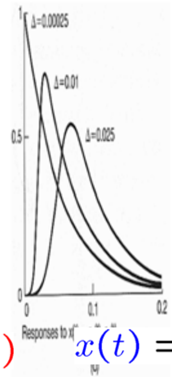
$x(t) = \delta_\triangle(t)$



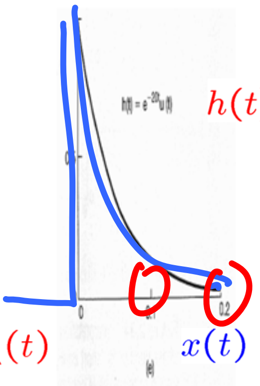
$x(t) = r_\triangle(t)$



$x(t) = \delta_\triangle(t) * r_\triangle(t)$



$x(t) = r_\triangle(t) * r_\triangle(t)$



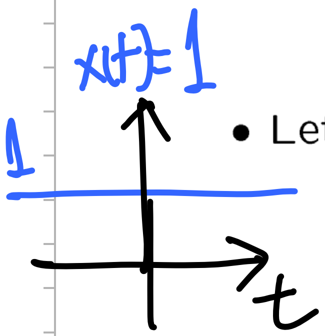
$h(t) = e^{-20t}\,u(t)$

$x(t) = \delta(t)$

Defining the Unit Impulse through Convolution:

$$x(t) = x(t) * \delta(t)$$

δ(t) ?



- Let $x(t) = 1$,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} \delta(\tau) \underbrace{x(t-\tau)}_{=1} d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- So that the unit impulse has unit area

Defining Unit Impulse through Convolution:

- Alternatively, consdier an arbitrary signal $g(t)$,

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau) d\tau$$

t=0

$$g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau) d\tau$$

- Define $x(t-\tau) = \underbrace{g(\tau)}_{=0}$

$$x(t) = g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau) d\tau = \boxed{x(t) * \delta(t)}$$

Defining Unit Impulse through Convolution:

- Consider the signal $f(t)\delta(t)$ $\rightarrow$ $f(0)\delta(0)$

$$\int_{-\infty}^{\infty} g(\tau) f(\tau)\delta(\tau) d\tau = g(0) f(0)$$

- On the other hand, consider the signal $f(0)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau) f(0)\delta(\tau) d\tau = g(0) f(0)$$

- Therefore,

$$f(t)\delta(t) = f(0)\delta(t)$$

Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt} x(t)$$

$$x(t) \rightarrow \boxed{D} \rightarrow y(t) = \frac{dx}{dt}$$

$\delta(t)$ $\qquad u_1(t)$

$h(t) = u_1(t)$

$\Rightarrow$ The unit impulse response of the system is the derivative of the unit impulse, which is called the unit doublet $u_1(t)$

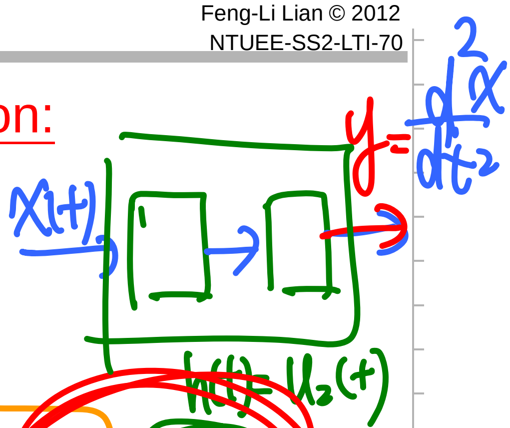
- That is, from $x(t) = x(t) * \delta(t)$, we have

$$\boxed{\frac{d}{dt} x(t) = x(t) * u_1(t)}$$

Unit Doublets of Derivative Operation:

- Similarly,

$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$



- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = \left(x(t) * u_1(t)\right) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

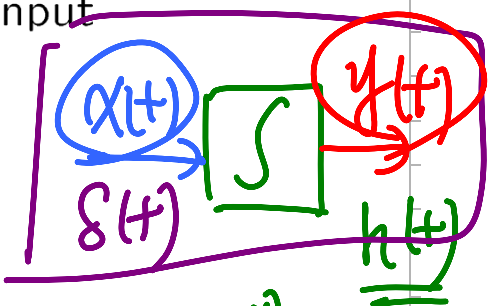
$u_k(t)$, $k > 0$, the kth derivative of $\delta(t)$

$$u_k(t) = u_1(t) * \cdots * u_1(t) \qquad k$$

Unit Doublets of Integration Operation:

- A system: Output is the integral of input

$$y(t) = \int_{-\infty}^{\infty} x(\tau)d\tau$$



- Therefore,

$$u(t) = \int_{-\infty}^{\infty} \delta(\tau)d\tau$$

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau)\, d\tau = u(t)$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

In Sys Out

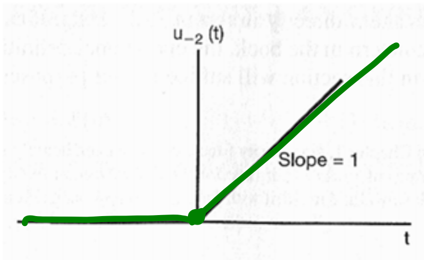
Unit Doublets of Integration Operation:

- Similarly,

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

- That is,

$$u_{-2}(t) = t\ u(t)$$ the unit ramp function



Unit Doublets of Integration Operation:

- Moreover,

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \left(\int_{-\infty}^{t} x(\sigma)d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)d\sigma \right) d\tau$$

- In general,

$$u_{-k}(t) = \overbrace{u(t) * \cdots * u(t)}^{k-1} = \int_{-\infty}^{t} u_{-(k-1)}(\tau)d\tau$$

$$\boxed{u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t)}$$

$\frac{d^k \delta(t)}{dt^k}$ $k>0$

- In Summary

$\delta(t) = u_0(t)$ $\qquad \delta(t) \to \boxed{D} \to \boxed{D} \to \cdots \to \boxed{D} \to u_k$

$u(t) = u_{-1}(t)$ $\qquad \delta(t) \to \boxed{\int} \to \boxed{\int} \to \cdots \to \boxed{\int} \to u_k$

$k<0$

$u_k(t)$

$k > 0$,

Impulse response of a cascade of k differentiators

$\underbrace{\int\int\int}_{k} \delta(t)$

$k < 0$,

Impulse response of a cascade of $|k|$ integrators

$\underbrace{\int\int\int}_{k-1} u(t)$

$u(t) * u_1(t) = \delta(t)$ or, $u_{-1}(t) * u_1(t) = u_0(t)$

$$\Rightarrow \boxed{u_k(t) * u_r(t) = u_{k+r}(t)}$$

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- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $\quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $\quad y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $\quad x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $\quad x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $\quad a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $\quad h(t) = 0$ for $t \neq 0$
  5. Invertibility $\quad h_2(t) * h_1(t) = \delta(t)$
  6. Causality $\quad h(t) = 0$, for $t < 0$
  7. Stability $\quad$ if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

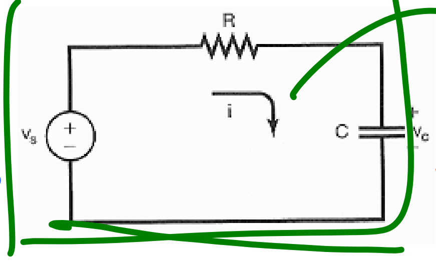
$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Linear Constant-Coefficient Differential Equations

- e.x., RC circuit

$h(t)$

Input signal: $v_s(t)$



Output signal: $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

$\delta(t)$

$x(t) \rightarrow$ RC Circuit $\rightarrow y(t)$

$$\Rightarrow \frac{d}{dt} y(t) + a\, y(t) = b\, x(t)$$

$\Rightarrow h(t) = y(t)$

- Provide an implicit specification of the system
- You have learned how to solve the equation in Diff Eqn

Linear Constant-Coefficient Differential Equations

- For a general CT LTI system, with N-th order,

$x(t) \rightarrow$ CT LTI $\rightarrow y(t)$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$\Rightarrow h(t) = ?$

$x(t) = \delta(t) \Rightarrow h(t) = y(t)$

$x(t) = f(t) \Rightarrow y(t) = f(t) * h(t)$

Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$x[n] = \delta[n] \Rightarrow h[n] = y[n]$

K

$$\Rightarrow \; h[n] = ?$$

Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$\delta[n]$

$$\Rightarrow y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k \, x[n-k] - \sum_{k=1}^{N} a_k \, y[n-k] \right\}$$

Recursive Equation:

$\delta[n] \rightarrow$ LTI $\rightarrow h[n]$

$x[n] \rightarrow$ LTI $\rightarrow y[n]$

- For example, (Example 2.15)

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

$y[n] = 0, \quad$ for $n \leq -1$

$x[n] = K \, \delta[n]$

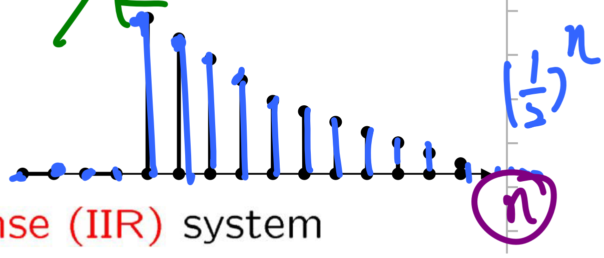
$$\Rightarrow \begin{cases} y[0] = x[0] + \frac{1}{2} y[-1] & = K \\ y[1] = x[1] + \frac{1}{2} y[0] & = K \, \frac{1}{2} \\ y[2] = x[2] + \frac{1}{2} y[1] & = K \left(\frac{1}{2}\right)^2 \\ \vdots & \\ y[n] = x[n] + \frac{1}{2} y[n-1] & = K \left(\frac{1}{2}\right)^n \end{cases}$$

$1 \cdot K$ $=0$ $=K$ $=\frac{1}{2}K$

$y[n] =$

$\Longleftrightarrow \; = h[n]$

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n]$$



$\Rightarrow$ an Infinite Impulse Response (IIR) system

Nonrecursive Equation:

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$y[n]$

- When N = 0,

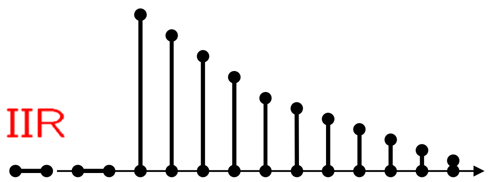
$$\Rightarrow y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

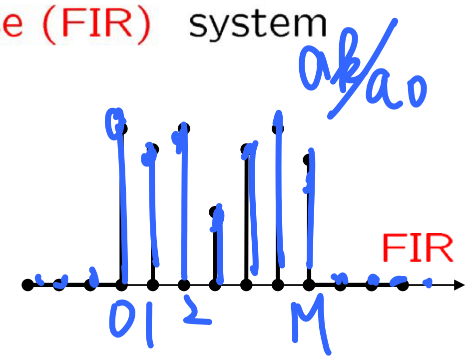
$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$0 \Longleftrightarrow M$

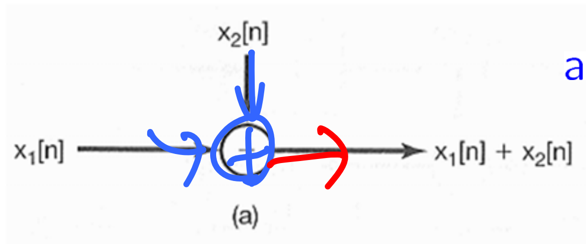
$$\Rightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ a Finite Impulse Response (FIR) system





Block Diagram Representations:



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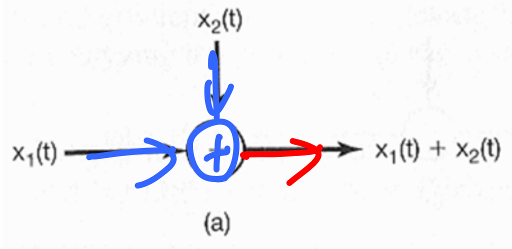
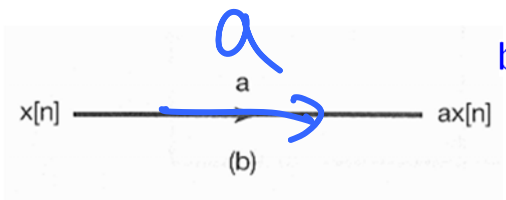
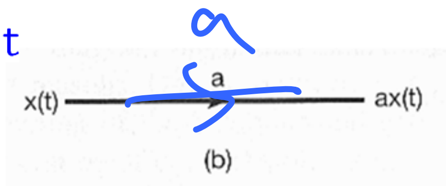
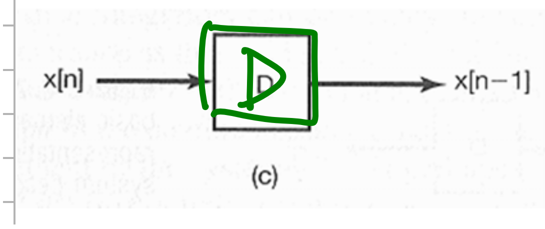


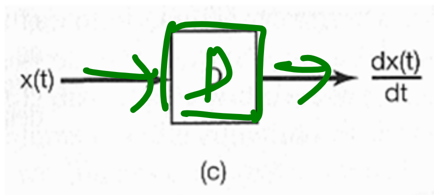


multiplication
by a coefficient





a unit delay/
differentiator



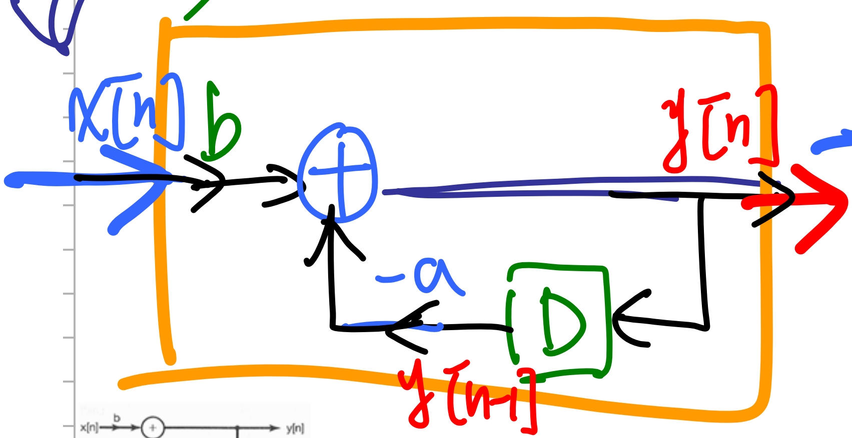
Block Diagram Representations:

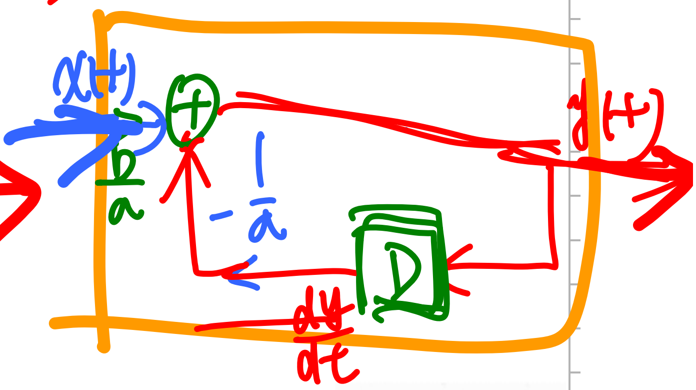
$$y[n] + ay[n-1] = bx[n]$$

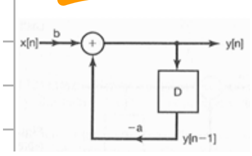
$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

$$y[n] = -ay[n-1] + bx[n]$$

$$y(t) = -\frac{1}{a}\frac{d}{dt}y(t) + \frac{b}{a}x(t)$$

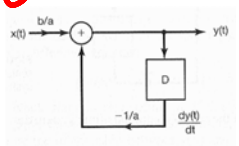






$D \Longleftrightarrow z^{-1}$

$D \Longleftrightarrow s$



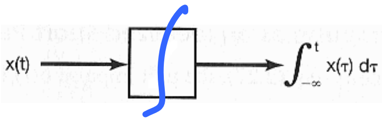
Block Diagram Representations:

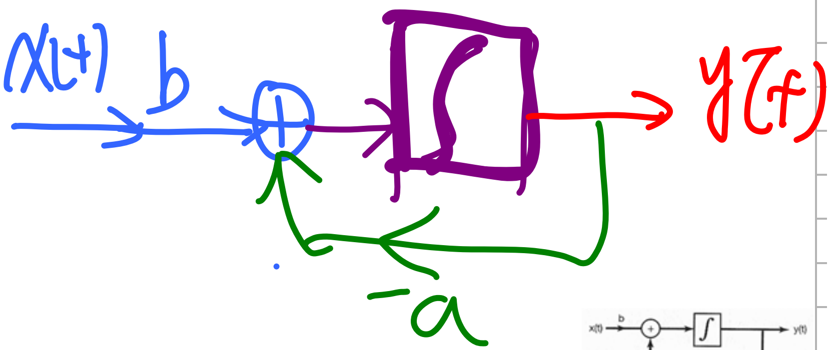
$$\frac{d}{dt}y(t) = bx(t) - ay(t)$$

$$\Rightarrow \; y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau$$

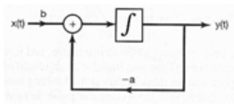
$$\Rightarrow \; y(t) = y(t_0) + \int_{t_0}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau$$

→0





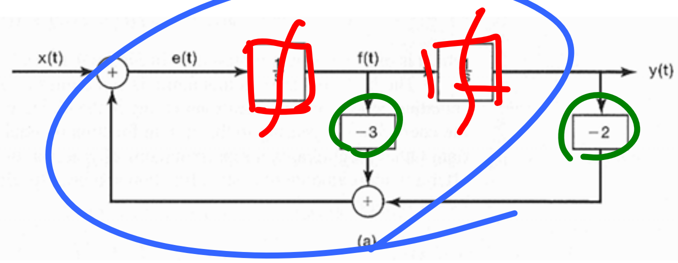
$$\int \Longleftrightarrow \frac{1}{s}$$

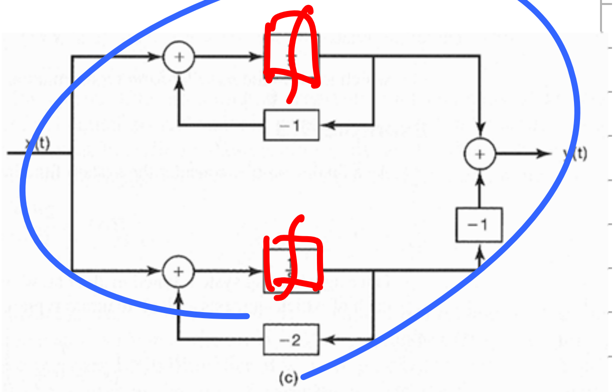


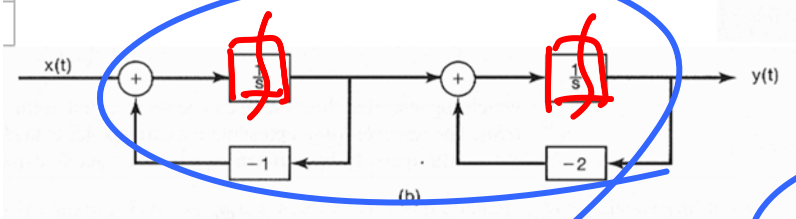
Block Diagram Representations:

$$\int \Longleftrightarrow \frac{1}{s}$$

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$







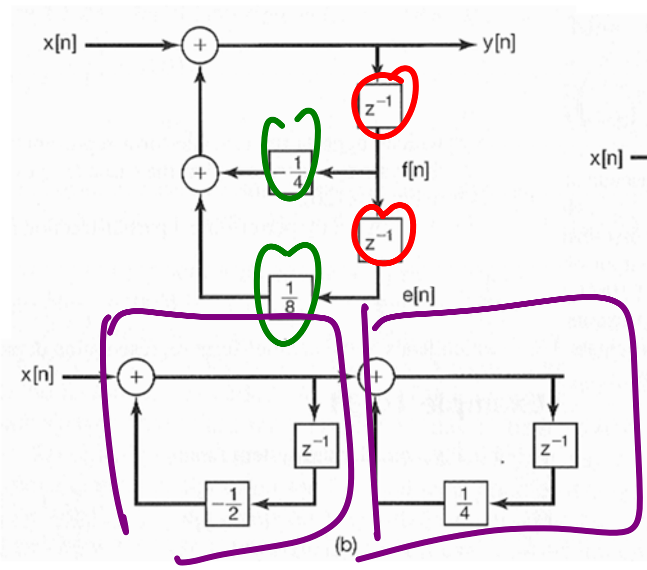
- Example 9.30 (pp.711)

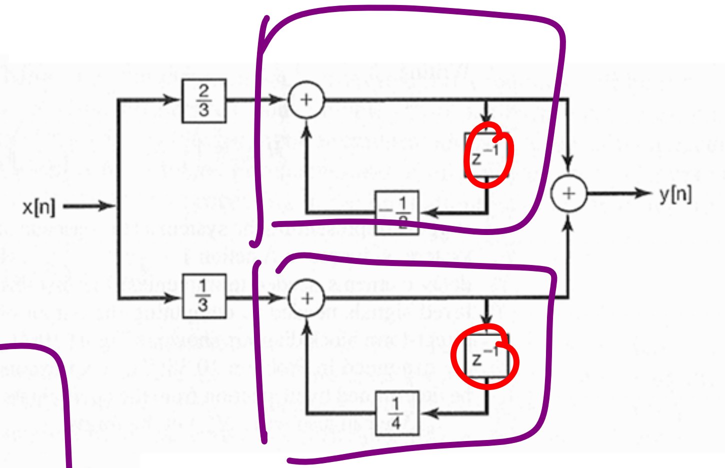
h(t)

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Block Diagram Representations:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$





$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

- Example 10.30 (pp.786)

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$
$$y[n] = x[n] * h[n]$$

- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$
$$y(t) = x(t) * h(t)$$

- Properties of Linear Time-Invariant Systems
  1. Commutative property
  2. Distributive property
  3. Associative property
  4. With or without memory
  5. Invertibility
  6. Causality
  7. Stability
  8. Unit step response

$$x(t) * h(t) = h(t) * x(t)$$
$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$
$$a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$$

$h[n] = 0$ for $n \neq 0$ $\quad$ $h(t) = 0,$ for $t < 0$

$h_2(t) * h_1(t) = \delta(t)$ $\quad$ if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations

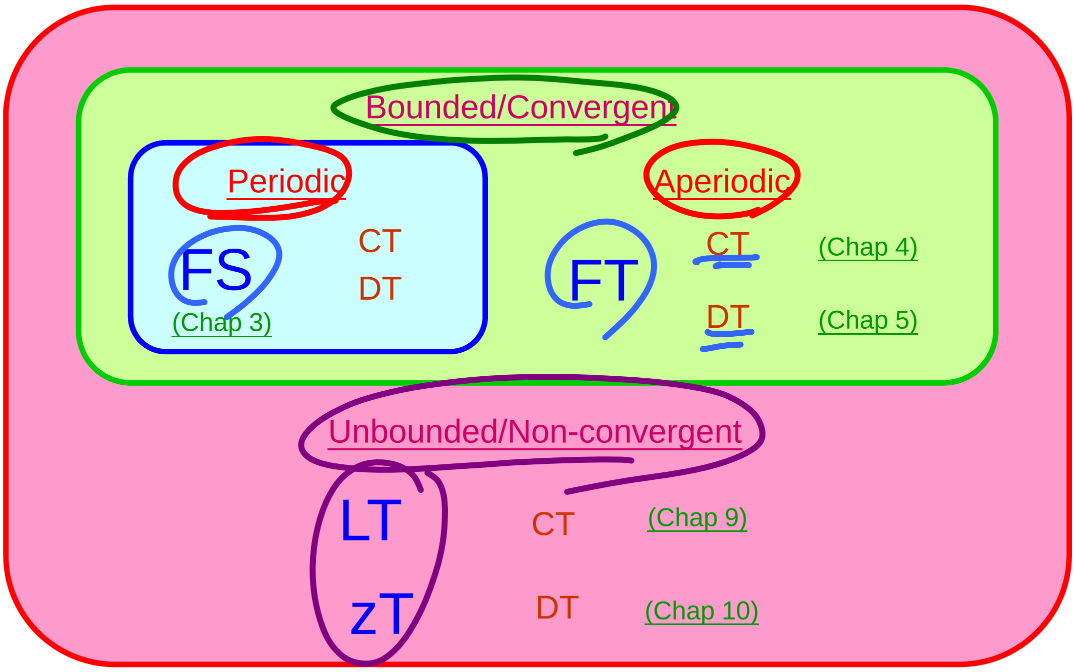
$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Flowchart

Signals & Systems (Chap 1) LTI & Convolution (Chap 2)



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Time-Frequency (Chap 6) Communication (Chap 8)

CT-DT (Chap 7) Control (Chap 11)