SPRING 2010

即時控制系統設計
Design of Real-Time Control Systems

Lecture 24
Controller Design of Digital Control Systems

Feng-Li Lian

NTU-EE

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- Design of Digital Control Systems
  - Transfer Function Design Methods
    - > Design by Emulation
    - > Discrete Design
  - State-Space Design Methods

- Two basic digital controller design techniques:

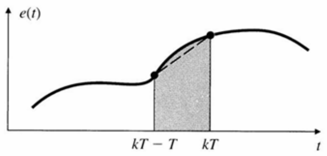
1. Emulation:
   - Design a continuous compensation D(s) by using any CT controller design methods
   - Approximate that D(s) by using any approximation methods such as Tustin's Method
2. Discrete Design:
   - Model in DT (difference equations)
   - Design in DT (z-transform)

Franklin et al. 2002

- Design by emulation:
  1. Design a continuous compensation
  2. Digitalize the continuous compensation
  3. Use discrete analysis, simulation, or experimentation
- Design techniques:
  1. Tustin's Method or bilinear approximation
  2. Matched Pole-Zero method (MPZ)
  3. Modified Matched Pole-Zero method (MMPZ)

Franklin et al. 2002

Tustin's method:



$$\frac{U(s)}{E(s)} = D(s) = \frac{1}{s}$$

$$\Rightarrow \; u[kT] = \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt$$

$$= u[kT-T] + \text{ area under } e(t) \text{ over last } T$$

$$\Rightarrow \; u[k] = u[k-1] + \frac{T}{2}\left[e[k-1] + e[k]\right]$$

$$\Rightarrow \; \frac{U(z)}{E(z)} = D(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Franklin et al. 2002

Tustin's method:

$$\frac{U(s)}{E(s)} = D(s) = \frac{a}{s+a}$$

$$\Rightarrow \; \frac{U(z)}{E(z)} = D(z) = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

$$\Rightarrow \; s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

for every occurance of s in any $D(s)$
yields a $D(z)$ on the trapezoidal integration

Franklin et al. 2002

Tustin's method:

$$D(s) = 10\,\frac{\frac{s}{2}+1}{\frac{s}{10}+1}$$

$$w_s = 25 \times w_{\text{BW}} = 25 \times 10 = 250 \text{ rad/sec}$$

$$f_s = w_s/(2\pi) \approx 40 \text{ Hz}$$

$$T = \frac{1}{f_s} = \frac{1}{40} = 0.025 \text{ sec}$$

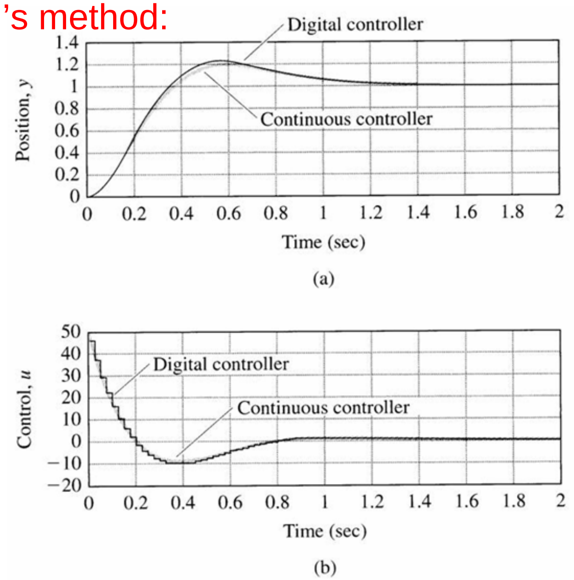
```
sysDs = tf( 10*[0.5 1], [0.1 1] );
sysDd = c2d( sysDc, 0.025, 'tustin');
```

$$\Rightarrow \; D(z) = \frac{45.56 - 43.33z^{-1}}{1 - 0.7778z^{-1}}$$

$$\Rightarrow \; u[k] = 0.7778u[k-1] + 45.56e[k] - 43.33e[k-1]$$

Franklin et al. 2002

Tustin's method:



Franklin et al. 2002

- Matched Pole-Zero (MPZ) method:

$$z = e^{sT}$$

1. Map poles and zeros according to the relation
2. If the numerator is of lower order than the denominator,
   add powers of (z+1) to the numerator
   until numerator and denominator are of equal order
3. Set the DC or low-frequency gain of D(z) = that of D(s)

Franklin et al. 2002

- Matched Pole-Zero (MPZ) method:
  - Case 1:

$$D(s) = K_c \frac{s+a}{s+b}$$

$$\Rightarrow \quad D(z) = K_d \frac{z - e^{-aT}}{z - e^{-bT}}$$

- By the Final Value Theorem:

$$K_c \frac{a}{b} = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

$$\text{or} \quad K_d = K_c \frac{a}{b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

Franklin et al. 2002

- Matched Pole-Zero (MPZ) method:
  - Case 2:

$$D(s) = K_c \frac{s+a}{s(s+b)}$$

$$\Rightarrow \quad D(z) = K_d \frac{(z+1)(z - e^{-aT})}{(z-1)(z - e^{-bT})}$$

- By the Final Value Theorem:

$$\Rightarrow \quad K_d = K_c \frac{a}{2b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

Franklin et al. 2002

- Matched Pole-Zero (MPZ) method:
  - The same power of z in the num & den of D(z):

$$\frac{U(z)}{E(z)} = D(z) = K_d \frac{1 - \alpha z^{-1}}{1 - \beta z^{-1}}$$

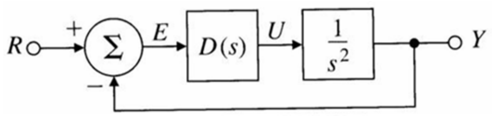
$$\alpha = e^{-aT} \quad \& \quad \beta = e^{-bT}$$

$$\Rightarrow \quad u[k+1] = \beta u[k] + K_d \Big[e[k+1] - \alpha e[k] \Big]$$

Franklin et al. 2002

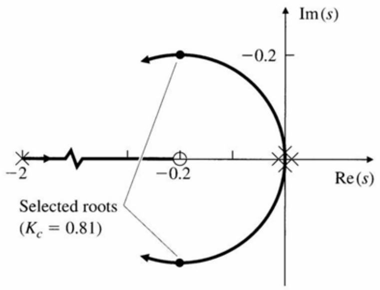
Matched Pole-Zero (MPZ) method:

- Space station attitude digital controller



$$w_n \approx 0.3 \text{ rad/sec}$$
$$\zeta = 0.7$$

$$\Rightarrow D(s) = 0.81 \frac{s+0.2}{s+2}$$



Franklin et al. 2002

Matched Pole-Zero (MPZ) method:

- Space station attitude digital controller

$$w_s = 0.3 \times 20 = 6 \text{ rad/sec}$$

$$\Rightarrow T \approx 1 \text{ sec}$$

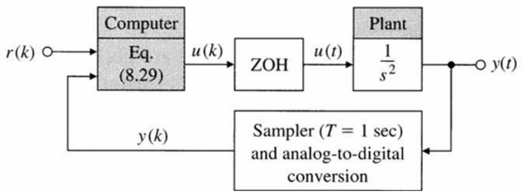
$$\Rightarrow D(z) = 0.389 \frac{z-0.82}{z-0.135} = \frac{0.389 - 0.319z^{-1}}{1-0.135z^{-1}}$$

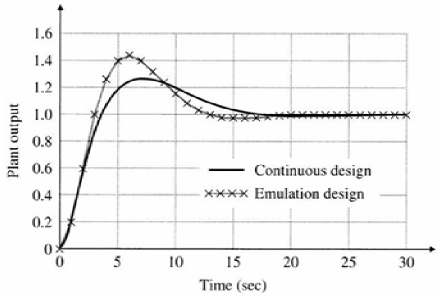
$$\Rightarrow u[k+1] = 0.135u[k] + 0.389e[k+1] - 0.319e[k]$$

Franklin et al. 2002

Matched Pole-Zero (MPZ) method:

- Space station attitude digital controller





Franklin et al. 2002

Modified Matched Pole-Zero (MMPZ) method:

- u[k+1] depends only on e[k], but not e[k+1]

$$D(s) = K_c \frac{s+a}{s(s+b)}$$

$$\Rightarrow D(z) = K_d \frac{(z-e^{-aT})}{(z-1)(z-e^{-bT})}$$

$$\Rightarrow K_d = K_c \frac{a}{b} \left(\frac{1-e^{-bT}}{1-e^{-aT}} \right)$$

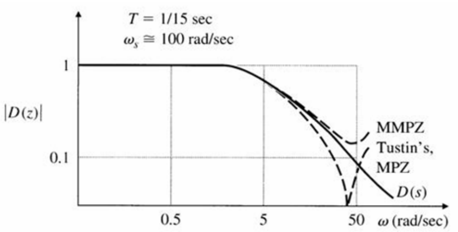
$$\Rightarrow u[k+1] = (1+e^{-bT})u[k] - e^{-bT}u[k-1] + K_d \left[e[k] - e^{-aT}e[k-1] \right]$$

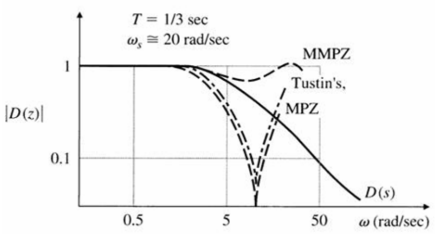
Franklin et al. 2002

Comparison of Digital Approximation Methods:

$$D(s) = \frac{5}{s+5}$$

Method	w_s 100 rad/sec	w_s 20 rad/sec
MPZ	$0.143\frac{z+1}{z-0.715}$	$0.405\frac{z+1}{z-0.189}$
MMPZ	$0.285\frac{1}{z-0.715}$	$0.811\frac{1}{z-0.189}$
Tustin's	$0.143\frac{z+1}{z-0.713}$	$0.454\frac{z+1}{z-0.0914}$





Franklin et al. 2002

Design of Digital Control Systems

- Transfer Function Design Methods
  - > Design by Emulation
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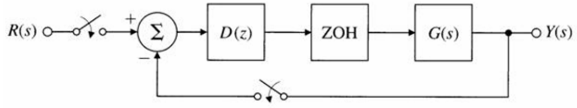
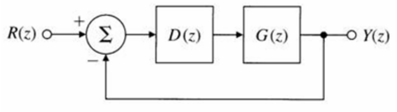
- State-Space Design Methods

The Exact Discrete Equivalent:





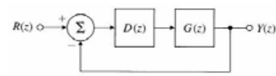
$$G(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

$$\Rightarrow \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1+D(z)G(z)}$$

Franklin et al. 2002

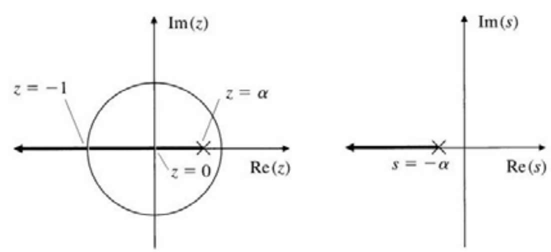
Discrete Root Locus: $G(s) = \frac{a}{s+a}$ & $D(z) = K$

$$\Rightarrow G(z) = (1-z^{-1})\mathcal{Z}\left\{\frac{a}{s(s+a)}\right\}$$



$$= (1-z^{-1})\left[\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}\right]$$

$$= \frac{1-e^{-aT}}{z-e^{-aT}}$$



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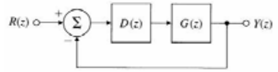

▪ Feedback Properties:

- Proportional: $u[k] = K\, e[k] \quad \Rightarrow \quad D(z) = K$
- Derivative: $u[k] = K\, T_D \left[e[k] - e[k-1]\right]$

$$\Rightarrow \quad D(z) = K\, T_D \left(1 - z^{-1}\right) = k_D \frac{z-1}{z}$$

- Integral: $u[k] = u[k-1] + \frac{K_p}{T_I} e[k]$

$$\Rightarrow \quad D(z) = \frac{K}{T_I} \left(\frac{1}{1 - z^{-1}}\right) = k_I \frac{z}{z-1}$$

- Lead Compensation:

$$u[k+1] = \beta u[k] + K \left[e[k+1] - \alpha e[k]\right]$$

$$\Rightarrow \quad D(z) = K \frac{1 - \alpha z^{-1}}{1 - \beta z^{-1}}$$

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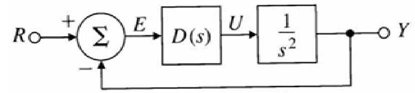
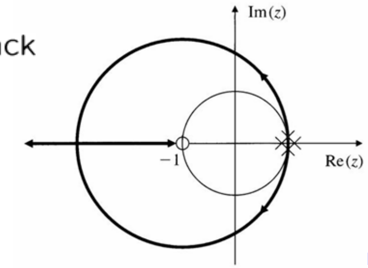
▪ Space Station Digital Controller example:



$$T = 1$$

$$G(z) = \frac{T^2}{2} \left[\frac{z+1}{(z-1)^2}\right] = \frac{1}{2} \left[\frac{z+1}{(z-1)^2}\right]$$

with proportional feedback



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▪ Space Station Digital Controller example:

- P + D feedback

$$\Rightarrow \quad U(z) = K \left[1 + T_D \left(1 - z^{-1}\right)\right] E(z)$$

$$\Rightarrow \quad D(z) = K \frac{z - \alpha}{z}$$

$$\omega_n \approx 0.3 \text{ rad/sec}, \quad \zeta = 0.7 \quad \Rightarrow \quad z = 0.78 \pm 0.18\, j$$

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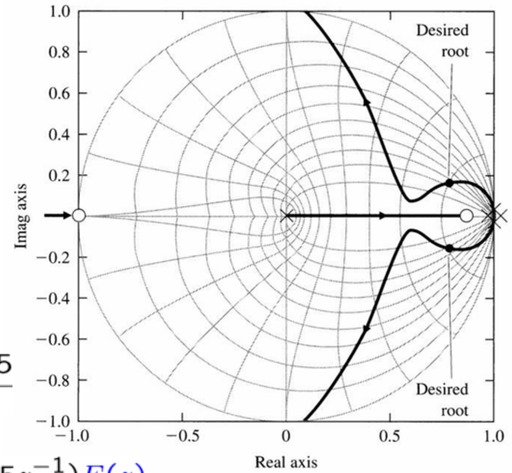
▪ Space Station Digital Controller example:

- P + D feedback



$$\Rightarrow \quad D(z) = 0.374 \frac{z - 0.85}{z}$$

$$\Rightarrow \quad U(z) = 0.374(1 - 0.85 z^{-1}) E(z)$$

$$\Rightarrow \quad u[k+1] = 0.374 e[k+1] - 0.318 e[k]$$

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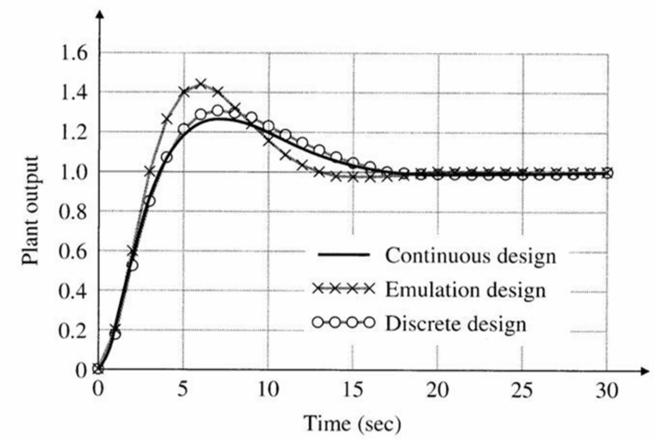
- Step Response of the continuous & digital systems:



ζ	w_n
0.705	0.324
0.645	0.441
0.733	0.306

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- Design of Digital Control Systems
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- State-Space Model:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t)$$
$$y(t) = \mathbf{H}\mathbf{x}(t) + Ju(t)$$

$$\Rightarrow \mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau)d\tau$$

Let $t = kT + T$ & $t_0 = kT$

$$\Rightarrow \mathbf{x}(kT+T) = e^{\mathbf{F}(T)}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

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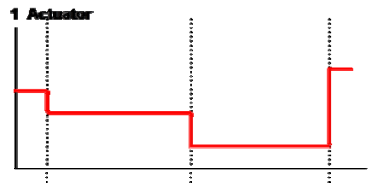
- State-Space Model:

Let $u(\tau)$ be piecewise constant through T



$$u(\tau) = u(kT), \quad kT \le \tau < kT + T$$

Let $\eta = kT + T - \tau$

$$\Rightarrow \mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \left(\int_0^T e^{\mathbf{F}\eta}d\eta\right)\mathbf{G}u(kT)$$

Let $\mathbf{\Phi} = e^{\mathbf{F}T}$ & $\mathbf{\Gamma} = \left(\int_0^T e^{\mathbf{F}\eta}d\eta\right)\mathbf{G}$

Then,
$$\mathbf{x}[k+1] = \mathbf{\Phi}\mathbf{x}[k] + \mathbf{\Gamma}u[k]$$
$$y[k] = \mathbf{H}\mathbf{x}[k] + Ju[k]$$

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Discrete Transfer Function:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{\Phi}\mathbf{x}[k] + \mathbf{\Gamma}u[k] \\ y[k] &= \mathbf{H}\mathbf{x}[k] \end{aligned}$$

$$\begin{aligned} z\mathbf{X}(z) &= \mathbf{\Phi}\mathbf{X}(z) + \mathbf{\Gamma}U(z) \\ Y(z) &= \mathbf{H}\mathbf{X}(z) \end{aligned}$$

$$\left(z\mathbf{I} - \mathbf{\Phi}\right)\mathbf{X}(z) = \mathbf{\Gamma}U(z)$$

$$\Rightarrow \frac{Y(z)}{U(z)} = G(z) = \mathbf{H}\left(z\mathbf{I} - \mathbf{\Phi}\right)^{-1}\mathbf{\Gamma}$$

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Discrete SS Model of 1/s^2:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \end{aligned}$$

$$\begin{aligned} \mathbf{\Phi} &= e^{\mathbf{F}T} = \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}T + \frac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 T^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \end{aligned}$$

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Discrete SS Model of 1/s^2:

$$\begin{aligned} \mathbf{\Gamma} &= \left(\int_0^T e^{\mathbf{F}\eta}d\eta\right)\mathbf{G} = \sum_{k=0}^{\infty}\frac{\mathbf{F}^kT^{k+1}}{(k+1)!}\mathbf{G} \\ &= \left(\mathbf{I} + \mathbf{F}\frac{T}{2!}\right)T\mathbf{G} \\ &= \left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{T^2}{2}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \end{aligned}$$

$$\Rightarrow G(z) = \frac{Y(z)}{U(z)} = \mathbf{H}\left(z\mathbf{I} - \mathbf{\Phi}\right)^{-1}\mathbf{\Gamma} = \frac{T^2}{2}\left[\frac{z+1}{(z-1)^2}\right]$$

Franklin et al. 2002

Pole Placement by State Feedback:

$$\alpha(z) = \det\left(z\mathbf{I} - \mathbf{\Phi}\right)$$

If the system is controllable

$$\mathcal{C} = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Phi}\mathbf{\Gamma} & \mathbf{\Phi}^2\mathbf{\Gamma} & \cdots & \mathbf{\Phi}^{n-1}\mathbf{\Gamma} \end{bmatrix} \quad \text{is full-rank}$$

$$u[k] = -\mathbf{K}\mathbf{x}[k]$$

$$\Rightarrow \det\left(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}\right) = \alpha_c(z)$$

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Discrete Full-Order Estimator:

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma u[k]$$
$$y[k] = \mathbf{H}\mathbf{x}[k]$$

$$\bar{\mathbf{x}}[k+1] = \Phi\bar{\mathbf{x}}[k] + \Gamma u[k] + \mathbf{L}\left[y[k] - \mathbf{H}\bar{\mathbf{x}}[k]\right]$$

$$\tilde{\mathbf{x}}[k+1] = \left(\Phi - \mathbf{L}\mathbf{H}\right)\tilde{\mathbf{x}}[k] \qquad (\tilde{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}})$$

If the system is observable, $\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H}\Phi \\ \mathbf{H}\Phi^2 \\ \vdots \\ \mathbf{H}\Phi^{n-1} \end{bmatrix}$ is full-rank

$$\Rightarrow \det\left(z\mathbf{I} - \Phi + \mathbf{L}\mathbf{H}\right) = \alpha_e(z)$$

Franklin et al. 2002

Hardware Characteristics:

- A/D Converters
- D/A Converters
- Anti-Alias Prefilters
- The Computer
- Word-Size Effects
- Sample-Rate Selection

Franklin et al. 2002