SPRING 2010

即時控制系統設計
Design of Real-Time Control Systems

Lecture 23
Dynamic Analysis of Digital Control Systems

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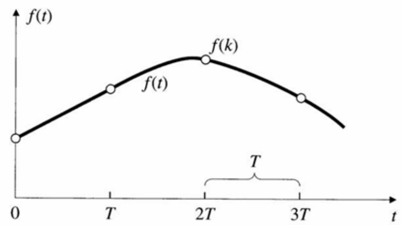
- Analysis of Discrete-Time Control Systems
  - By Transform methods (by transfer function)
    - Poles, Zeros
    - Stability
    - Transient Response, Steady-State Response
    - Impulse/step response
    - Root locus, Bode plot
  - By State-Variable methods (by state-space model)
    - CT -> DT
    - Linear, Nonlinear
    - Stability analysis, Sensitivity Analysis
    - Controllability, Observability
  - The z-transform
  - Relationship between s and z



- Laplace transform:

$$\mathcal{L}\left\{f(t)\right\} = F(s) \triangleq \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{L}\left\{\dot{f}(t)\right\} = s\, F(s)$$

- z-Transform:

$$\mathcal{Z}\left\{f[k]\right\} = F(z) \triangleq \sum_{k=0}^{\infty} f[k]\, z^{-k}$$

$$\mathcal{Z}\left\{f[k-1]\right\} = z^{-1}\, F(z)$$

Franklin et al. 2002

- Analysis of discrete systems by the z-transform:

$$y[k] + a_1\, y[k-1] + a_2\, y[k-2] = b_0\, u[k] + b_1\, u[k-1] + b_2\, u[k-2]$$

$$\Rightarrow\ Y(z) + a_1\, z^{-1}\, Y(z) + a_2\, z^{-2}\, Y(z) = b_0\, U(z) + b_1\, z^{-1}\, U(z) + b_2\, z^{-2}\, U(z)$$

$$\Rightarrow\ \frac{Y(z)}{U(z)} = \frac{b_0 + b_1\, z^{-1} + b_2\, z^{-2}}{1 + a_1\, z^{-1} + a_2\, z^{-2}} = \frac{b_0\, z^2 + b_1\, z + b_2}{z^2 + a_1\, z + a_2}$$

Franklin et al. 2002

- The z-transform inversion:
  - Long division

$$\frac{Y(z)}{U(z)} = \frac{1}{1 - a\,z^{-1}}$$

- For a unit pulse input: $u[0] = 1, \;\; \& \;\; u[k] = 0, k \neq 0$

$$\Rightarrow U(z) = 1$$

$$\Rightarrow Y(z) = \frac{1}{1 - a\,z^{-1}}$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

$$= y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + \cdots$$

Franklin et al. 2002

- The z-transform inversion:
  - The z-transform table

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Oppenheim et al. 1997

Laplace Transforms and z-Transforms of Simple Discrete Time Functions

F(s) is the Laplace transform of $f(t)$, and $F(z)$ is the z-transform of $f(kT)$. *Note:* $f(t) = 0$ for $t = 0$.

No.	F(s)	$f(kT)$	$F(z)$
1		$1, k = 0; 0, k \neq 0$	1
2		$1, k = k_o; 0, k \neq k_o$	z^{-k_o}
3	$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(kT)^2$	$\frac{T^2}{2}\left[\frac{z(z+1)}{(z-1)^3}\right]$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(kT)^3$	$\frac{T^3}{6}\left[\frac{z(z^2+4z+1)}{(z-1)^4}\right]$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$
8	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(kT)^2e^{-akT}$	$\frac{T^2}{2}e^{-aT}z\frac{(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-akT}\right)$	$\frac{(-1)^{m-1}}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}\right)$

Franklin et al. 2002

12	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(akT-1+e^{-akT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1-akT)e^{-akT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1-e^{-akT}(1+akT)$	$\frac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bkT}-ae^{-akT}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
18	$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
19	$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
20	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-akT}\cos bkT$	$\frac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
21	$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
22	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-akT}\left(\cos bkT+\frac{a}{b}\sin bkT\right)$	$\frac{z(Az+B)}{(z-1)[z^2-2e^{-aT}(\cos bT)z+e^{-2aT}]}$ $A = 1-e^{-aT}\cos bT-\frac{a}{b}e^{-aT}\sin bT$ $B = e^{-2aT}+\frac{a}{b}e^{-aT}\sin bT-e^{-aT}\cos bT$

Franklin et al. 2002

Properties of the z-transform:

1. Definition:

$$F(z) = \sum_{k=0}^{\infty} f[k] z^{-k}$$

2. Inversion:

$$f[k] = \frac{1}{2\pi i} \oint F(z) z^{k-1} dz$$

3. Linearity:

$$\mathcal{Z}\{af + bg\} = a\mathcal{Z}\{f\} + b\mathcal{Z}\{g\}$$

4. Time shift:

$$\mathcal{Z}\{q^{-n} f\} = z^{-n} F$$

$$\mathcal{Z}\{q^{n} f\} = z^{n}(F - F_1)$$

where $F_1(z) = \sum_{j=0}^{n-1} f[j] z^{-j}$

5 Initial-value theorem:

$$f[0] = \lim_{z \to \infty} F(z)$$

6 Final-value theorem:

If $(1 - z^{-1})F(z)$ does not have any poles on or outside the unit circle:

$$\lim_{k \to \infty} f[k] = \lim_{z \to 1} (1 - z^{-1}) F(z)$$

7 Convolution:

$$\mathcal{Z}\{f * g\} = (\mathcal{Z}f)(\mathcal{Z}g)$$

$$f * g = \sum_{n=0}^{k} f[n] g[k-n]$$

Astrom & Wittenmark 1997

Properties of the z-transform:

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \to \infty} X(z)$			

Oppenheim et al. 1997

Relationship between s and z:

$$f(t) = e^{-at}, \quad t > 0 \quad \Rightarrow \quad F(s) = \frac{1}{s+a}$$

$$\Rightarrow \quad \text{Pole: } s = -a$$

$$f[kT] = e^{-akT}, \quad k \in N \quad \Rightarrow \quad F(z) = \mathcal{Z}\left\{e^{-akT}\right\}$$

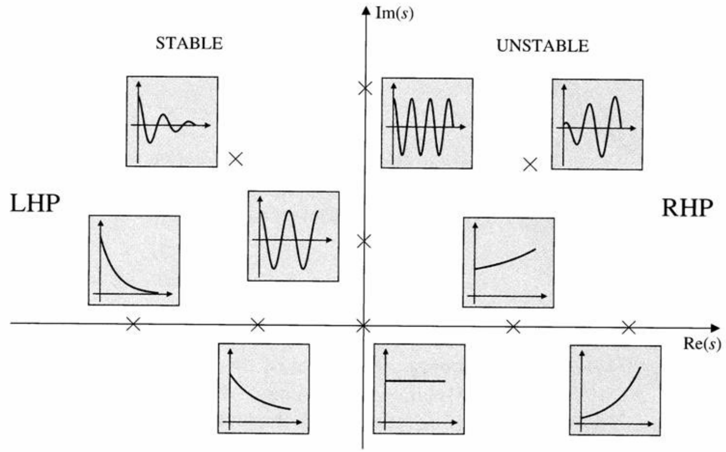
$$= \frac{z}{z - e^{-aT}}$$

$$\Rightarrow \quad \text{Pole: } z = e^{-aT}$$

$$\Rightarrow \quad z = e^{sT}$$

Franklin et al. 2002

Relationship between s and z:



Franklin et al. 2002

▪ Relationship between s and z:

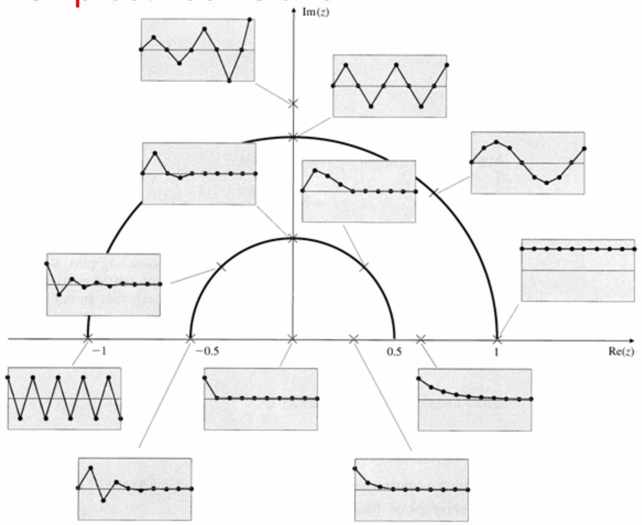
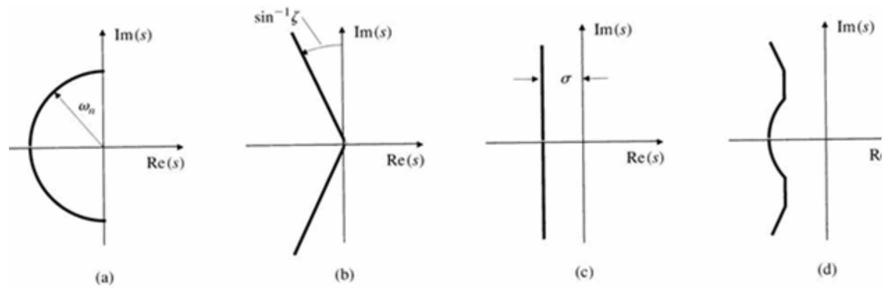


Figure 8.5 Time sequences associated with points in the z-plane

Franklin et al. 2002

▪ Relationship between s and z:



Franklin et al. 2002

▪ Relationship between s and z:

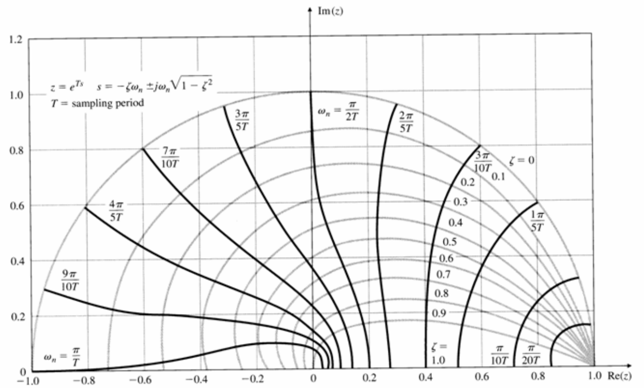
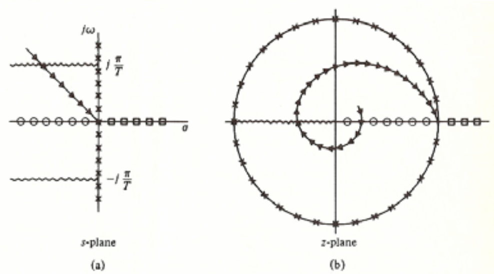


Figure 8.4 Natural frequency (solid color) and damping loci (light color) in the z-plane; the portion below the Re(z)-axis (not shown) is the mirror image of the upper half shown.

Franklin et al. 2002

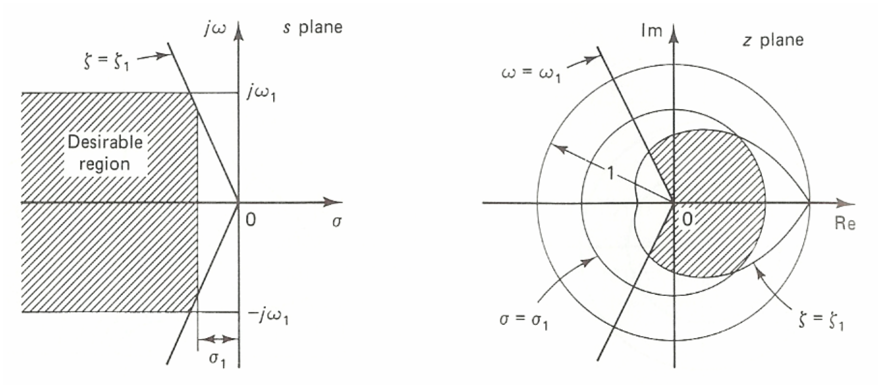
Description of corresponding lines in s-plane and z-plane

s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z\rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r, 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}.$ $= -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma,$	$\sigma \leq 0$	$z = -r$



Franklin et al. 2002

▪ Relationship between s and z:



Ogata 1995

▪ Final Value Theorem:

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} s\,X(s)$$

If all the poles of $s\,X(s)$ are in LHP

$$\lim_{k\to\infty} x[k] = x_{ss} = \lim_{z\to 1} (1 - z^{-1})\,X(z)$$

If all the poles of $(1 - z^{-1})\,X(z)$ are inside the unit circle

Franklin et al. 2002