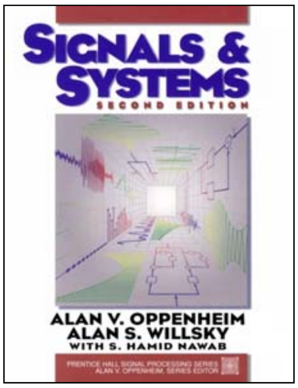
Spring 2015

信號與系統
Signals and Systems

Chapter SS-2
Linear Time-Invariant Systems



Feng-Li Lian
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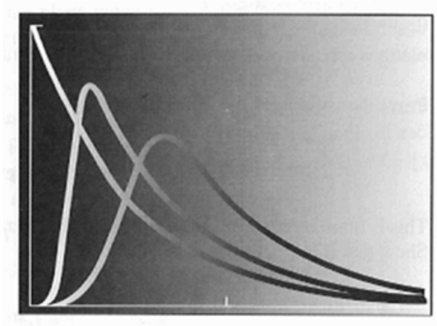
Feb15 – Jun15



Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

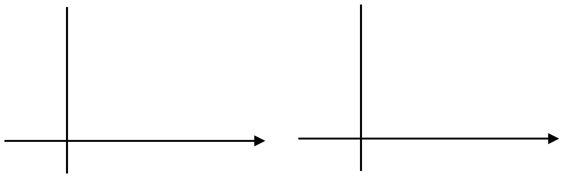
Outline

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$ $x(t) \rightarrow$ h(t) $\rightarrow y(t)$

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$ $\quad x(t) \rightarrow$ h(t) $\rightarrow y(t)$

Signals



Systems

$$\Rightarrow \frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\Rightarrow y[n] + ay[n-1] = bx[n]$$

✓
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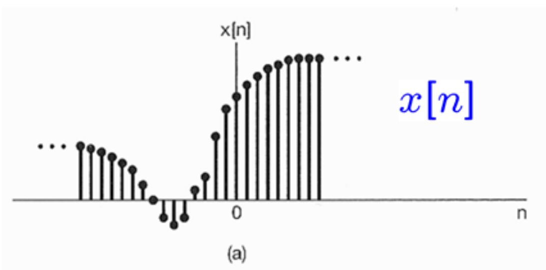
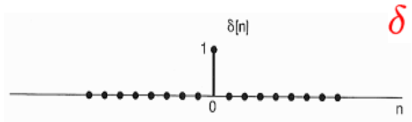
- Sample by Unit Impulse
- For x[n]

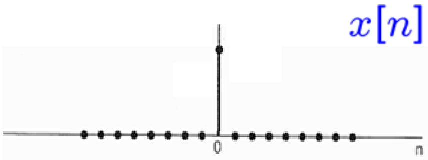
$$x[n]\delta[n] = x[0]\delta[n]$$



$x[n]$



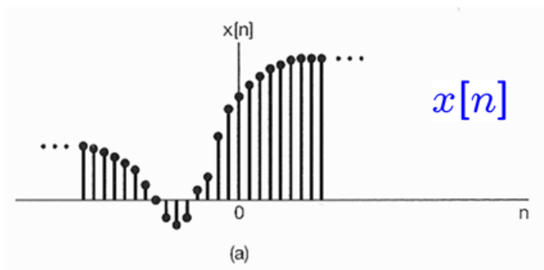
$\delta[n]$



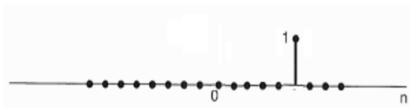
$x[n]\delta[n]$

- More generally,

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

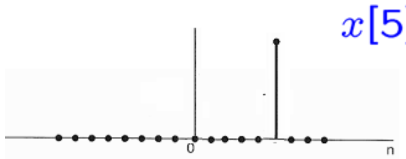


$x[n]$



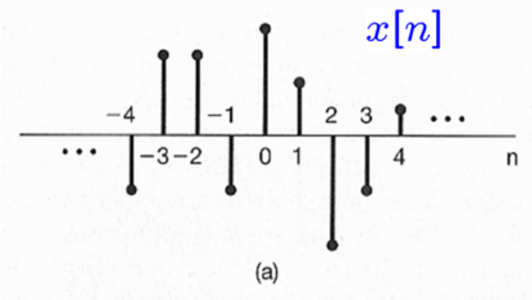
$\delta[n-5]$

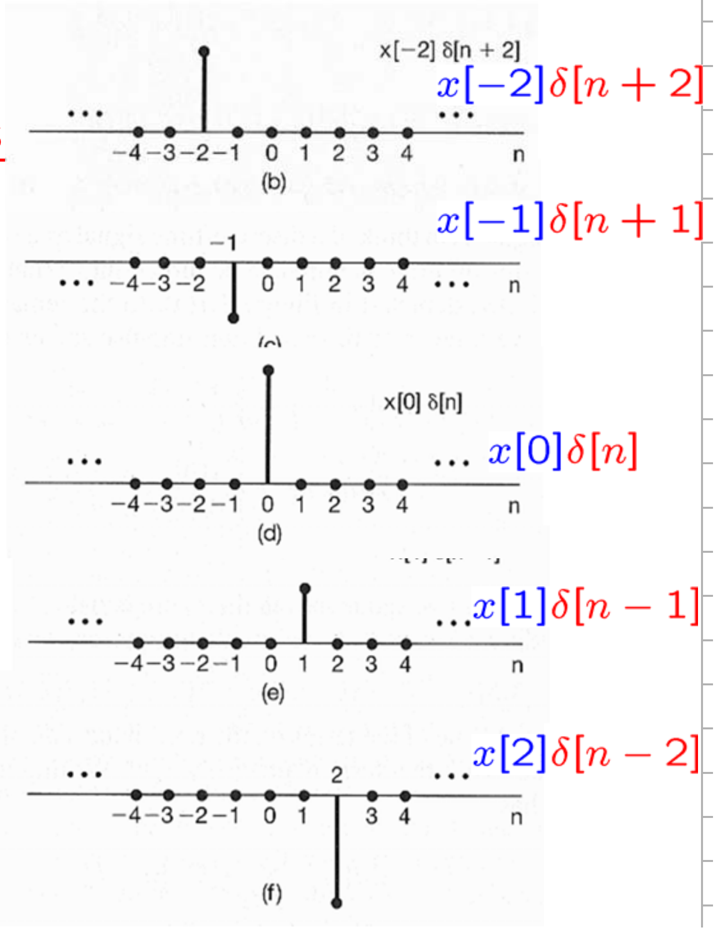
5



$x[5]\delta[n-5]$

- Representation of DT Signals by Impulses





- Representation of DT Signals by Impulses:
  - More generally,

$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2]$$

$$+ x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1]$$

$$+ x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

- The sifting property of the DT unit impulse
- x[n] = a superposition of scaled versions of shifted unit impulses δ[n-k]

▪ DT Unit Impulse Response & Convolution Sum:

input → Linear System → output

$\delta[n] \longrightarrow$ Linear System $\longrightarrow h_0[n]$

$\delta[n-1] \longrightarrow$ Linear System $\longrightarrow h_1[n]$

$\delta[n-2] \longrightarrow$ Linear System $\longrightarrow h_2[n]$

⋮

$\delta[n-k] \longrightarrow$ Linear System $\longrightarrow h_k[n]$

▪ DT Unit Impulse Response & Convolution Sum:

$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

$x[0] \cdot \delta[n] \longrightarrow$ Linear System $\longrightarrow h_0[n] \cdot x[0]$

$x[1] \cdot \delta[n-1] \longrightarrow$ Linear System $\longrightarrow h_1[n] \cdot x[1]$

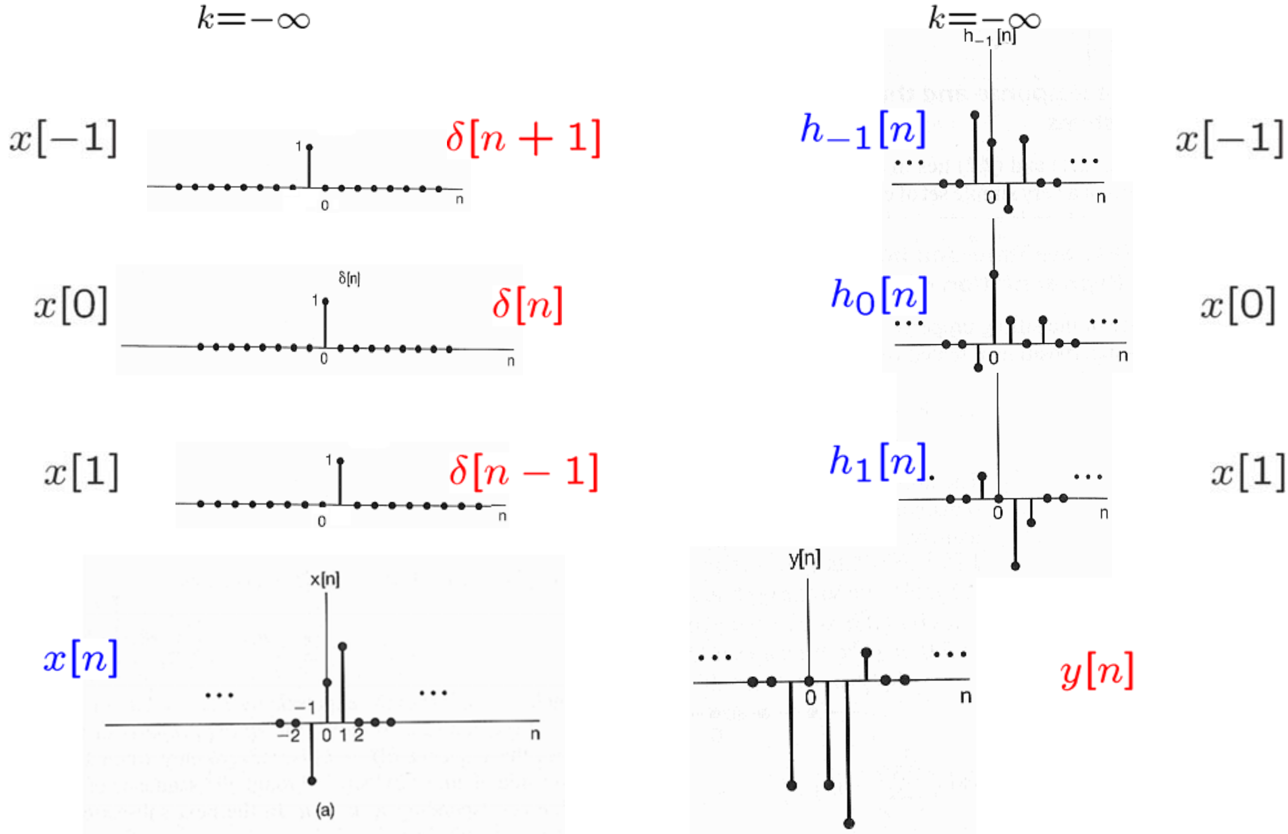
$x[2] \cdot \delta[n-2] \longrightarrow$ Linear System $\longrightarrow h_2[n] \cdot x[2]$

⋮

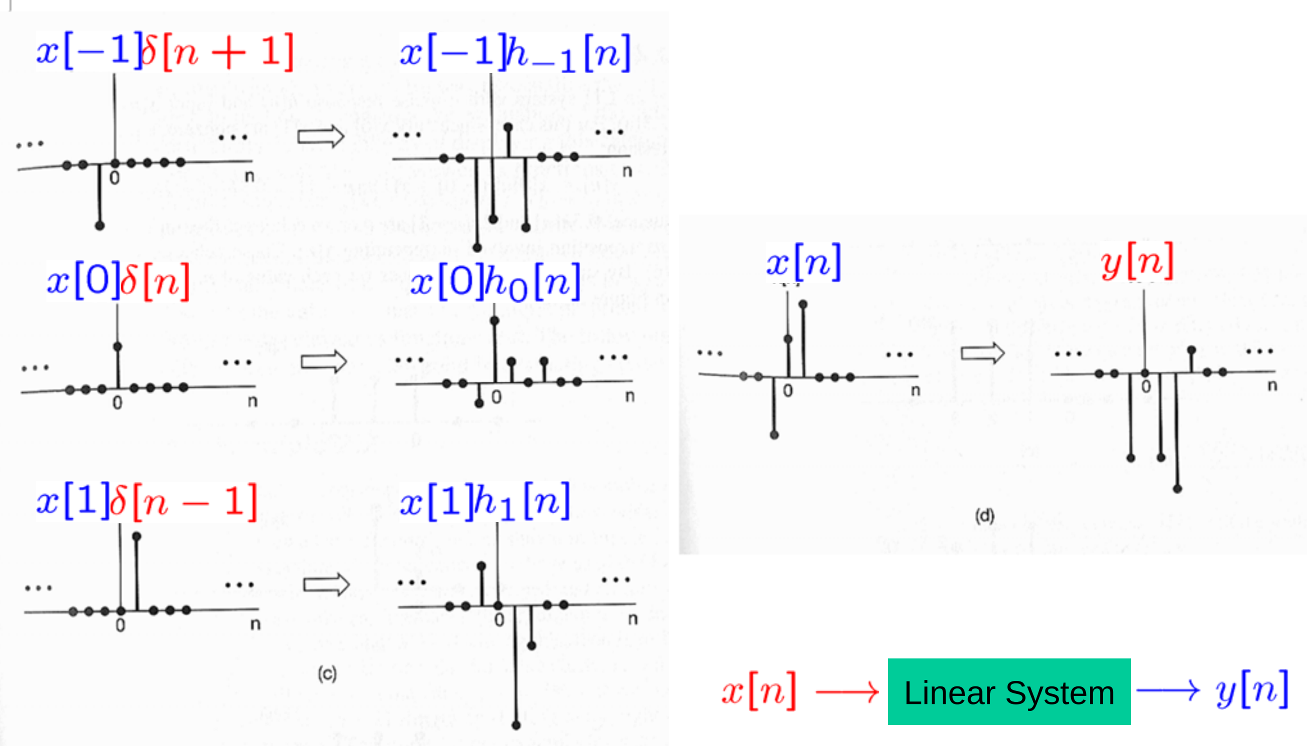
$x[k] \cdot \delta[n-k] \longrightarrow$ Linear System $\longrightarrow h_k[n] \cdot x[k]$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

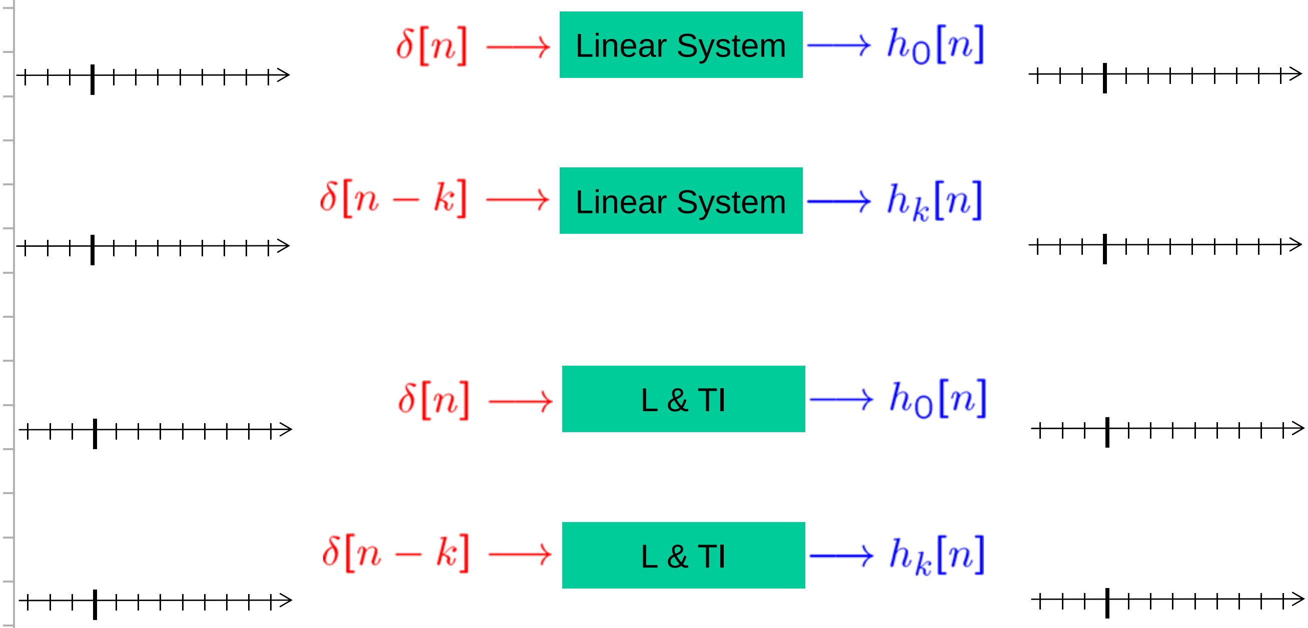


$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$



$x[n] \longrightarrow$ Linear System $\longrightarrow y[n]$

- If the linear system (L) is also time-invariant (TI)



- Then,

$$h_k[n] \;=\; h_0[n-k] \;=\; h[n-k]$$

- Hence, for an LTI system,

$$y[n] \;=\; \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

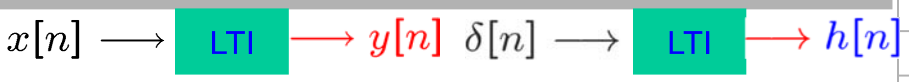
$$\Longrightarrow \quad y[n] \;=\; \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \;=\; \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

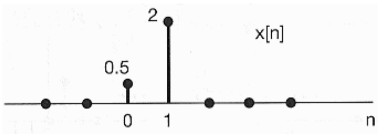
  - Known as the convolution of x[n] & h[n]
  - Referred as the convolution sum or superposition sum

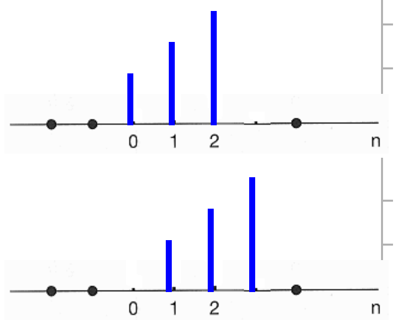
- Symbolically,

$$y[n] \;=\; x[n] * h[n] \;=\; h[n] * x[n]$$

- Example 2.1m:

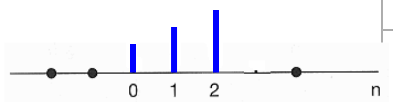






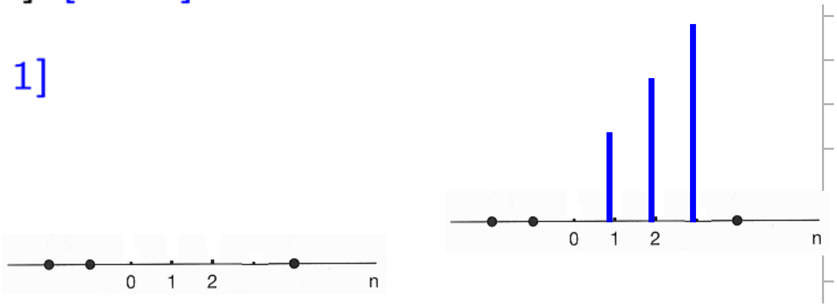
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

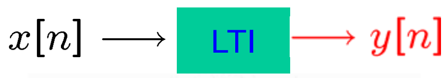


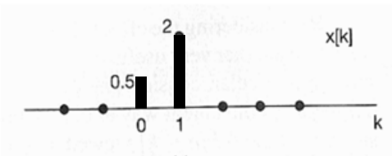
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

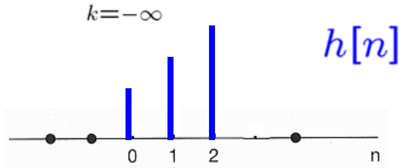
$$= 0.5h[n] + 2h[n-1]$$

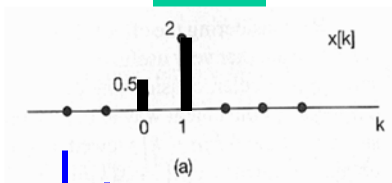


- Example 2.2m: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

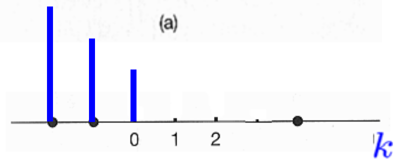






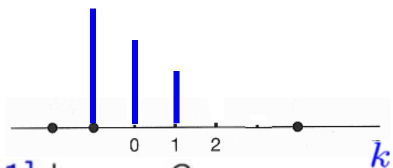


$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$



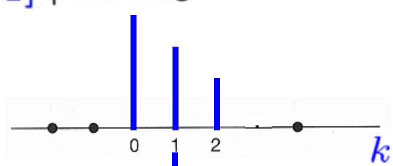
$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k]$$

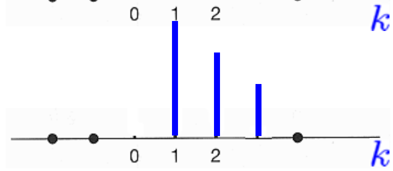


$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 3$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 5.5$$

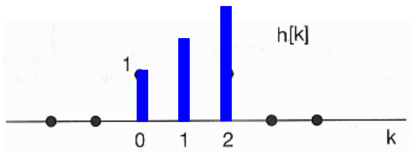


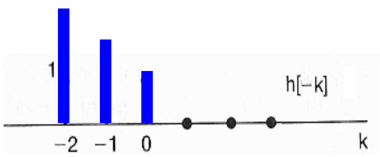
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 6.0$$

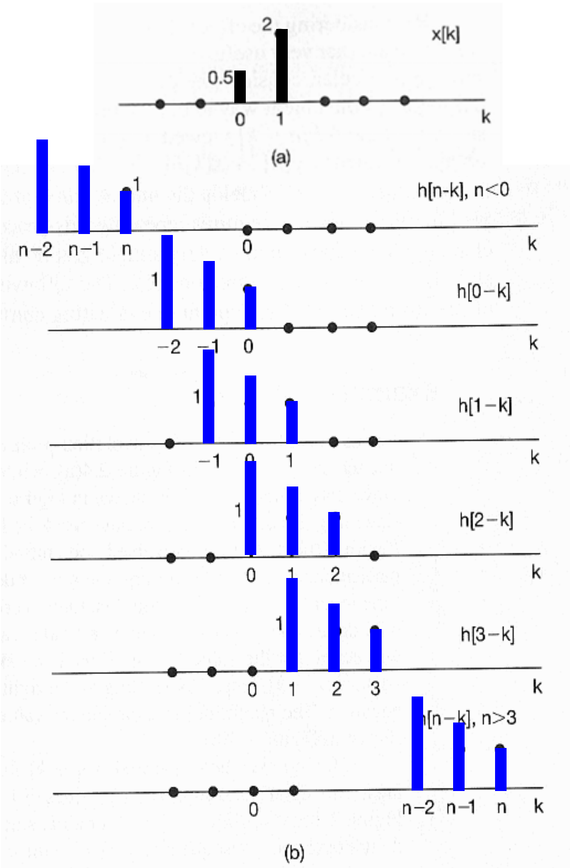


Example 2.2m:

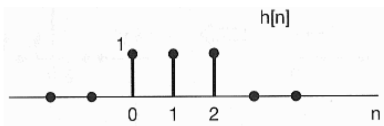
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$







Example 2.1o: $x[n] \longrightarrow$ LTI $\longrightarrow y[n]$ $h[n]$

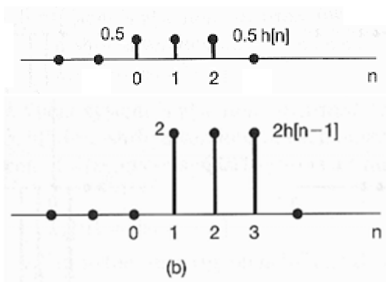


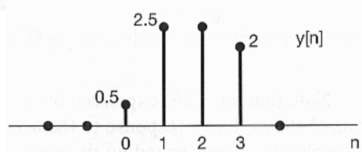
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

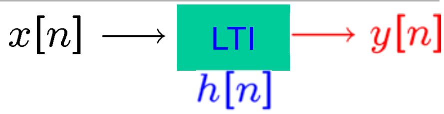
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

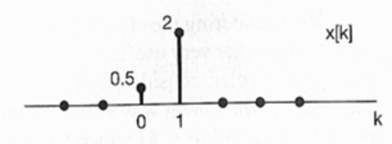
$$= 0.5h[n] + 2h[n-1]$$

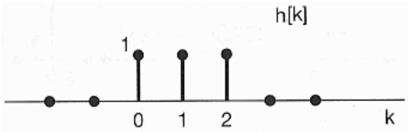


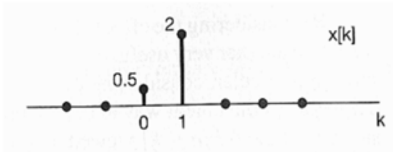


▪ Example 2.2o: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



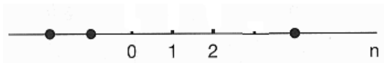






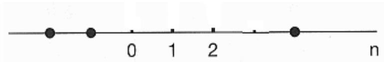
(a)

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$



$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$



$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 2.5$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

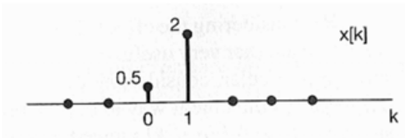
$y[n] = 0$ for $n < 0$

$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

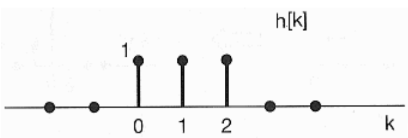
$y[n] = 0$ for $n > 3$

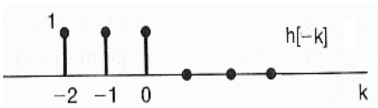
▪ Example 2.2o:

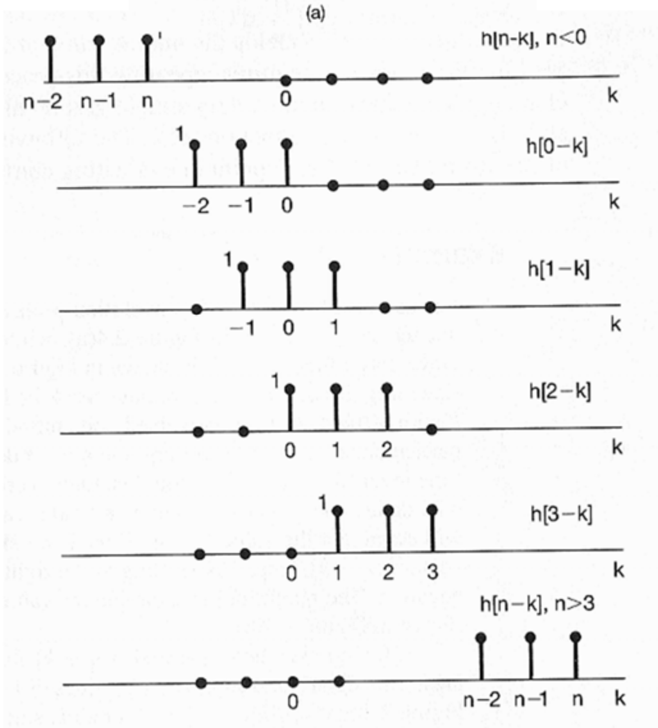
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



(a)



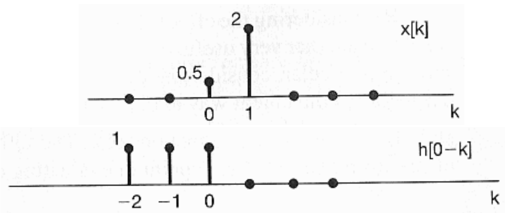


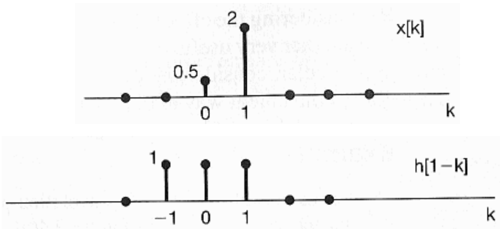


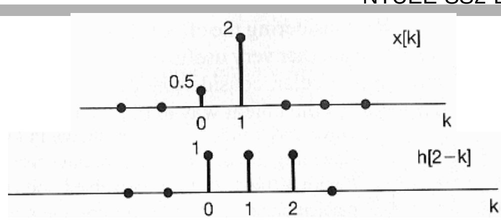
(b)

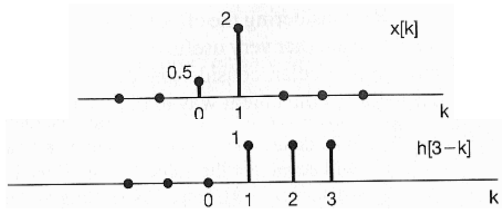
- Example 2.2o:

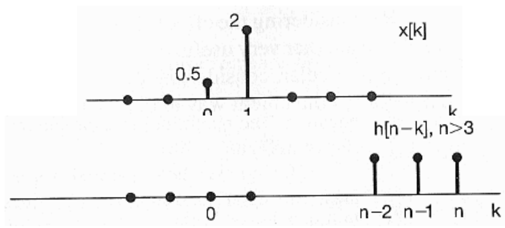
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

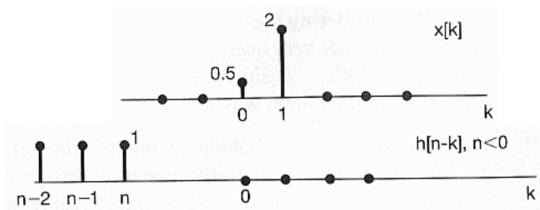




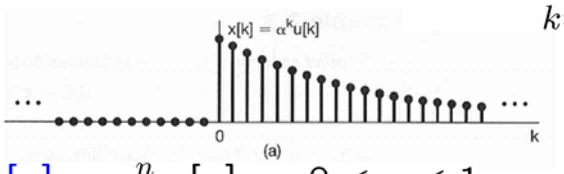




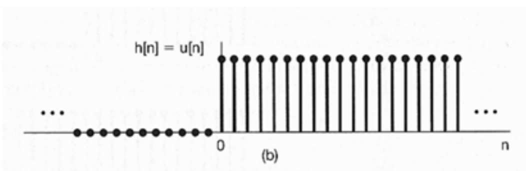




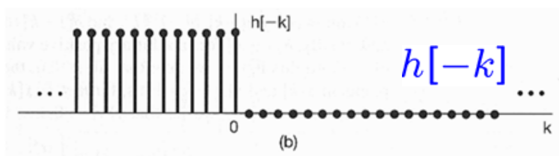
- Example 2.3: $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$



$x[n] = \alpha^n \, u[n], \quad 0 < \alpha < 1$

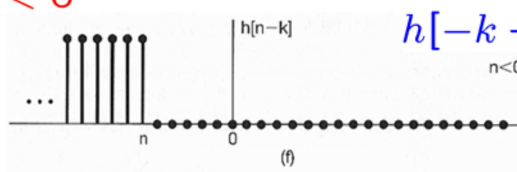


$h[n] = u[n]$



$h[-k]$

$n < 0$

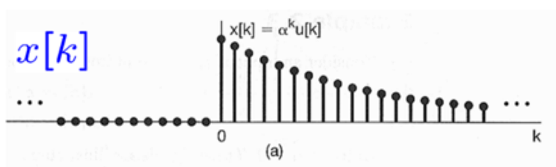


$h[-k+n]$

for $n < 0, \quad x[k] \; h[n-k] = 0$

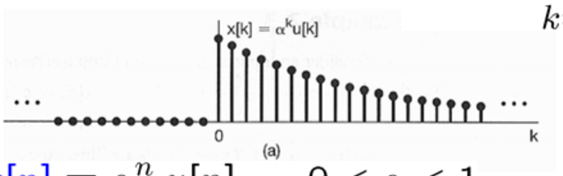
$\Rightarrow \; y[n] = 0$

$x[k]$

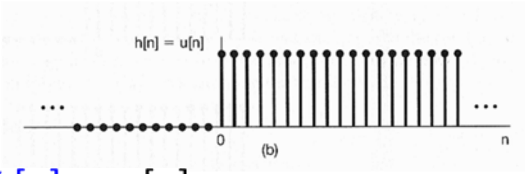


Example 2.3:

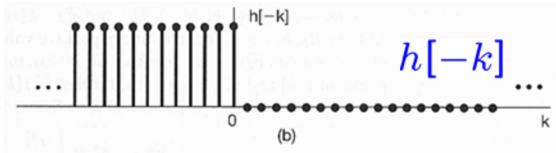
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



$x[n] = \alpha^n \, u[n], \qquad 0 < \alpha < 1$



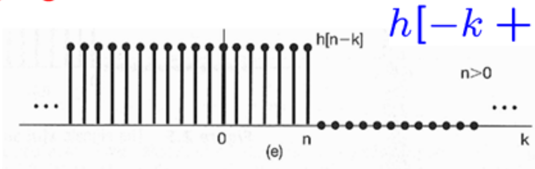
$h[n] = u[n]$



$h[-k]$

$n > 0$

$h[-k+n]$

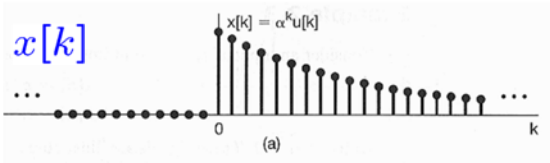


for $n \geq 0$,

$$x[k]\; h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

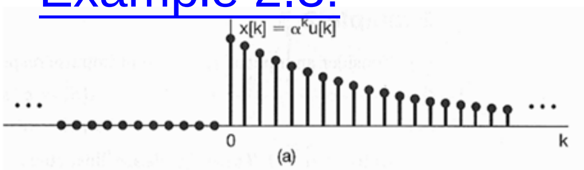
$$\Rightarrow \; y[n] = \sum_{k=0}^{n} \alpha^k \quad = \frac{1-\alpha^{n+1}}{1-\alpha}$$

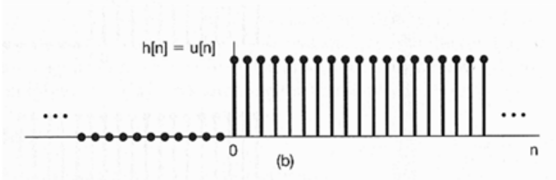
$x[k]$



Example 2.3:

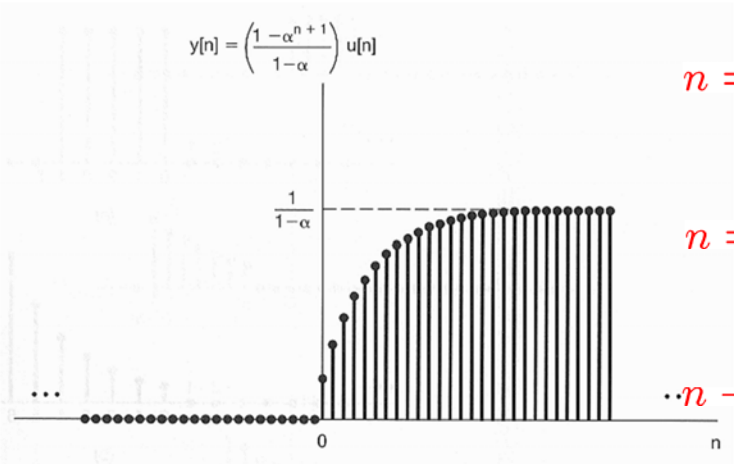
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$





for all n, $\quad y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$

$$\alpha = \frac{7}{8}$$



$n = 0 \qquad y[0] = \dfrac{1-\frac{7}{8}}{1-\frac{7}{8}} = 1$

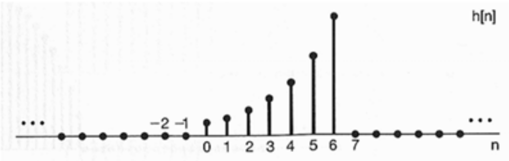
$n = 1 \qquad y[1] = \dfrac{1-(\frac{7}{8})^2}{1-\frac{7}{8}} = \dfrac{15}{8}$

$n \to \infty \qquad y[n] = \dfrac{1-0}{1-\frac{7}{8}} = 8$

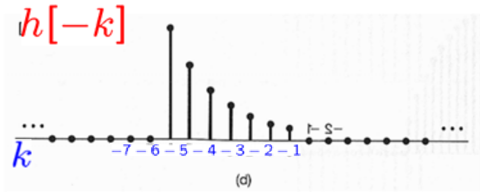
DT LTI Systems: Convolution Sum

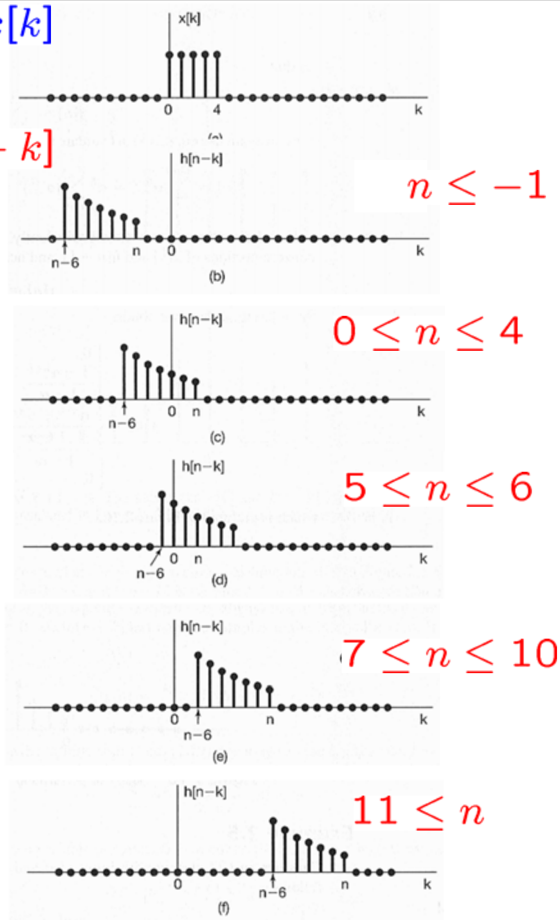
- **Example 2.4:**

$x[n] = 1, \; 0 \le n \le 4$

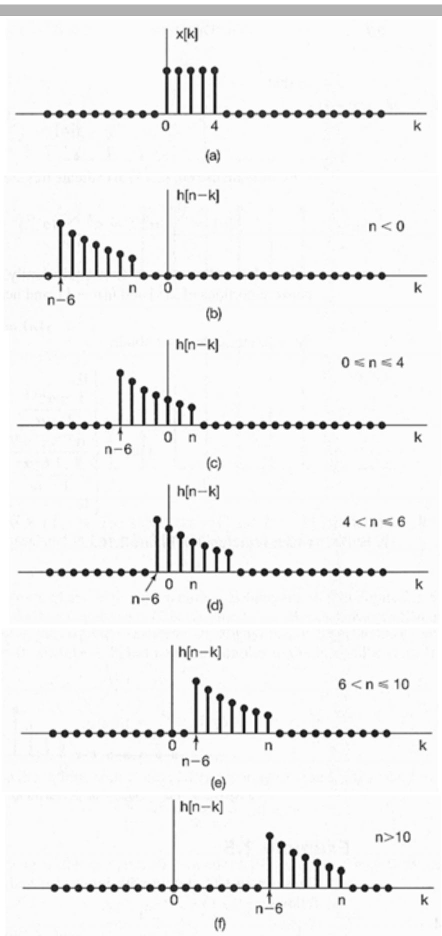


$h[n] = \alpha^n, \; 0 \le n \le 6$





DT LTI Systems: Convolution Sum



for $n < 0$, $\quad x[k]\,h[n-k] = 0 \;\Rightarrow\; y[n] = 0$

for $0 \le n \le 4$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{n} \alpha^{n-k} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

for $5 \le n \le 6$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}$$

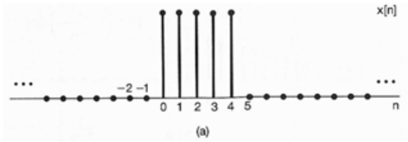
for $7 \le n \le 10$, $\quad x[k]\,h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \le k \le 4 \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=n-6}^{4} \alpha^{n-k} = \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}$$

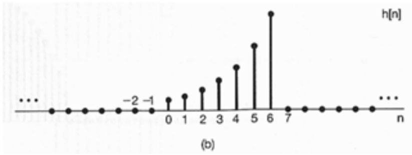
for $11 \le n$, $\quad \Rightarrow y[n] = 0$

$x[n] \longrightarrow$ LTI $\longrightarrow y[n]$

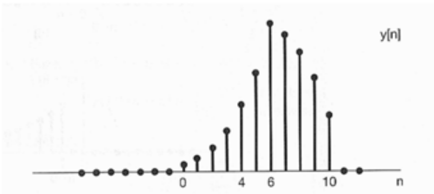
$h[n]$



$$x[n] = 1, \; 0 \leq n \leq 4$$



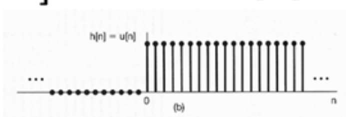
$$h[n] = \alpha^n, \; 0 \leq n \leq 6$$



$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha}, & 0 \leq n \leq 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha}, & 5 \leq n \leq 6 \\ \frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}, & 7 \leq n \leq 10 \\ 0, & 11 \leq n \end{cases}$$

- **Example 2.5:** $x[n] = 2^n u[-n]$

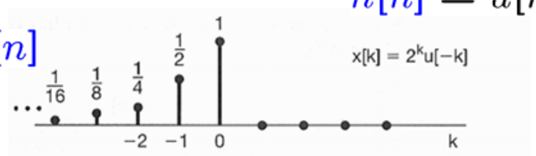
$h[n] = u[n]$



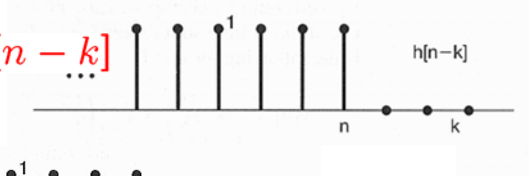
$x[n] \longrightarrow$ LTI $\longrightarrow y[n]$

$h[n]$

$x[n]$



$h[n-k]$



for $n \geq 0$, $y[n] = \sum_{k=-\infty}^{0} x[k]\, h[n-k] = \sum_{k=-\infty}^{0} 2^k$

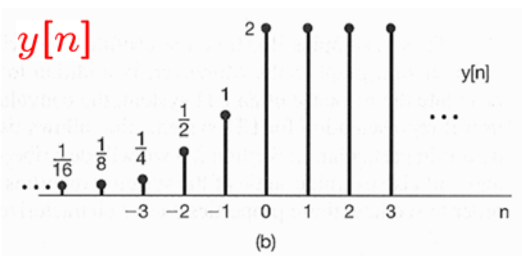
$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1-(1/2)} = 2$$

for $n < 0$, $y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] = \sum_{k=-\infty}^{n} 2^k$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

$y[n]$



- For an LTI system, $$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$
- The convolution of finite-duration discrete-time signals may be expressed as the product of a matrix and a vector.

$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k] = \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots$$
$$+ x[M-1]h[-(M-1)] + x[M]h[-M] + \cdots$$
$$+ x[L-1]h[-(L-1)] + x[L]h[-L] + \cdots$$

$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots$$

$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = \cdots + x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0] + \cdots$$

$$y[L+M-2] = \sum_{k=-\infty}^{+\infty} x[k]h[L+M-2-k]$$
$$= \cdots + x[-1]h[L+M-1] + x[0]h[L+M-2] + x[1]h[L+M-3] + x[2]h[L+M-4] + \cdots$$
$$+ x[L-2]h[M] + x[L-1]h[M-1] + x[L]h[M-2] + \cdots$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

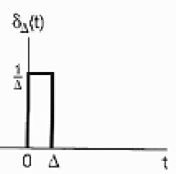
$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[L+M-2] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ h[1] & h[0] & 0 & \cdots & \cdots & \cdots & 0 \\ h[2] & h[1] & h[0] & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\ h[M-1] & \cdots & \cdots & \cdots & h[0] & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & h[M-1] & \cdots & \cdots & h[1] & h[0] \\ 0 & \cdots & 0 & h[M-1] & \cdots & h[2] & h[1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & h[M-1] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[L-1] \end{bmatrix}$$

$$\mathbf{y} = \mathbf{H}\,\mathbf{x}$$

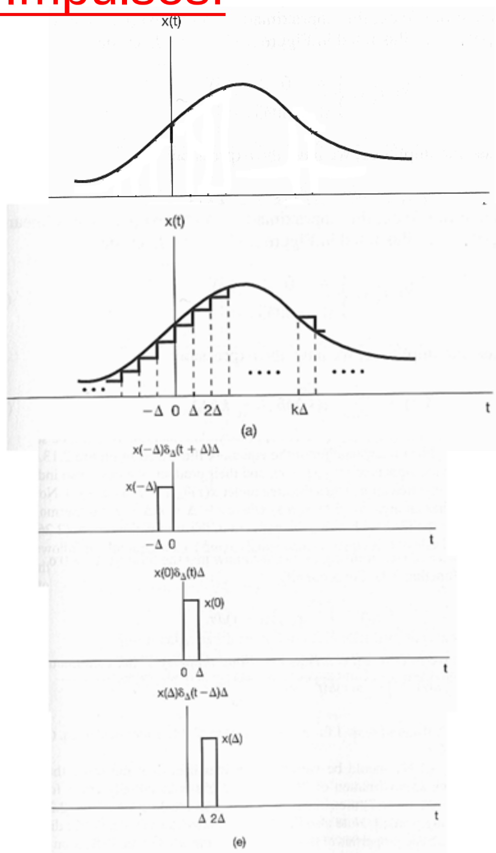
- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

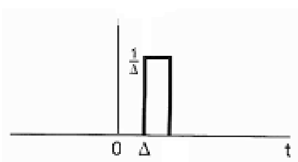
- Representation of CT Signals by Impulses:

$\delta_\Delta(t)$

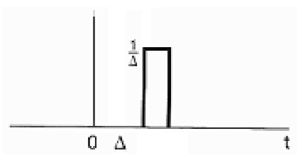


$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

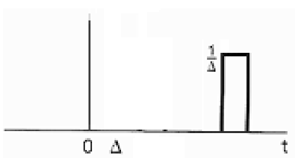




$\delta_\Delta(t-\Delta)$

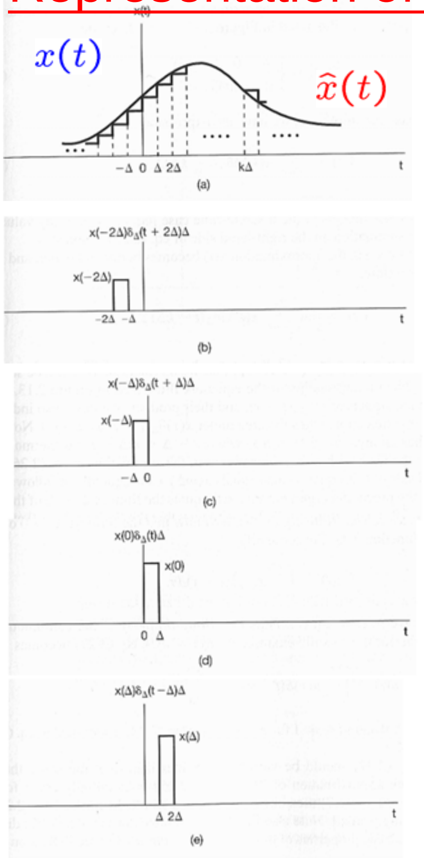


$\delta_\Delta(t-2\Delta)$



$\delta_\Delta(t-k\Delta)$

Representation of CT Signals by Impulses:



$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

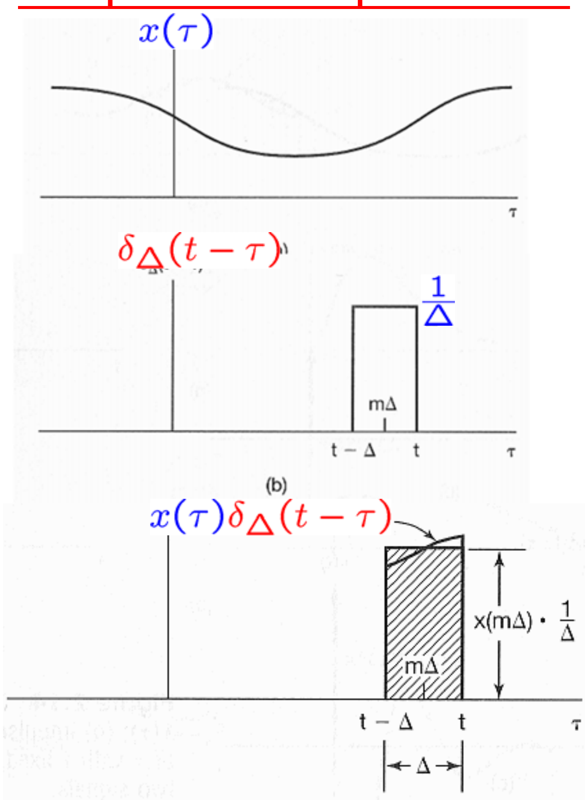
$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

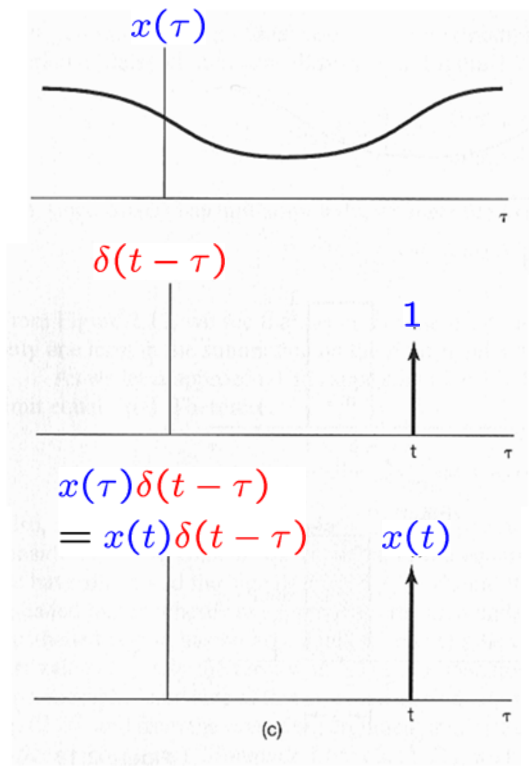
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

the sifting property of CT impulse

$x(t)$ = an integral of weighted, shifted impulses

Graphical interpretation:





▪ CT Impulse Response & Convolution Integral:

input $\longrightarrow$ Linear System $\longrightarrow$ output

$\delta_\Delta(t) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{0\Delta}(t)$

$\delta_\Delta(t-1\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{1\Delta}(t)$

$\delta_\Delta(t-2\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{2\Delta}(t)$

$\vdots$

$\delta_\Delta(t-k\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{k\Delta}(t)$

▪ CT Impulse Response & Convolution Integral:

input $\longrightarrow$ Linear System $\longrightarrow$ output

$x(0\Delta)$ $\delta_\Delta(t) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{0\Delta}(t)$ $x(0\Delta)$

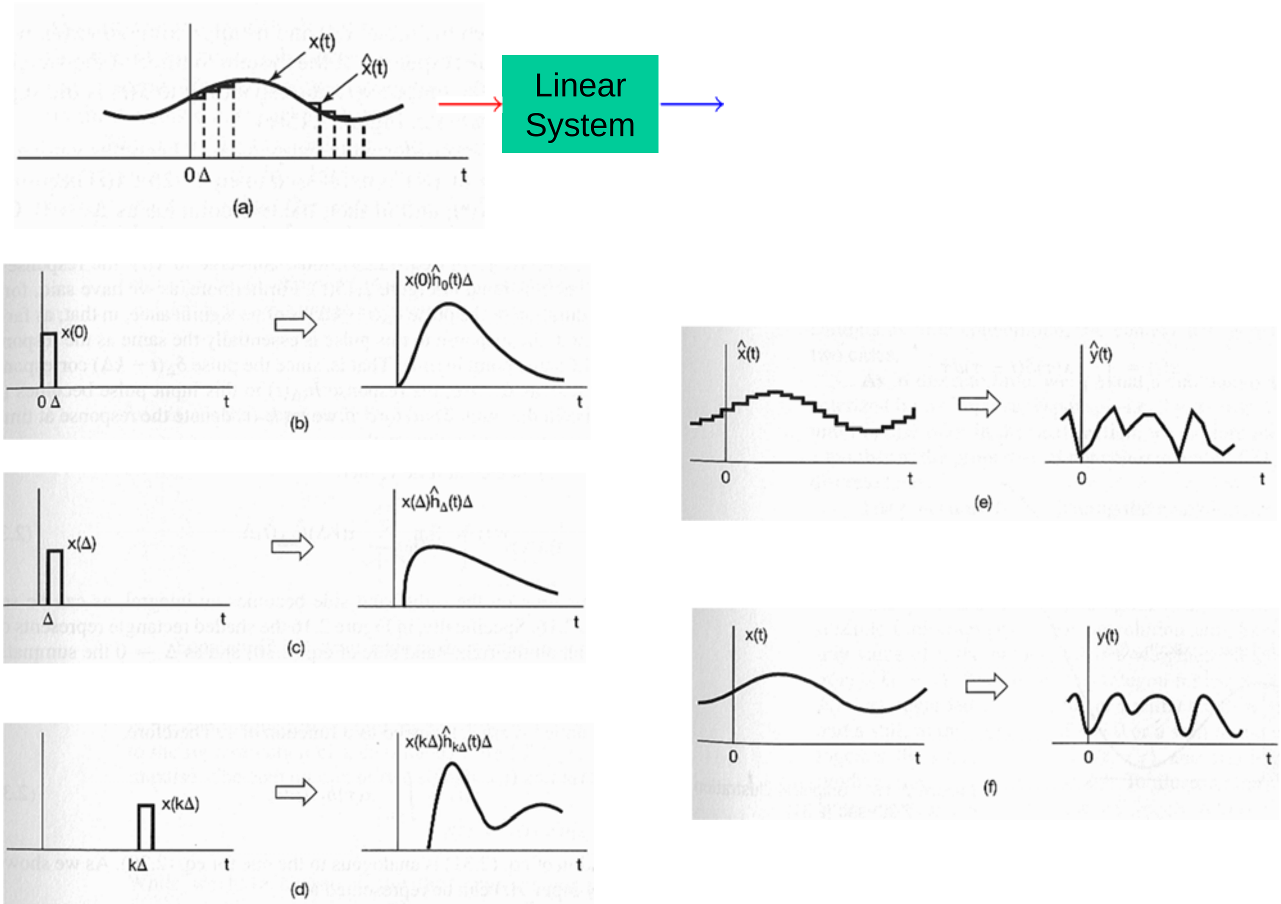
$x(1\Delta)$ $\delta_\Delta(t-1\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{1\Delta}(t)$ $x(1\Delta)$

$x(2\Delta)$ $\delta_\Delta(t-2\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{2\Delta}(t)$ $x(2\Delta)$

$\vdots$

$x(k\Delta)$ $\delta_\Delta(t-k\Delta) \longrightarrow$ Linear System $\longrightarrow \hat{h}_{k\Delta}(t)$ $x(k\Delta)$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta \Longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta$$



- CT Unit Impulse Response & Convolution Integral:

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \hat{h}_{k\Delta}(t)\, \Delta$$

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\, \hat{h}_{k\Delta}(t)\, \Delta$$

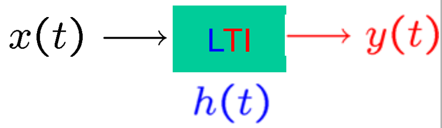
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h_\tau(t)\, d\tau$$

$\delta(t-\tau) \longrightarrow$ Linear System $\longrightarrow h_\tau(t)$

$x(t) \longrightarrow$ Linear System $\longrightarrow y(t)$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \quad \Longrightarrow \quad y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau$$

- If the linear system (L) is also time-invariant (TI)
  - Then,



$$h_\tau(t) = h_0(t-\tau) = h(t-\tau)$$

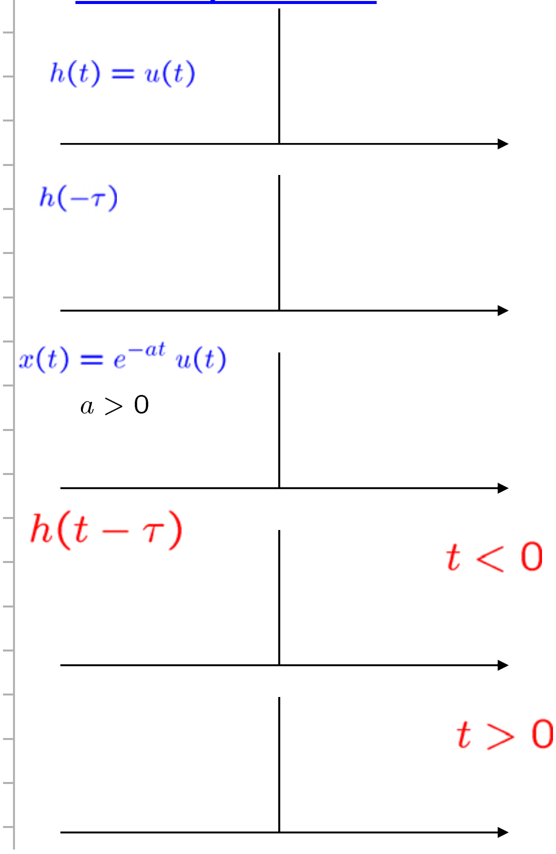
- Hence, for an LTI system,

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

  - Known as the convolution of x(t) & h(t)
  - Referred as the convolution integral or the superposition integral

- Symbolically, $y(t) = x(t) * h(t) \qquad = h(t) * x(t)$

- Example 2.6: $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$



for $t < 0$, $x(\tau)\, h(t-\tau) = 0$

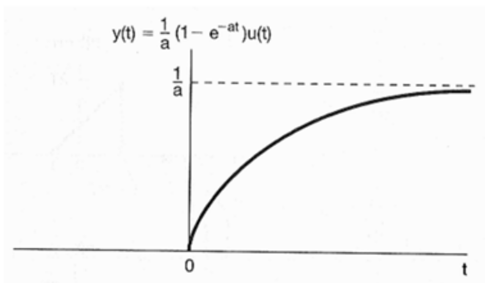
$$\Rightarrow y(t) = \int_{-\infty}^{t} 0\, d\tau = 0$$

for $t \geq 0$, $x(\tau)\, h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y(t) = \int_0^t e^{-a\tau} d\tau$$

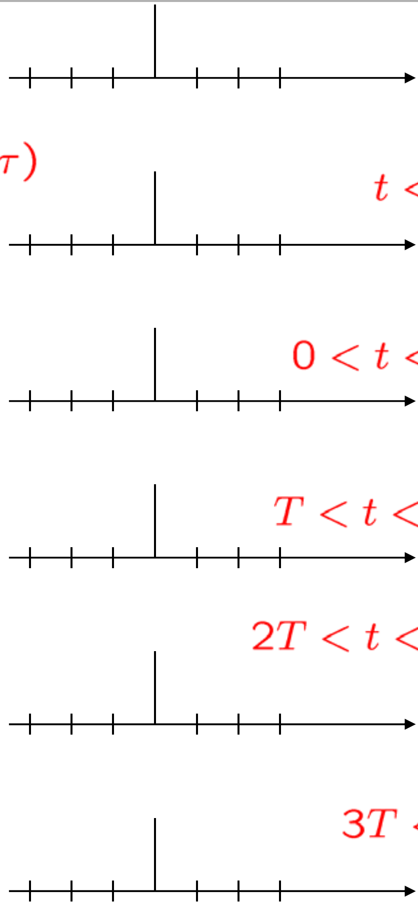
$$= -\frac{1}{a} e^{-a\tau}\Big|_0^t$$

$$= \frac{1}{a}(1 - e^{-at})$$



- **Example 2.7:** $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

$x(\tau)$

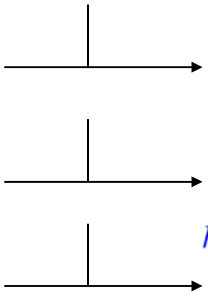


$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$

$h(\tau)$

$h(-\tau)$



$h(t-\tau)$

$t < 0$

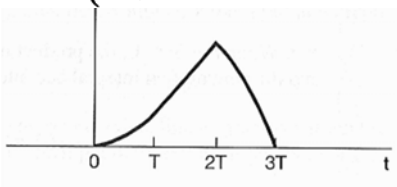
$0 < t < T$

$T < t < 2T$

$2T < t < 3T$

$3T < t$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

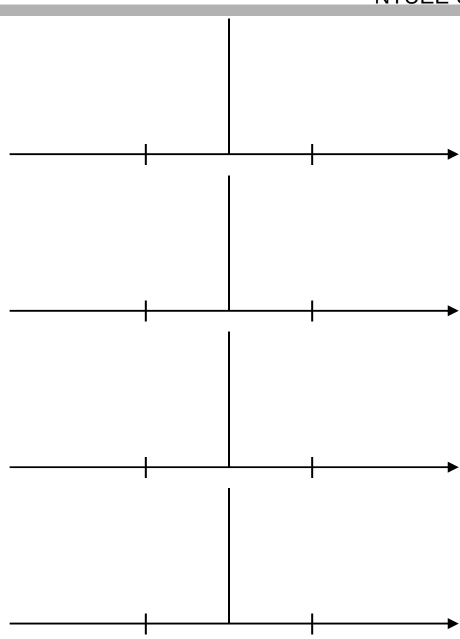


- **Example 2.8:** $x(t) = e^{2t}u(-t)$

$h(t) = u(t-3)$

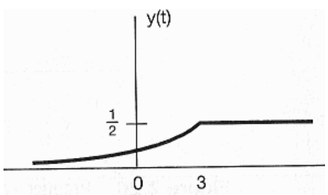
$h(-\tau)$

$h(t-\tau)$



for $t-3 \le 0$, $y(t) = \int_{-\infty}^{t-3} e^{2\tau}d\tau \quad = \frac{1}{2}e^{2(t-3)}$

for $t-3 \ge 0$, $y(t) = \int_{-\infty}^{0} e^{2\tau}d\tau \quad = \frac{1}{2}$

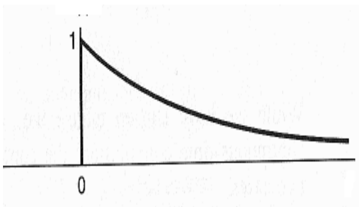


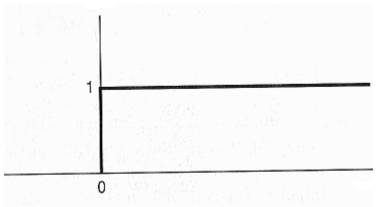
Signal and System:

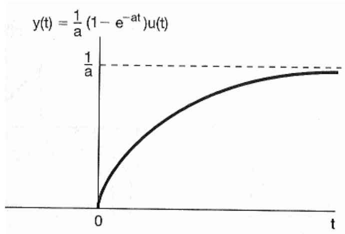
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = x[n] * h[n]$$

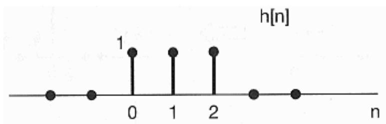
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = x(t) * h(t)$$

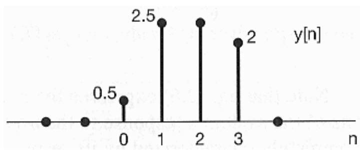
$x(t) \rightarrow$ LTI: h(t) $\rightarrow y(t)$











- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions

▪ Convolution Sum & Integral of LTI Systems:

$$y[n] \;=\; \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \;=\; x[k] * h[n]$$

$$y(t) \;=\; \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \;=\; x(t) * h(t)$$

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$

$x(t) \rightarrow$ h(t) $\rightarrow y(t)$

▪ Properties of LTI Systems

$$y[n] = x[k] * h[n]$$
$$y(t) = x(t) * h(t)$$

1. Commutative property
2. Distributive property
3. Associative property

$$a \times b = b \times a$$
$$a + b = b + a$$
$$a \times (b + c) = a \times b + a \times c$$
$$a \times (b \times c) = (a \times b) \times c$$
$$= \cdots = a \times b \times c$$

4. With or without memory
5. Invertibility
6. Causality
7. Stability

$x[n] \rightarrow$ h[n] $\rightarrow y[n]$

$x(t) \rightarrow$ h(t) $\rightarrow y(t)$

$$\forall x[n] \;\rightarrow\; \forall y[n] \qquad h[n] = ?$$

8. Unit step response

$a \times b \;=\; b \times a$

- Commutative Property: $n - k = r$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \qquad = \sum_{r=+\infty}^{-\infty} x[n-r]h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r]x[n-r] \qquad = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad \begin{aligned} t - \tau &= \sigma \\ -d\tau &= d\sigma \end{aligned}$$

$$= \int_{+\infty}^{-\infty} x(t-\sigma)h(\sigma)(-d\sigma) \qquad = \int_{-\infty}^{+\infty} x(t-\sigma)h(\sigma)d\sigma$$

$$= \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma \qquad = h(t) * x(t)$$

$a \times (b + c) \;=\; a \times b + a \times c$

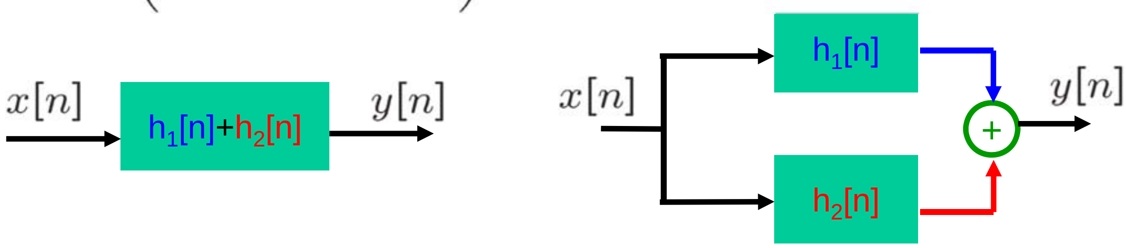
- Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x[n] * \left(h_1[n] + h_2[n]\right) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$



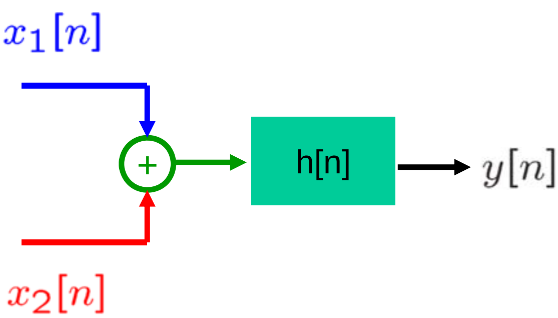
- **Distributive Property:**

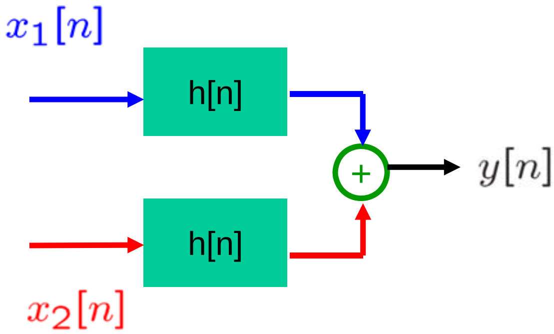
$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$\Big(x_1[n] + x_2[n]\Big) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$\Big(x_1(t) + x_2(t)\Big) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

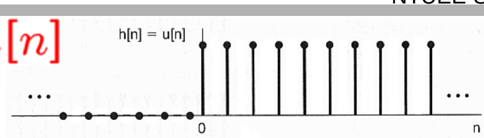


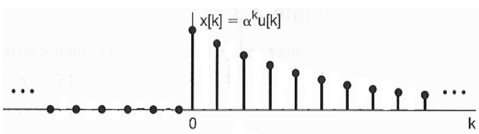


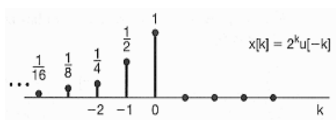
- **Example 2.10**

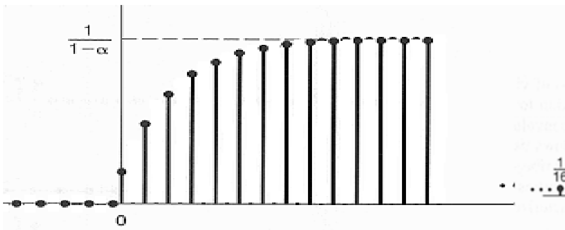
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n\, u[-n]$$

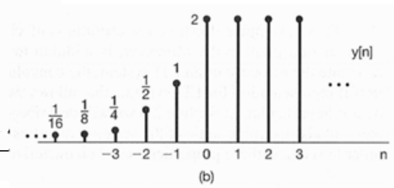
$h[n]$







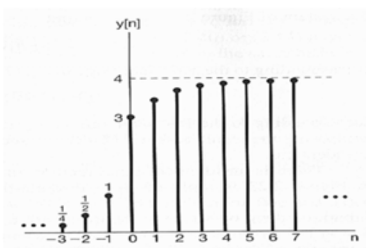




$$y[n] = x[n] * h[n]$$

$$= \Big(x_1[n] + x_2[n]\Big) * h[n]$$

$$= x_1[n] * h[n] + x_2[n] * h[n]$$



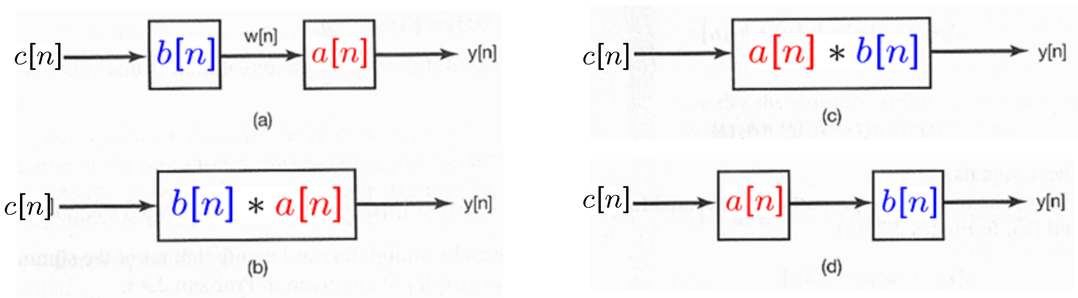
- Associative Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$a[n] * \Big(b[n] * c[n]\Big) = \Big(a[n] * b[n]\Big) * c[n]$$

$$a(t) * \Big(b(t) * c(t)\Big) = \Big(a(t) * b(t)\Big) * c(t)$$



$$a[n] * \Big(b[n] * c[n]\Big) = \Big(a[n] * b[n]\Big) * c[n]$$

$$= a[n] * \Big(c[n] * b[n]\Big)$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$= a[n] * \left(\sum_{k=-\infty}^{+\infty} c[k]\ b[n-k]\right)$$

$$= \sum_{m=-\infty}^{+\infty} a[m] \left(\sum_{k=-\infty}^{+\infty} c[k]\ b[n-m-k]\right)$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} a[m]\ c[k]\ b[n-m-k]$$

$$= \sum_{k=-\infty}^{+\infty} c[k] \sum_{m=-\infty}^{+\infty} a[m]\ b[n-k-m]$$

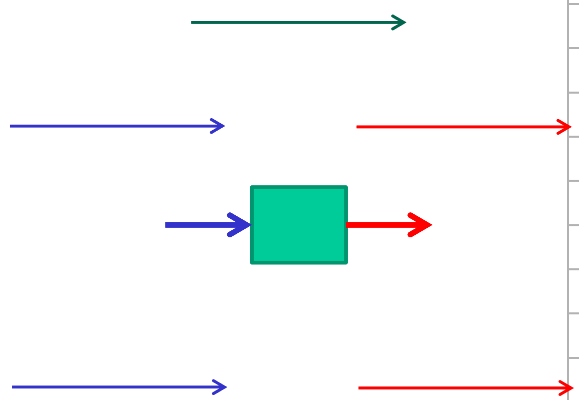
$$= c[n] * \left(\sum_{m=-\infty}^{+\infty} a[m]\ b[n-m]\right) \qquad = c[n] * \Big(a[n] * b[n]\Big)$$

Systems with or without memory

- Memoryless systems
  - Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t)$$ (resistor)



- Systems with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ (accumulator)

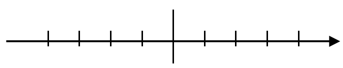
$$y[n] = x[n-1]$$ (delay)

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

Memoryless:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- A DT LTI system is memoryless if $h[n] = 0$ for $n \neq 0$



- The impulse response: $h[n] = K\delta[n], \quad K = h[0]$

- From the convolution sum:

$$y[n] = x[n] * h[n]$$

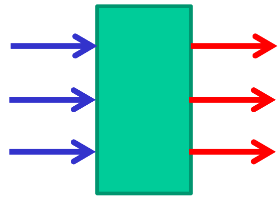
$$= Kx[n]$$

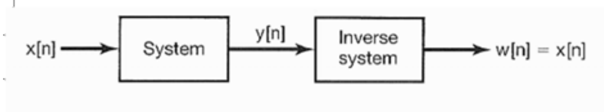
- Similarly, for CT LTI system: $y(t) = x(t) * h(t) = Kx(t)$

Invertibility & Inverse Systems

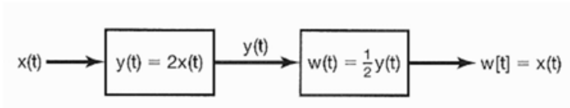
Invertible systems

- Distinct inputs lead to distinct outputs

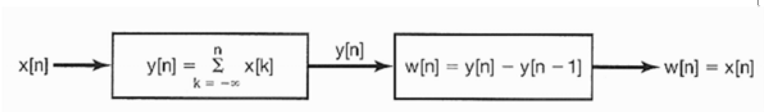




(a)



(b)

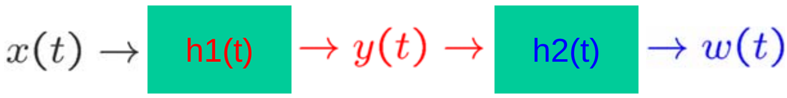


(c)

$y(t) = x(t)^2$ is not invertible

Invertibility:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$



$$y(t) = x(t) * h_1(t) \qquad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

$$x(t) = x(t) * \delta(t)$$

$$\Longrightarrow h_2(t) * h_1(t) = \delta(t)$$

▪ Example 2.11: Pure time shift

$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$

$x(t) \rightarrow$ h1(t) $\rightarrow y(t) \rightarrow$ h2(t) $\rightarrow w(t)$

- $y(t) = x(t - t_0)$
  - delay if $t_0 > 0$
  - advance if $t_0 < 0$

$$\Rightarrow \quad h_1(t) = \delta(t - t_0) \quad \Rightarrow \quad x(t) * \delta(t - t_0) = x(t - t_0)$$

- $w(t) = x(t) = y(t + t_0)$

$$\Rightarrow \quad h_2(t) = \delta(t + t_0) \quad \Rightarrow \quad y(t) * \delta(t + t_0) = y(t + t_0)$$

$$\Rightarrow \quad h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

▪ Example 2.12

$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$

$x[n] \rightarrow$ h1[n] $\rightarrow y[n] \rightarrow$ h2[n] $\rightarrow w[n]$

$$h_1[n] = u[n]$$

$$\Rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] \, u[n-k] \quad = \sum_{k=-\infty}^{n} x[k]$$

$\Rightarrow$ a running-sum operation

- Its inverse is a first difference operation:

$$w[n] = y[n] - y[n-1] \quad \Rightarrow \quad h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow \quad h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

Causality:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

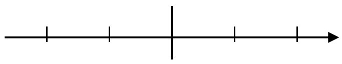
- The output of a causal system depends only on the present and past values of the input to the system
- Specifically, y[n] must not depend on x[k], for k > n

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

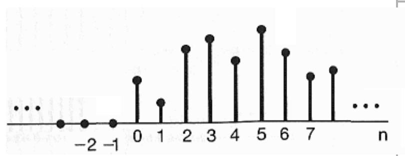
$$h[n-k] = 0, \quad \text{for } k > n$$

$$h[m] = 0, \quad \text{for } m = n-k < 0$$

$$h[n] = 0, \quad \text{for } n < 0$$



- It implies that the system is initially rest



- A CT LTI system is causal if $h(t) = 0, \quad \text{for } t < 0$

Convolution Sum & Integral

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} x[k]\ h_k[n] \\ &= \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k] \\ &= \sum_{k=-\infty}^{n} x[k]\ h[n-k] \\ &= \sum_{m=\infty}^{0} x[n-m]\ h[m] \\ &= \sum_{m=0}^{\infty} h[m]\ x[n-m] \\ &= \sum_{k=0}^{\infty} h[k]\ x[n-k] \end{aligned}$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)\ h_\tau(t)\ d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau \\ &= \int_{-\infty}^{t} x(\tau)\ h(t-\tau)\ d\tau \\ &= \int_{\infty}^{0} x(t-\sigma)\ h(\sigma)\ (-d\sigma) \\ &= \int_{0}^{\infty} x(t-\sigma)\ h(\sigma)\ d\sigma \\ &= \int_{0}^{\infty} h(\tau)\ x(t-\tau)\ d\tau \end{aligned}$$

- Stability
  - Stable systems
    - Small inputs lead to responses that do not diverge
    - Every bounded input excites a bounded output
      - Bounded-input bounded-output stable (BIBO stable)
      - For all $|x(t)| < a$, then $|y(t)| < b$, for all t

- Balance in a bank account? $y[n] = 1.01y[n-1] + x[n]$

- Stability: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$
  - A system is stable
    if every bounded input produces a bounded output

$x[n] \rightarrow$ Stable LTI $\rightarrow y[n]$

$\left|x[n]\right| < B$ for all n

$$\left|y[n]\right| = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$\Rightarrow y[n] \le \sum_{k=-\infty}^{+\infty} h[k] \quad x[n-k]$$

$$\Rightarrow y[n] \le \left(\sum_{k=-\infty}^{+\infty} h[k] \right)$$

if $\sum_{k=-\infty}^{+\infty} \left|h[k]\right| < \infty$ then, $y[n]$ is bounded

absolutely summable

$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

- **Stability:**
  - For CT LTI stable system:

$$x(t) \rightarrow \boxed{\text{Stable LTI}} \rightarrow y(t)$$

$|x(t)| < B$ for all t $\qquad |y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \right|$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{+\infty} |h(\tau)||x(t-\tau)|d\tau$$

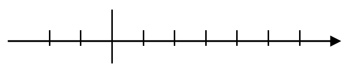
$$\Rightarrow |y(t)| \leq \left(\int_{-\infty}^{+\infty} |h[(\tau)|d\tau \right)$$

if $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$ then, $y(t)$ is bounded

absolutely integrable

- **Example 2.13: Pure time shift**

  - $y[n] = x[n-n_0]$ & $h[n] = \delta[n-n_0]$
  - $y(t) = x(t-t_0)$ & $h(t) = \delta(t-t_0)$



$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n-n_0]| = 1$$ absolutely summable

$$\Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| = \int_{-\infty}^{+\infty} |\delta(\tau-t_0)|d\tau = 1$$ absolutely integrable

$\Rightarrow$ A (CT or DT) pure time shift is stable

Example 2.13: Accumulator

- $y[n] = \sum_{k=-\infty}^{n} x[k]$ & $h[n] = u[n]$

- $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ & $h(t) = u(t)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left|h[n]\right| = \sum_{n=0}^{+\infty} \left|u[n]\right| = \infty$$ NOT absolutely summable

$$\Rightarrow \int_{-\infty}^{+\infty} \left|h(\tau)\right| = \int_{0}^{\infty} \left|u(\tau)\right| d\tau = \infty$$ NOT absolutely integrable

$\Rightarrow$ A accumulator or integrator is NOT stable

Unit Impulse and Step Responses:

$h[n] = \delta[n] * h[n]$

- For an LTI system, its unit impulse response is:

$\delta[n] \rightarrow$ DT LTI $\rightarrow h[n]$ $\delta(t) \rightarrow$ CT LTI $\rightarrow h(t)$

- Its unit step response is:

$u[n] \rightarrow$ DT LTI $\rightarrow s[n]$ $u(t) \rightarrow$ CT LTI $\rightarrow s(t)$

$$\Rightarrow s[n] = u[n] * h[n] = \sum_{k=-\infty}^{+\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k]$$

$$\Rightarrow h[n] = s[n] - s[n-1]$$

$$\Rightarrow s(t) = u(t) * h(t) = \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

$$\Rightarrow h(t) = \frac{ds(t)}{dt}$$

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $\quad y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $\quad y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t)*\left(h_1(t)+h_2(t)\right) = x(t)*h_1(t)+x(t)*h_2(t)$
  3. Associative property $a(t)*\left(b(t)*c(t)\right) = \left(a(t)*b(t)\right)*c(t)$
  4. With or without memory $h[n]=0$ for $n \neq 0$ $\quad h(t)=0$, for $t<0$
  5. Invertibility $h_2(t)*h_1(t)=\delta(t)$
  6. Causality
  7. Stability if $\int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $h[n] = \left(\frac{1}{2}\right)^n u[n]$

- Singularity Functions
  - CT unit impulse function is one of singularity functions

$$\delta(t) = u_0(t) \qquad u(t) = u_{-1}(t)$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

$$\frac{d}{dt}\delta(t) = u_1(t) \qquad \int_{-\infty}^{t} u(\tau)d\tau = u_{-2}(t)$$

$$\frac{d^2}{dt^2}\delta(t) = u_2(t) \qquad \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} u(\sigma)d\sigma\right) d\tau = u_{-3}(t)$$

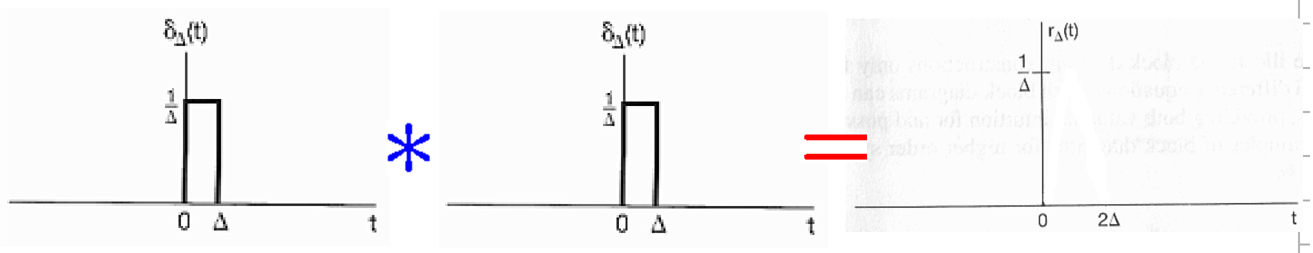
$$\frac{d^k}{dt^k}\delta(t) = u_k(t) \qquad \int_{-\infty}^{t} \cdots \left(\int_{-\infty}^{\tau} u(\sigma)d\sigma\right) \cdots d\tau = u_{-k}(t)$$

▪ Singularity Functions

$$x(t) = x(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t)$$

$$r_\triangle(t) = \delta_\triangle(t) * \delta_\triangle(t)$$



$$\lim_{\triangle \to 0} \delta_\triangle(t) = \delta(t) \quad \Rightarrow \quad \lim_{\triangle \to 0} r_\triangle(t) = \delta(t)$$

▪ Example 2.16

$$\frac{d}{dt}y(t) + 2\,y(t) = x(t)$$

with initial-rest condition

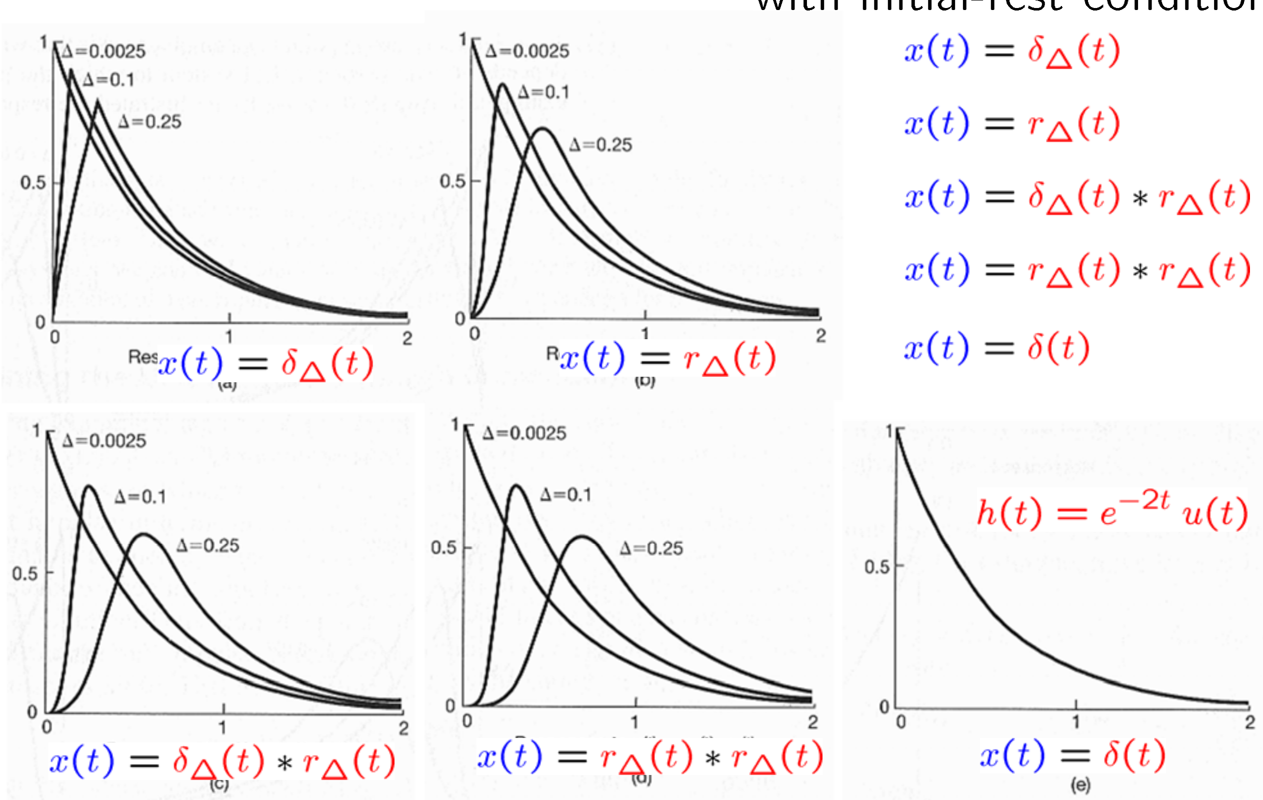
$$x(t) = \delta_\triangle(t)$$

$$x(t) = r_\triangle(t)$$

$$x(t) = \delta_\triangle(t) * r_\triangle(t)$$

$$x(t) = r_\triangle(t) * r_\triangle(t)$$

$$x(t) = \delta(t)$$



- Example 2.16 $\frac{d}{dt}y(t) + 20\,y(t) = x(t)$

with initial-rest condition

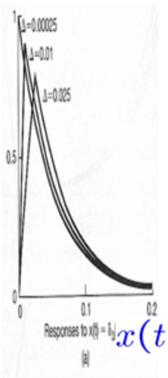
$x(t) = \delta_{\triangle}(t)$

$x(t) = r_{\triangle}(t)$

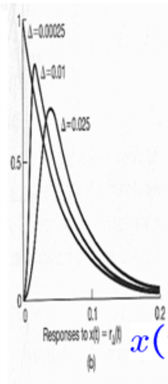
$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$

$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$

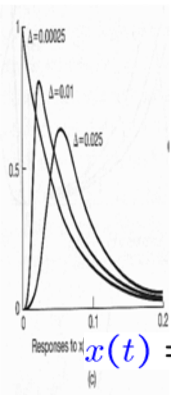
$x(t) = \delta(t)$



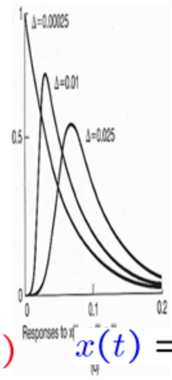
$x(t) = \delta_{\triangle}(t)$



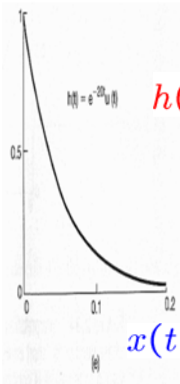
$x(t) = r_{\triangle}(t)$



$x(t) = \delta_{\triangle}(t) * r_{\triangle}(t)$



$x(t) = r_{\triangle}(t) * r_{\triangle}(t)$



$h(t) = e^{-20t}\,u(t)$

$x(t) = \delta(t)$

- Defining the Unit Impulse through Convolution:

  - Define: $x(t) = x(t) * \delta(t)$

  - Let $x(t) = 1$,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau)d\tau$$

  - So that the unit impulse has unit area

Defining the Unit Impulse through Convolution:

- Alternatively, consdier an arbitrary signal $g(t)$,
- Define: $g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$
- Define $x(t-\tau) = g(\tau)$

$$x(t) = g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau = x(t) * \delta(t)$$

$$x(t) = x(t) * \delta(t) \iff g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)d\tau$$

[hint: $g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau)d\tau$]

Defining the Unit Impulse through Convolution:

- Consider the signal $f(t)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(\tau)\delta(\tau)d\tau = g(0)f(0)$$

- On the other hand, consider the signal $f(0)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau)f(0)\delta(\tau)d\tau = g(0)f(0)$$

- Therefore,

$$f(t)\delta(t) = f(0)\delta(t)$$

Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt}x(t)$$

$\Rightarrow$ The unit impulse response of the system

is the derivative of the unit impulse,

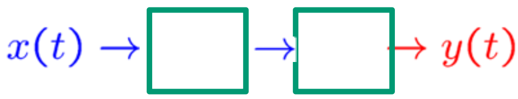
which is called the unit doublet $u_1(t)$

- That is, from $x(t) = x(t) * \delta(t)$, we have

$$\frac{d}{dt}x(t) = x(t) * u_1(t)$$

Unit Doublets of Derivative Operation:

- Similarly,



$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$

- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{d}{dt}x(t)\right) = \big(x(t) * u_1(t)\big) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

$u_k(t)$, $k > 0$, the kth derivative of $\delta(t)$

$$u_k(t) = u_1(t) * \cdots * u_1(t)$$

Unit Doublets of Integration Operation:

- A system: Output is the integral of input

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

- Therefore,

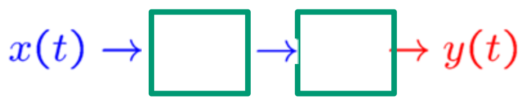
$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

Unit Doublets of Integration Operation:

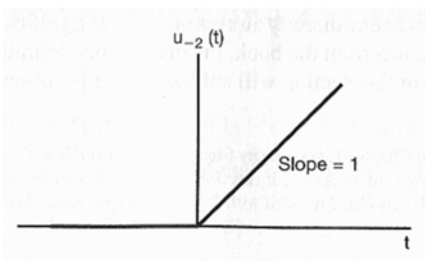
- Similarly,



$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

- That is,

$$u_{-2}(t) = t\ u(t)$$ the unit ramp function



Unit Doublets of Integration Operation:

- Moreover,

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \left(\int_{-\infty}^{t} x(\sigma) d\sigma \right) * u(t)$$

$$= \int_{-\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau$$

- In general,

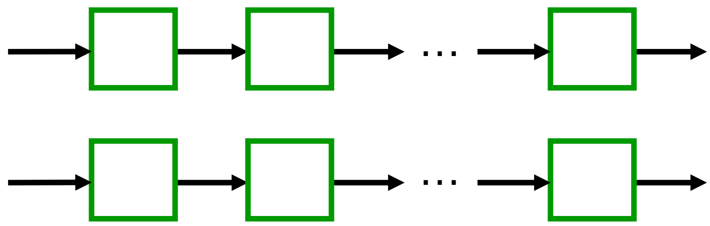
$$u_{-k}(t) = u(t) * \cdots * u(t) = \int_{-\infty}^{t} u_{-(k-1)}(\tau) d\tau$$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

In Summary

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$



$u_k(t)$

$k > 0,$ Impulse response of a cascade of k differentiators

$k < 0,$ Impulse response of a cascade of $|k|$ integrators

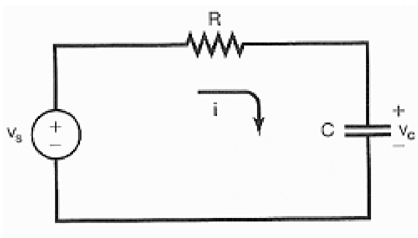
$$u(t) * u_1(t) = \delta(t) \qquad \text{or, } u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow \; u_k(t) * u_r(t) = u_{k+r}(t)$$

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ $y[n] = x[n] * h[n]$
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$ $y(t) = x(t) * h(t)$
- Properties of Linear Time-Invariant Systems
  1. Commutative property $x(t) * h(t) = h(t) * x(t)$
  2. Distributive property $x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$
  3. Associative property $a(t) * \left(b(t) * c(t)\right) = \left(a(t) * b(t)\right) * c(t)$
  4. With or without memory $h(t) = 0$ for $t \neq 0$
  5. Invertibility $h_2(t) * h_1(t) = \delta(t)$
  6. Causality $h(t) = 0,$ for $t < 0$
  7. Stability if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$
  8. Unit step response
- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations
- Singularity Functions $u_k(t) * u_r(t) = u_{k+r}(t)$

- Linear Constant-Coefficient Differential Equations
  - e.x., RC circuit

Input signal: $v_s(t)$



Output signal: $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

$x(t) \rightarrow$ RC Circuit $\rightarrow y(t)$

$$\Rightarrow \frac{d}{dt} y(t) + a\, y(t) = b\, x(t)$$

- Provide an implicit specification of the system
- You have learned how to solve the equation in Diff Eqn

Linear Constant-Coefficient Differential Equations

- For a general CT LTI system, with N-th order,

$x(t) \rightarrow$ CT LTI $\rightarrow y(t)$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$\Rightarrow h(t) = ?$ $\qquad x(t) = \quad \Rightarrow y(t) =$

$x(t) = \quad \Rightarrow y(t) =$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

$x[n] \rightarrow$ DT LTI $\rightarrow y[n]$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$\Rightarrow h[n] = ?$ $\qquad x[n] = \quad \Rightarrow y[n] =$

$x[n] = \quad \Rightarrow y[n] =$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k]$$

$$\Rightarrow \; y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k \, x[n-k] - \sum_{k=1}^{N} a_k \, y[n-k] \right\}$$

$$x[n] = \qquad \Rightarrow \; y[n] =$$

For example, Example 2.15

$\delta[n] \rightarrow$ LTI $\rightarrow h[n]$

$x[n] \rightarrow$ LTI $\rightarrow y[n]$

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

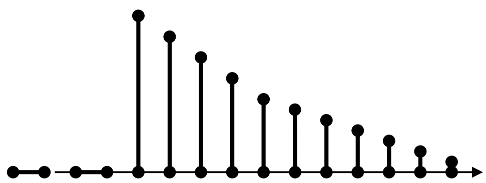
$$y[n] = 0, \quad \text{for } n \leq -1$$

$$x[n] = K \, \delta[n]$$

$$\Rightarrow \begin{cases} y[0] = x[0] + \frac{1}{2} y[-1] & = K \\ y[1] = x[1] + \frac{1}{2} y[0] & = K \, \frac{1}{2} \\ y[2] = x[2] + \frac{1}{2} y[1] & = K \left(\frac{1}{2}\right)^2 \\ \vdots & \vdots \\ y[n] = x[n] + \frac{1}{2} y[n-1] & = K \left(\frac{1}{2}\right)^n \end{cases}$$

$$\Rightarrow \; y[n] = K \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow \; h[n] = \left(\frac{1}{2}\right)^n u[n]$$



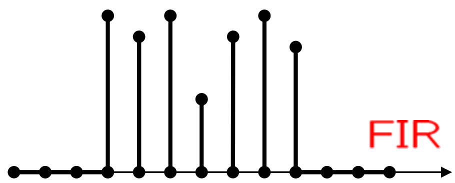
$\Rightarrow$ an Infinite Impulse Response (IIR) system

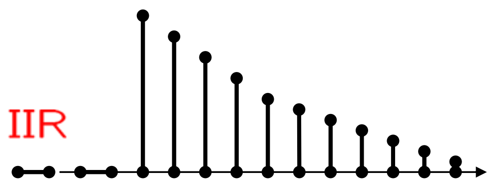
- **Nonrecursive Equation:** $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$
  - When N = 0,

$$\Rightarrow\ y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k] \qquad\qquad y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

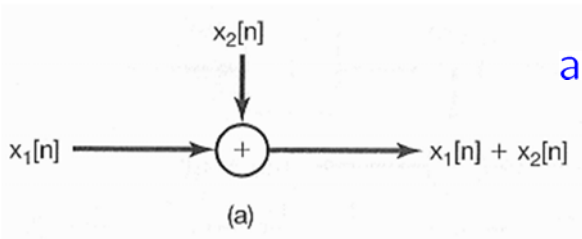
$$\Rightarrow\ h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ a Finite Impulse Response (FIR) system

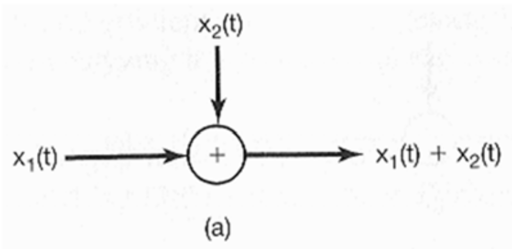




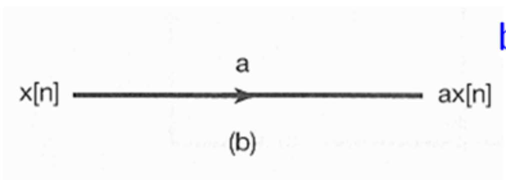
- **Block Diagram Representations:**

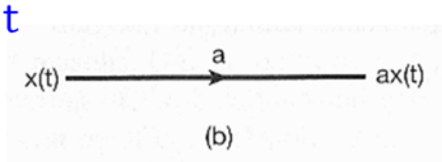


an adder

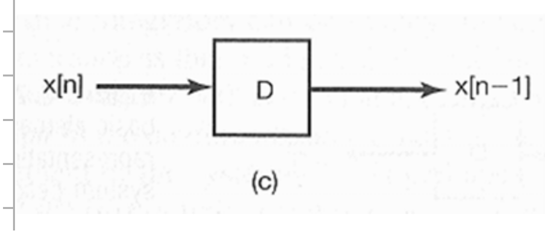


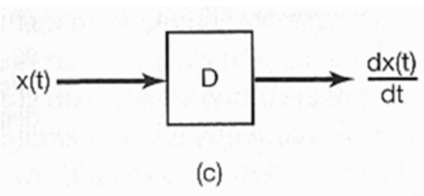
multiplication
by a coefficient





a unit delay/
differentiator





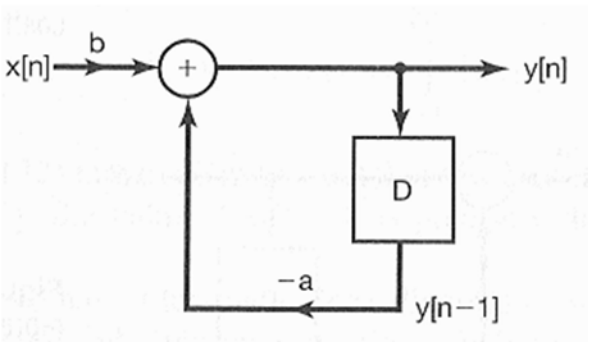
Block Diagram Representations:

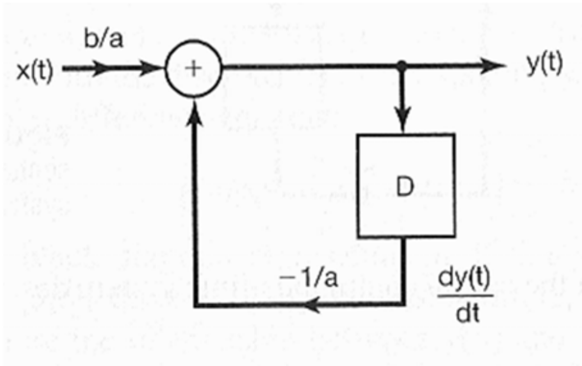
$$y[n] + a y[n-1] = b x[n]$$

$$\frac{d}{dt} y(t) + a y(t) = b x(t)$$

$$y[n] = -a y[n-1] + b x[n]$$

$$y(t) = -\frac{1}{a}\frac{d}{dt} y(t) + \frac{b}{a} x(t)$$





$$D \Longleftrightarrow z^{-1}$$

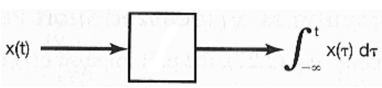
$$D \Longleftrightarrow s$$

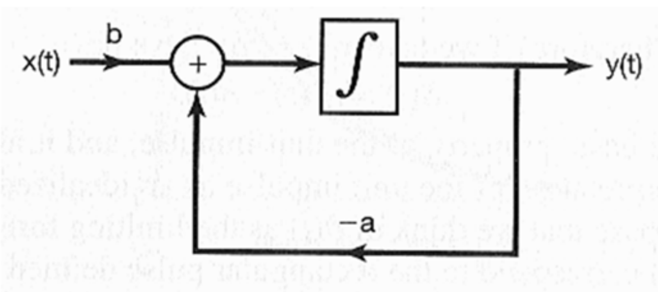
Block Diagram Representations:

$$\frac{d}{dt} y(t) = b x(t) - a y(t)$$

$$\Rightarrow \; y(t) = \int_{-\infty}^{t} \left[b x(\tau) - a y(\tau) \right] d\tau$$

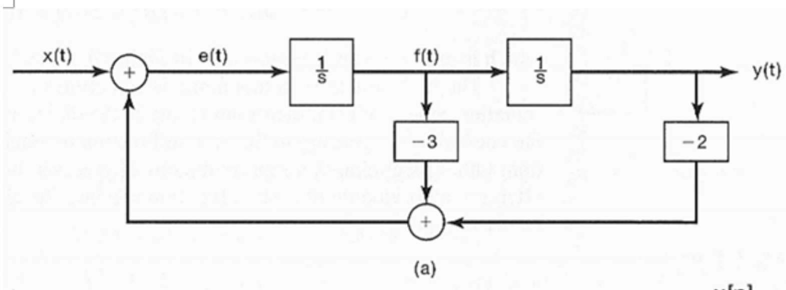
$$\Rightarrow \; y(t) = y(t_0) + \int_{t_0}^{t} \left[b x(\tau) - a y(\tau) \right] d\tau$$

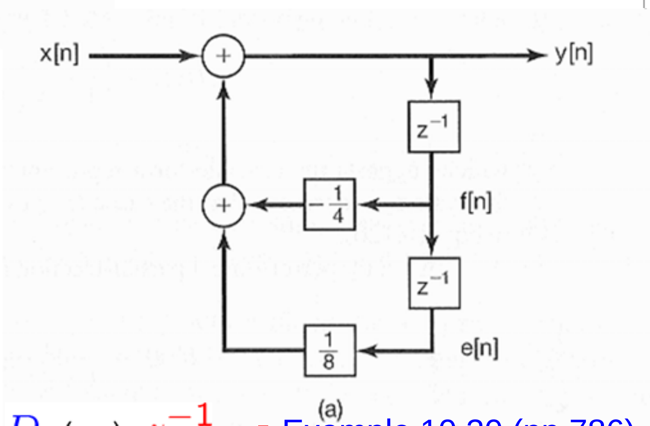




- Block Diagram Representations:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t) \qquad y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$





- Example 9.30 (pp.711) $D \Longleftrightarrow s$ $D \Longleftrightarrow z^{-1}$
- Example 10.30 (pp.786)

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$
$$y[n] = x[n] * h[n]$$

- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$
$$y(t) = x(t) * h(t)$$

- Properties of Linear Time-Invariant Systems
  1. Commutative property
  2. Distributive property
  3. Associative property
  4. With or without memory
  5. Invertibility
  6. Causality
  7. Stability
  8. Unit step response

$$x(t) * h(t) = h(t) * x(t)$$
$$x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$$
$$a(t) * \big(b(t) * c(t)\big) = \big(a(t) * b(t)\big) * c(t)$$

$h[n] = 0$ for $n \neq 0$ $\qquad h(t) = 0,$ for $t < 0$

$h_2(t) * h_1(t) = \delta(t)$ $\qquad$ if $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$

- Causal Linear Time-Invariant Systems Described by Differential & Difference Equations

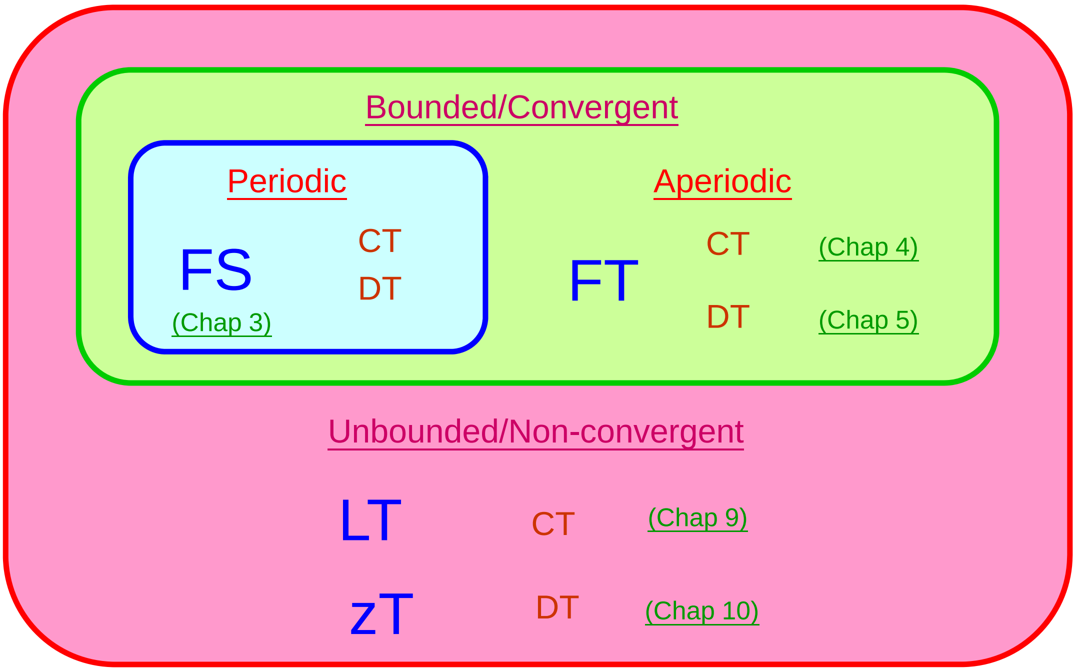
$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

- Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Flowchart

Signals & Systems (Chap 1)   LTI & Convolution (Chap 2)



Time-Frequency (Chap 6)   Communication (Chap 8)

CT-DT (Chap 7)   Control (Chap 11)