SPRING 2010

即時控制系統設計
Design of Real-Time Control Systems

Lecture 22
Sampling

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NTU-EE

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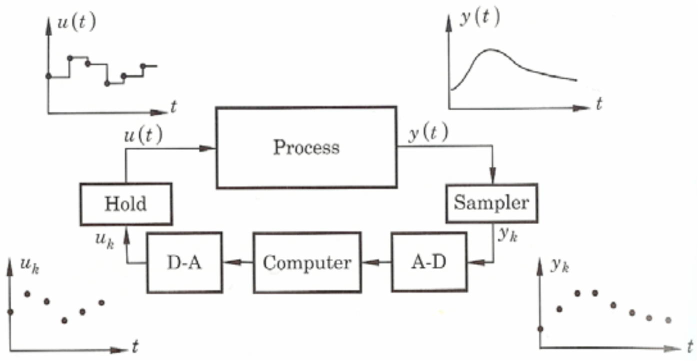
Figures and images used in these lecture notes are adopted from
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997
“Computer-Controlled Systems: Theory & Design” 3rd Ed. by KJ Astrom & B. Wittenmark, 1997

Outline

- Representation of of a CT Signal by Its Samples: The Sampling Theorem
- Reconstruction of of a Signal from Its Samples Using Interpolation
- The Effect of Under-sampling: Aliasing
- Discrete-Time Processing of Continuous-Time Signals

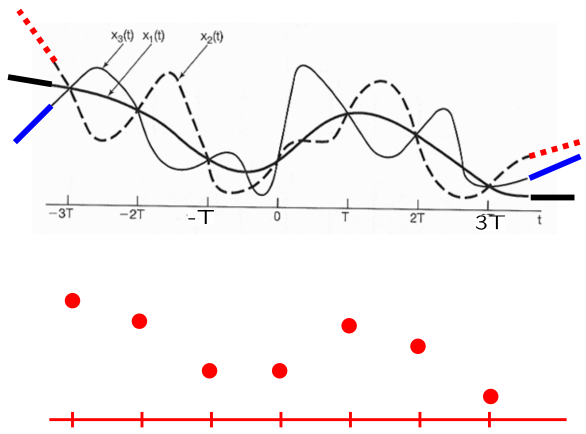
Computer-Controlled System (7.2)



1. Wait for a clock pulse
2. Perform A/D conversion
3. Compute control variable
4. Perform D/A conversion
5. Update regulator state
6. Go to step 1

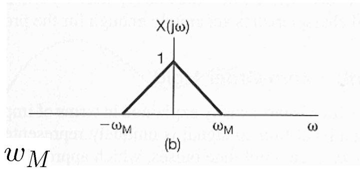
The Sampling Theorem

- Representation of CT Signals by its Samples



$$x_1(kT) = x_2(kT) = x_3(kT)$$

▪ The Sampling Theorem:

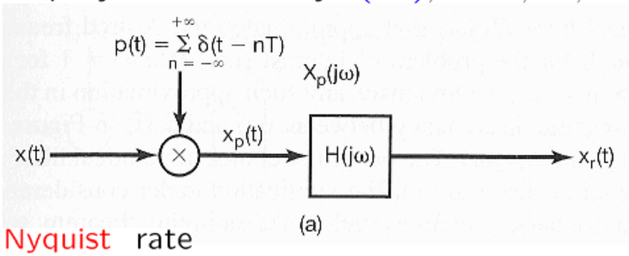


$x(t)$: a band-limited signal

with $X(jw) = 0$ for $|w| > w_M$

if $w_s > 2w_M$ where $w_s = \dfrac{2\pi}{T}$

$\Rightarrow$ $x(t)$ is uniquely determined by $x(nT), n = 0, \pm1, \pm2, ...,$



$\Rightarrow$ $2w_M$: Nyquist rate

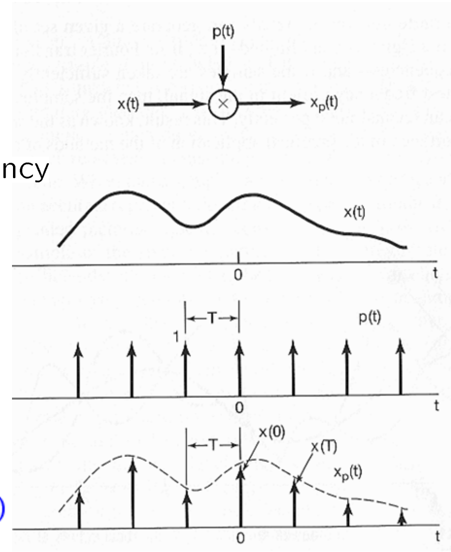
w_M : Nyquist frequency

▪ Impulse-Train Sampling:

$p(t)$: sampling function

T : sampling period

$w_s = \dfrac{2\pi}{T}$: sampling frequency

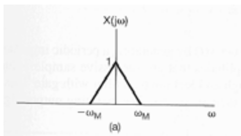


$\Rightarrow$ $x_p(t) = x(t)\, p(t)$

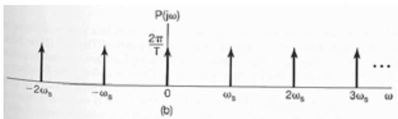
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

▪ Impulse-Train Sampling:



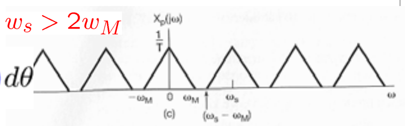
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(jw)$$



$$P(jw) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(w - kw_s)$$

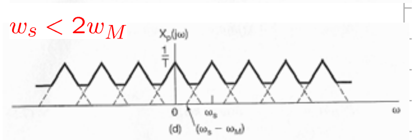
From multiplication property,

$w_s > 2w_M$



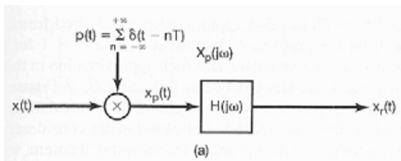
$$X_p(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(w-\theta))d\theta$$

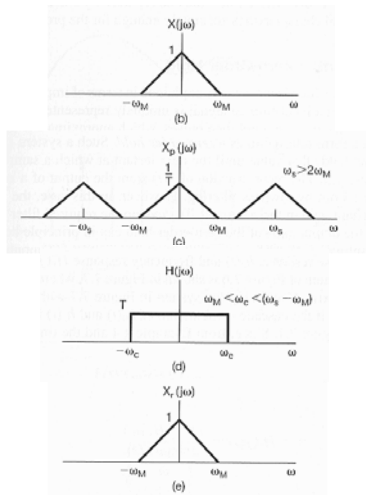
$w_s < 2w_M$



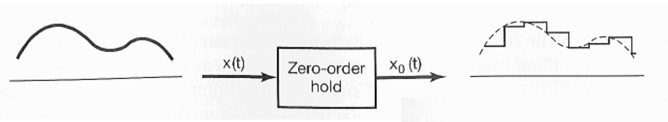
$$= \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(w - kw_s))$$

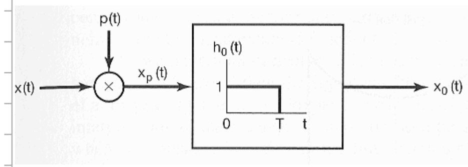
▪ Exact Recovery by an Ideal Lowpass Filter:

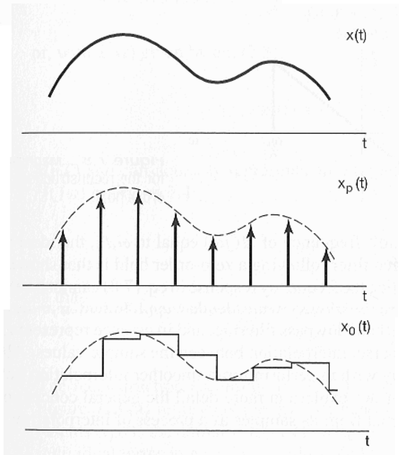




- Sampling with Zero-Order Hold:







- Representation of of a CT Signal by Its Samples: The Sampling Theorem
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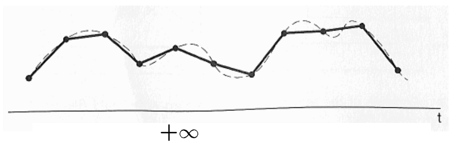
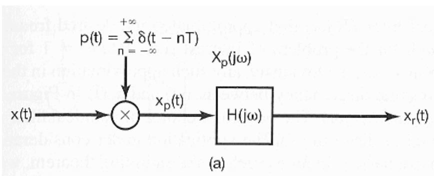
- Exact Interpolation:





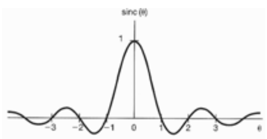
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

ideal lowpass filter

$$x_r(t) = x_p(t) * h(t)$$

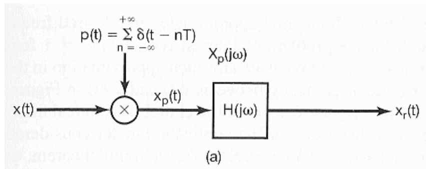
$$h(t) = \frac{w_c T \sin(w_c t)}{\pi w_c t}$$

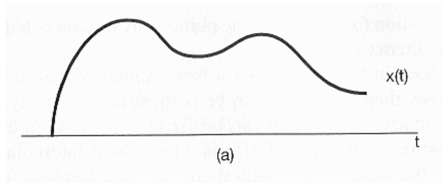
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) h(t - nT)$$



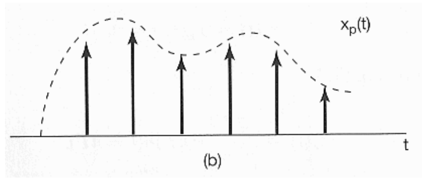
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$

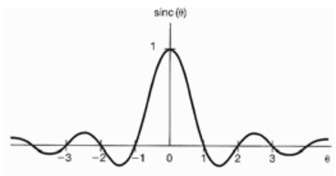
- Exact Interpolation:

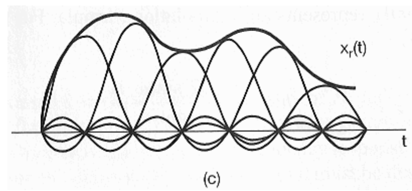




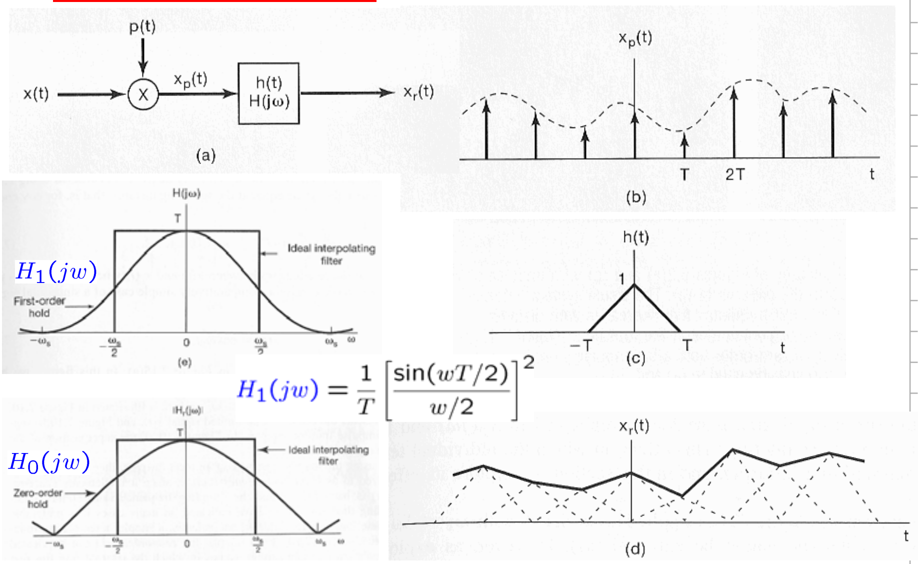
$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin(w_c(t - nT))}{w_c(t - nT)}$$







- Higher-Order Holds:



$$H_1(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$

- Representation of of a CT Signal by Its Samples: The Sampling Theorem
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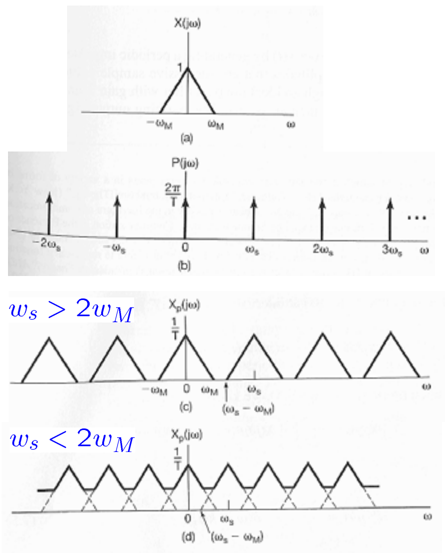
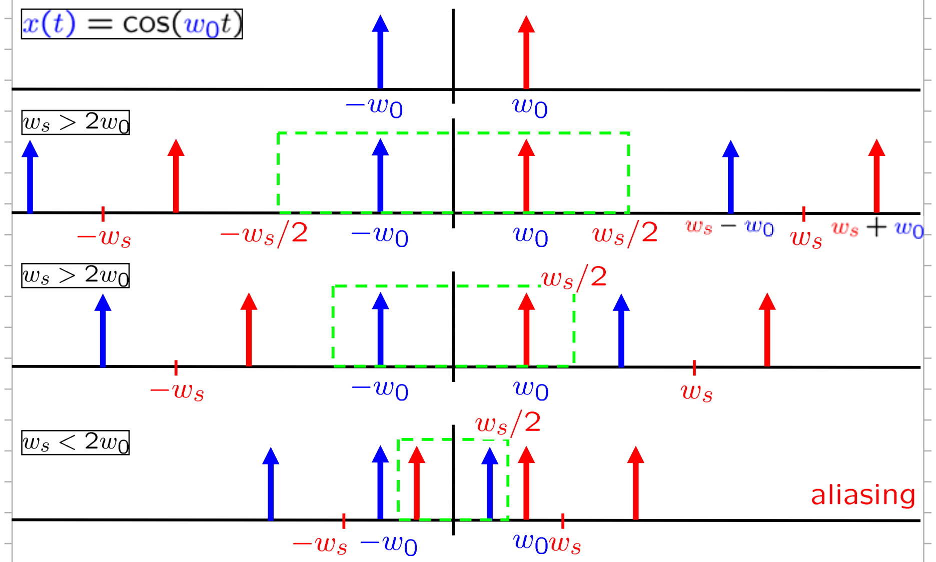
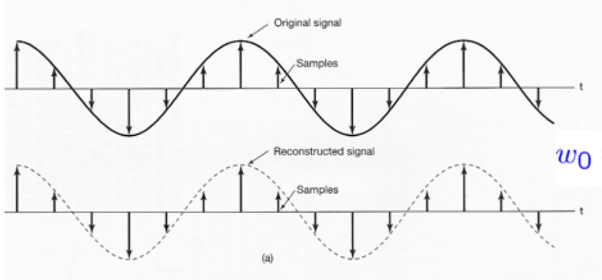
- Overlapping in Frequency-Domain: Aliasing



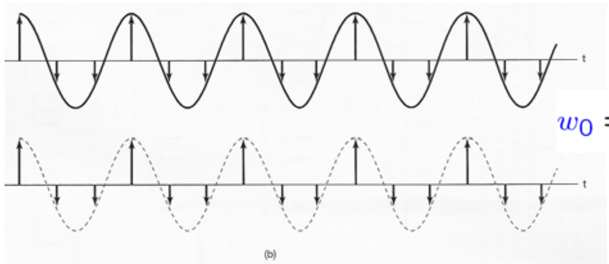
- Overlapping in Frequency-Domain: Aliasing



Overlapping in Frequency-Domain: Aliasing

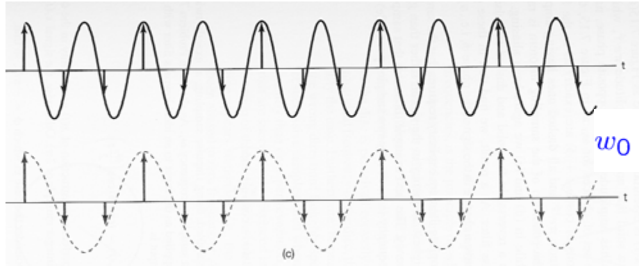


$$w_0 = \frac{w_s}{6}$$

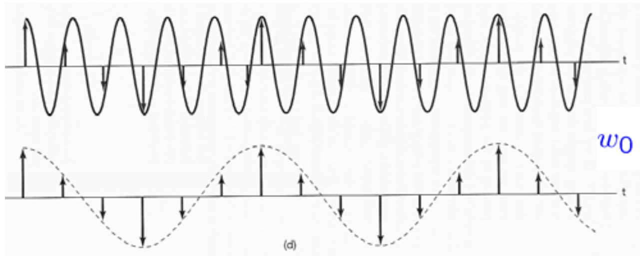


$$w_0 = \frac{2w_s}{6}$$

Overlapping in Frequency-Domain: Aliasing

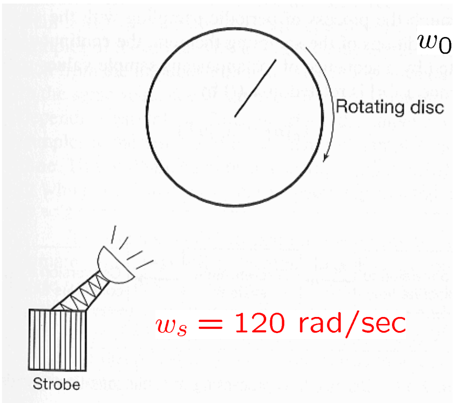


$$w_0 = \frac{4w_s}{6}$$



$$w_0 = \frac{5w_s}{6}$$

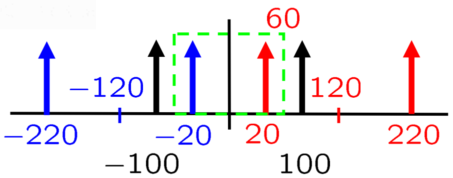
Strobe Effect:



$w_0 = 100$ rad/sec

$w_s = 120$ rad/sec

$$\Rightarrow w = \pm w_s \pm w_0$$
$$= +20, -20$$



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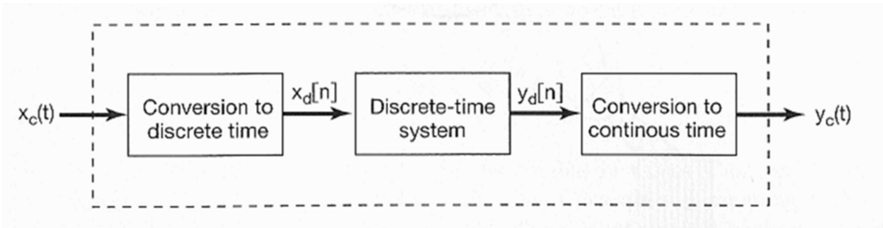
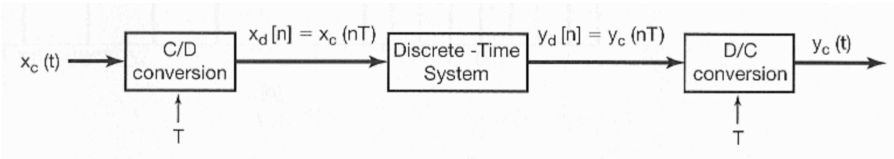
Discrete-Time Processing of CT Signals:



C/D or A-to-D (ADC) and D/C or D-to-A (DAC):



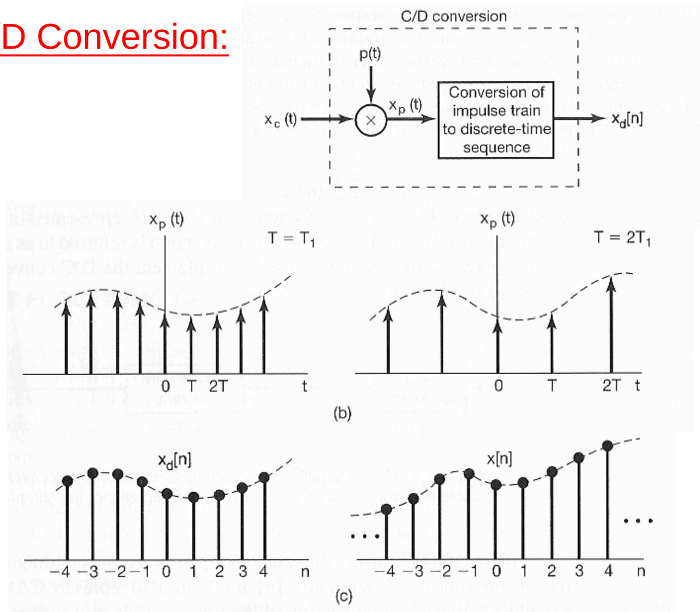
C/D: continous-to-discrete-time conversion

A-to-D: analog-to-digital converter

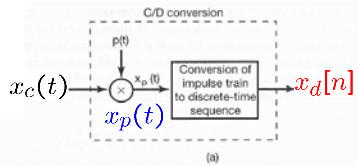
D/C: discrete-to-continous-time conversion

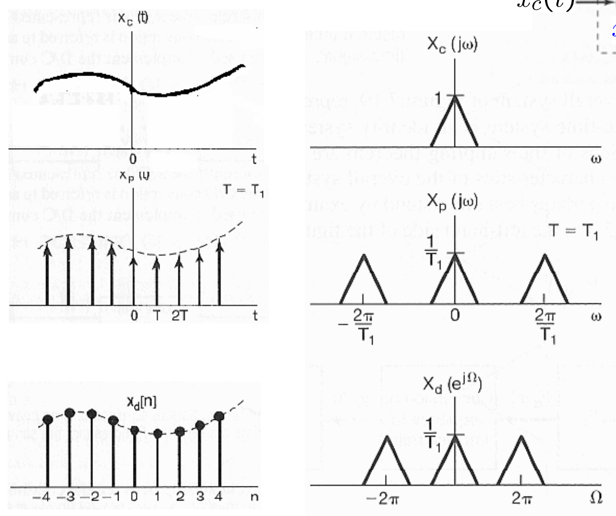
D-to-A: digital-to-analog converter

C/D Conversion:



C/D Conversion:



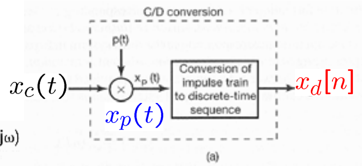


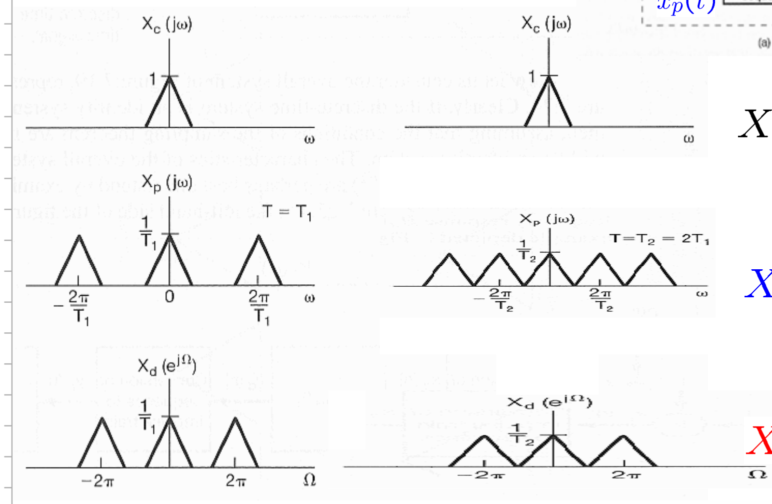
$X_c(jw)$

$X_p(jw)$

$X_d(e^{j\Omega})$

C/D Conversion:



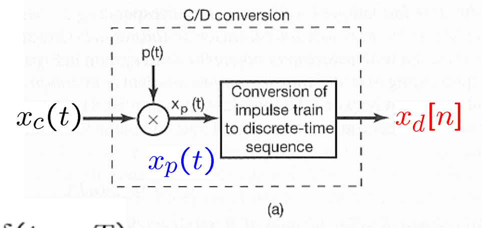


$X_c(jw)$

$X_p(jw)$

$X_d(e^{j\Omega})$

C/D Conversion:



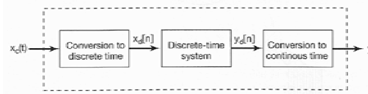
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT)$$

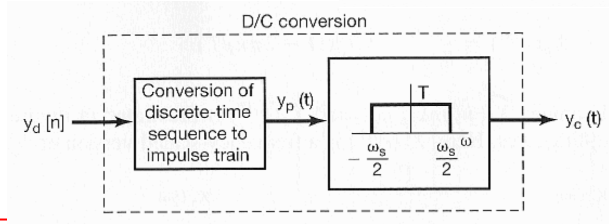
$$X_p(jw) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-jwnT} = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c(j(w-kw_s))$$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

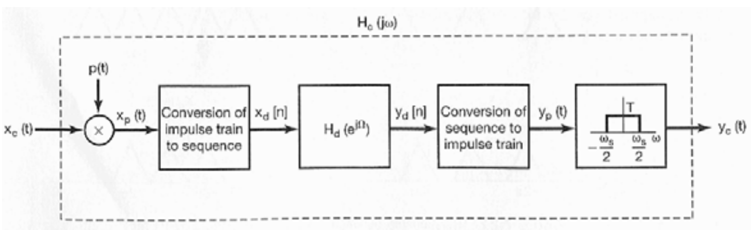
$$\Rightarrow X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) = \frac{1}{T}\sum_{K=-\infty}^{+\infty} X_c\left(j\left(\frac{\Omega}{T}-k\frac{2\pi}{T}\right)\right)$$

D/C Conversion:

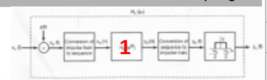


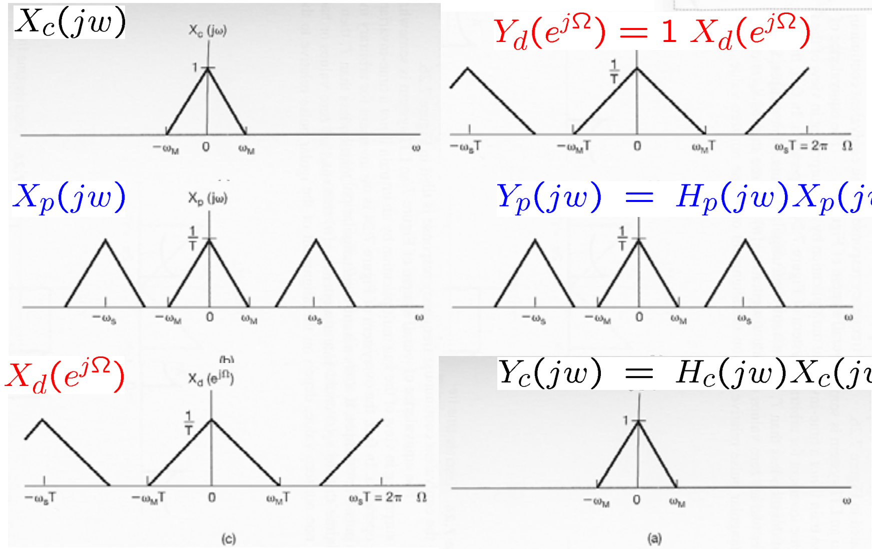


Overall



Frequency-Domain Illustration:





$X_c(jw)$

$Y_d(e^{j\Omega}) = 1\ X_d(e^{j\Omega})$

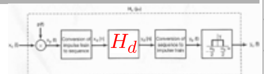
$X_p(jw)$

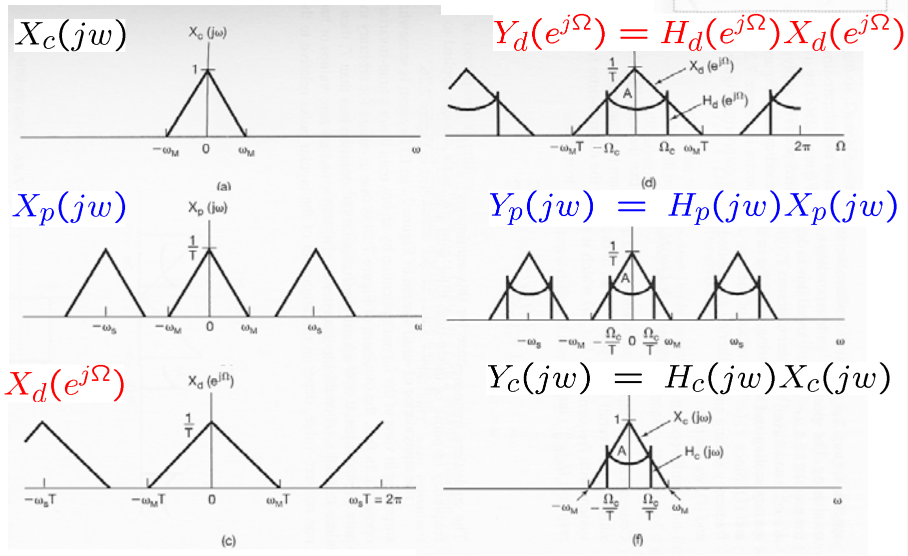
$Y_p(jw) = H_p(jw)X_p(jw)$

$X_d(e^{j\Omega})$

$Y_c(jw) = H_c(jw)X_c(jw)$

▪ Frequency-Domain Illustration:





$X_c(jw)$

$Y_d(e^{j\Omega}) = H_d(e^{j\Omega})X_d(e^{j\Omega})$

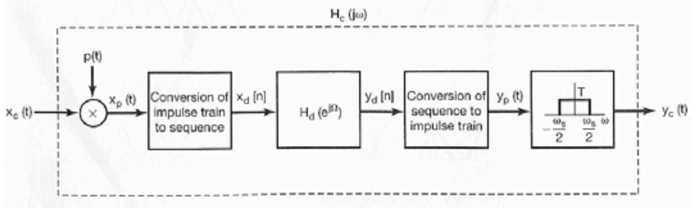
$X_p(jw)$

$Y_p(jw) = H_p(jw)X_p(jw)$

$X_d(e^{j\Omega})$

$Y_c(jw) = H_c(jw)X_c(jw)$

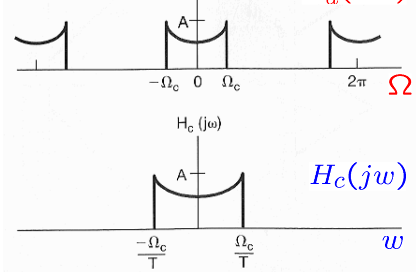
▪ CT & DT Frequency Responses:



$\Omega = wT$

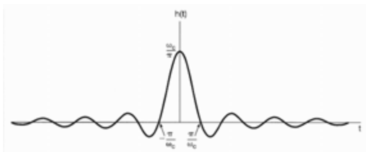
$Y_c(jw) = X_c(jw)H_d(e^{jwT})$

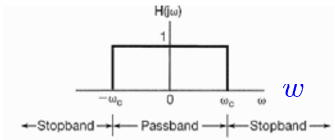
$$H_c(jw) = \begin{cases} H_d(e^{jwT}), & |w| < w_s/2 \\ 0, & |w| > w_s/2 \end{cases}$$



$H_d(e^{j\Omega})$

$H_c(jw)$

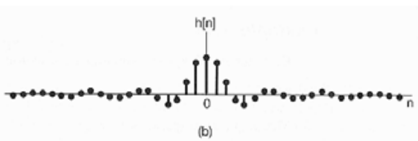


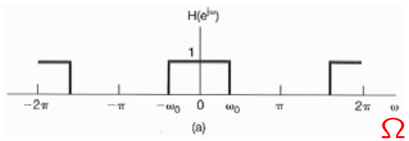


$h(t) \overset{C.T.F.T.}{\longleftrightarrow} H(jw)$

$w_s = \dfrac{2\pi}{T}$ $\qquad \Omega = wT$

$h[n] \overset{D.T.F.T.}{\longleftrightarrow} H(e^{j\Omega})$

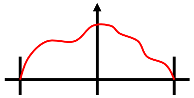




▪ The Sampling Theorem:

- If the sampling instants are sufficiently close, very little is lost by sampling a CT signal
- If the sampling points are too far apart, much of the information about a signal can be lost
- So, when a CT signal can be uniquely given by its sampled version?

Theorem 7.1: (Shannon, 1949)



- $f(t)$: a continuous-time signal
- $F(w)$: the Fourier transform of $f(t)$

  $\to F(w) = 0$ outside $(-w_0, w_0)$
- w_s: sampling frequency

$\Rightarrow$ If $w_s > 2w_0$

Then $f(t)$ can be computed by:

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t-kh)/2)}{w_s(t-kh)/2}$$

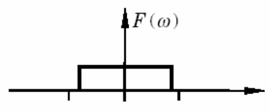
$\text{sinc}\frac{w_s(t-kh)}{2}$

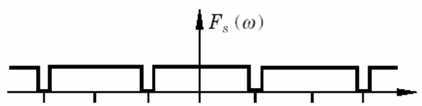
Note that:

- $w_N = w_s/2$: Nyquist freqency
- Reconstruction of signlas:

  $F(w) = 0$ when $w > w_N$

Reconstruction:





- $F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t) dt$
- $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} F(w) dw$
- $F_s(w) = \frac{1}{h} \sum_{k=-\infty}^{\infty} F(w + kw_s)$

  $= \sum_{k=-\infty}^{\infty} C_k e^{-ikhw}$ $\qquad C_k = \frac{1}{w_s} \int_0^{w_s} e^{ikhw} F_s(w) dw$

  $= \sum_{k=-\infty}^{\infty} f(kh) e^{-ikhw}$
- $F(w) = \begin{cases} hF_s(w) & |w| \le \frac{w_s}{2} \\ 0 & |w| > \frac{w_s}{2} \end{cases}$

Shannon Reconstruction:

- For periodic sampling of band-limited signals

$$f(t) = \sum_{k=-\infty}^{\infty} f(kh) \frac{\sin(w_s(t-kh)/2)}{w_s(t-kh)/2}$$

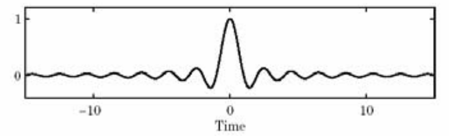
- However, it is NOT a caucal operator

Shannon Reconstruction:

- Let's look at the impulse response:

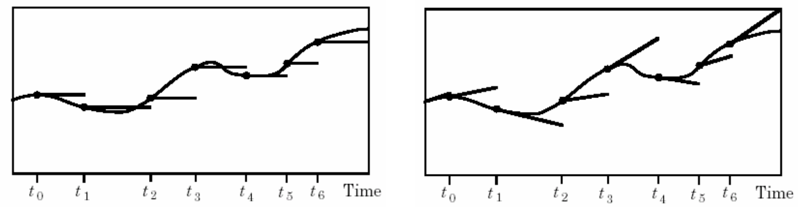
$$h(t) = \frac{\sin(w_s t/2)}{w_s t/2}$$

$h = 1$



- The weights are 10% after 3 samples
  $< 5\%$ after 6 samples
- This construction has a delay
  $\Rightarrow$ Not good for control
- Only applied to periodic sampling

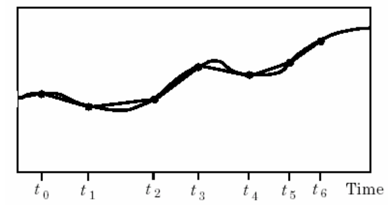
Zero-Order Hold (ZOH) & First-Order Hold (FOH)



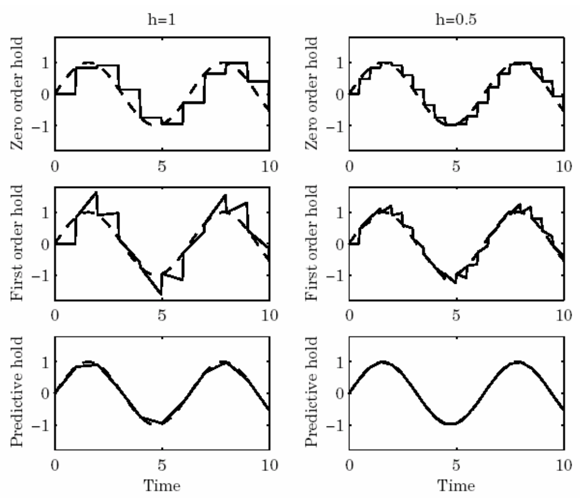
- They are caucal operators

Predictive FOH:

- It is NOT caucal
  But, can be replaced by model prediction

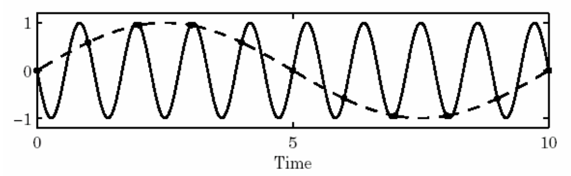


Sinusoidal signal with h = 1 and h = 0.5



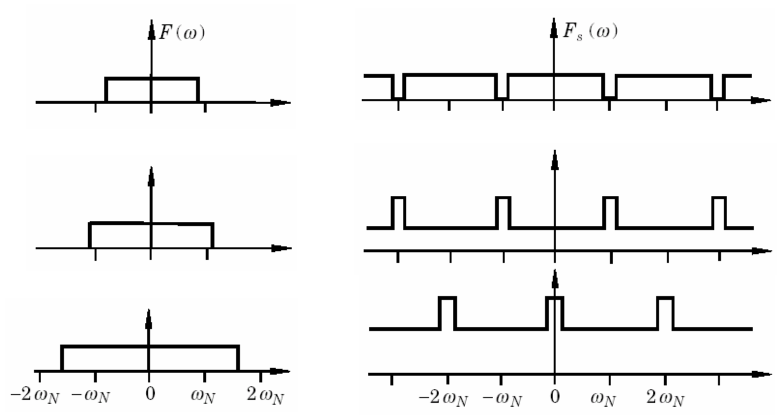
Aliasing:

- Two signals with frequency, 0.1 Hz and 0.9 Hz
- They have the same values at all sampling instants

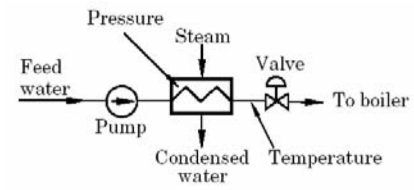


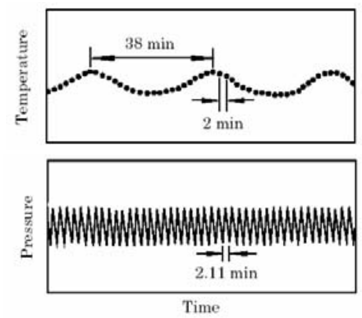
Fourier transform of sampled signal:

- $F(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)dt$
- $F_s(w) = \sum_{k=-\infty}^{\infty} f(kh) e^{-ikhw}$



Example 7.1: Feed-water heater in a ship boiler

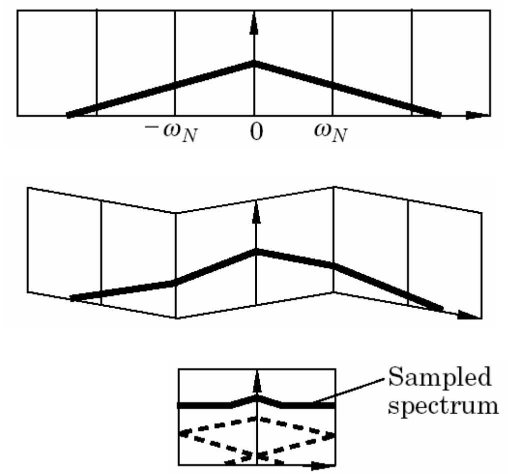




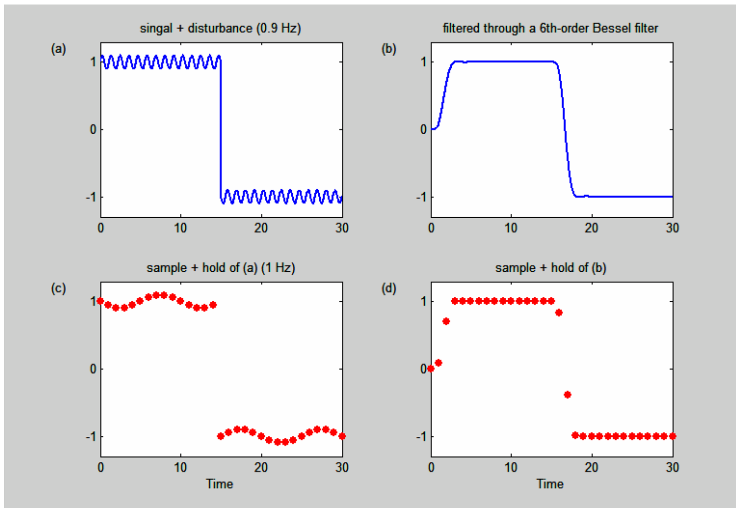
- $w_s = \frac{2\pi}{2} = 3.142$ rad/min
- $w_0 = \frac{2\pi}{2.11} = 2.978$ rad/min
- $w_s - w_0 = 0.1638$ rad/min

$\Rightarrow T_s = 38$ min

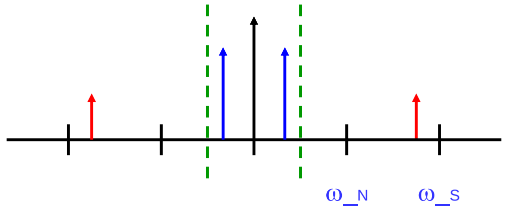
Frequency Folding



Pre-Sampling Filter in Example 7.2:

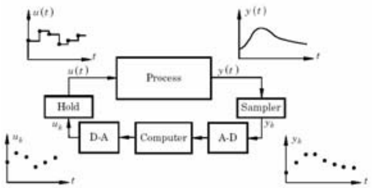


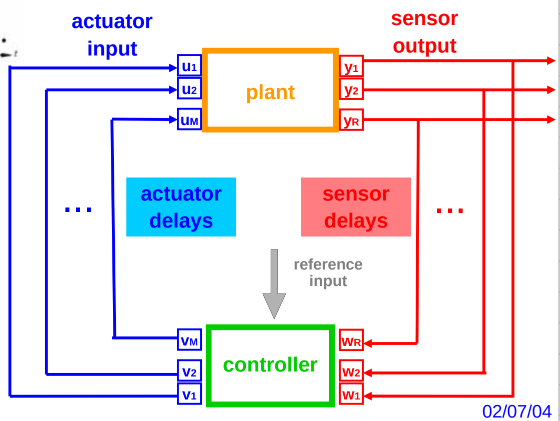
- Pre-Sampling Filter in Example 7.2:
  - With a sinusoidal perturbation (0.9Hz)
  - Sampling frequency = 1 Hz

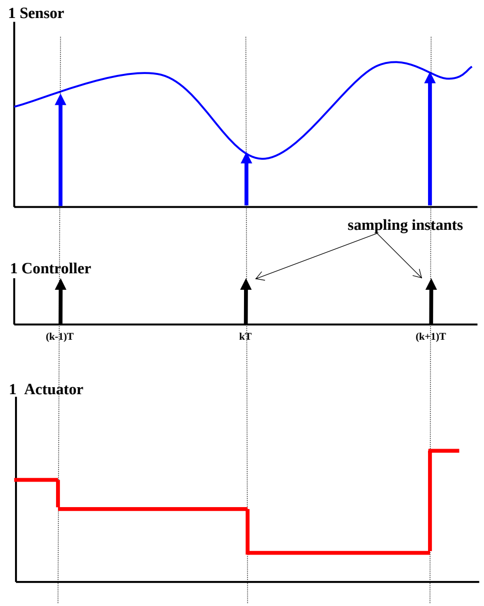


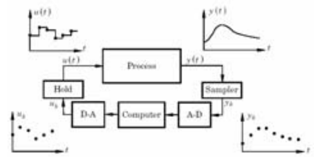
- Post-Sampling Filter:
  - Because signal from D/A is piecewise constant
    - May excite some oscillatory modes
    - So, use higher-order hold!
      such as piecewise linear signal

- Real-Time Control Systems:
  - Computing, Communication, and Control

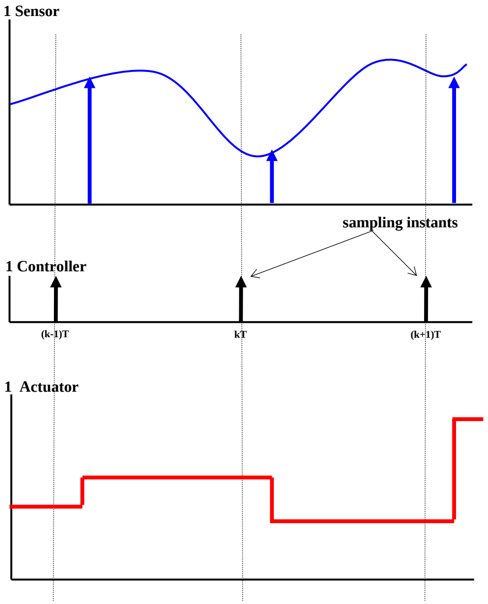


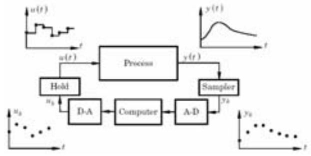




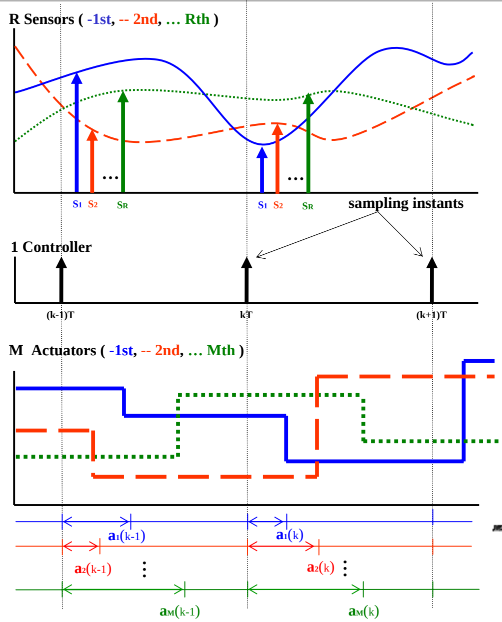


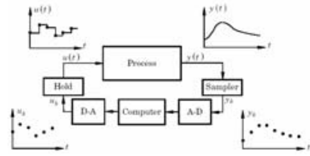
Timing Analysis

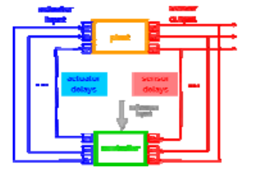


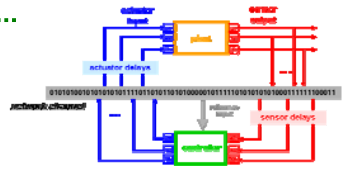


Timing Analysis



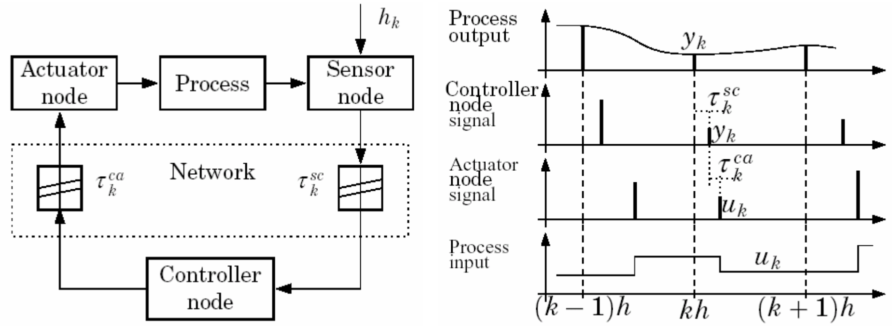






Further Readings

- Multiple time-delay systems
- Random time delays

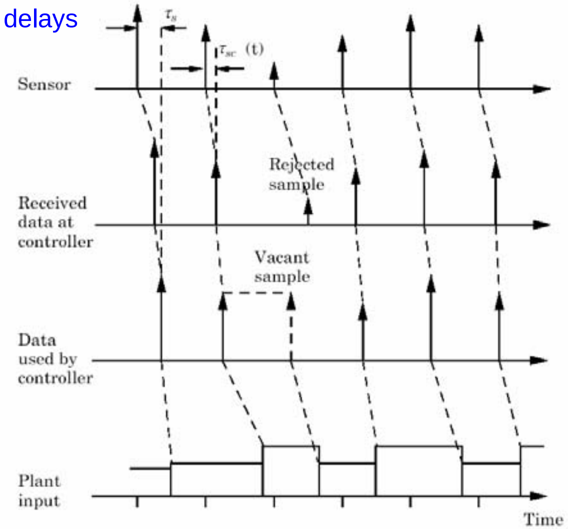


$$x_{k+1} = \Phi x_k + \Gamma_0(\tau_k^{sc}, \tau_k^{ca})u_k + \Gamma_1(\tau_k^{sc}, \tau_k^{ca})u_{k-1} + v_k$$

J. Nilsson, B. Bernhardsson, B. Wittenmark, Lund Institute of Technology

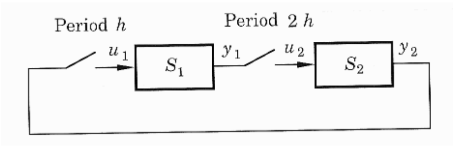
Further Readings

- Multiple time-delay systems
- Random time delays

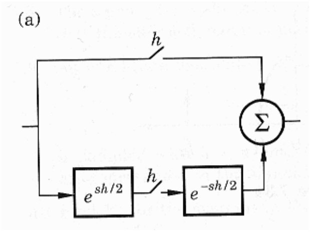


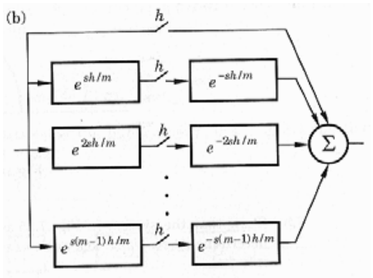
B. Wittenmark, J. Nilsson, M. Torngren, “Timing problems in real-time control systems,” ACC1995

- Multi-rate System:



- Switch Decomposition:





- Multi-rate systems

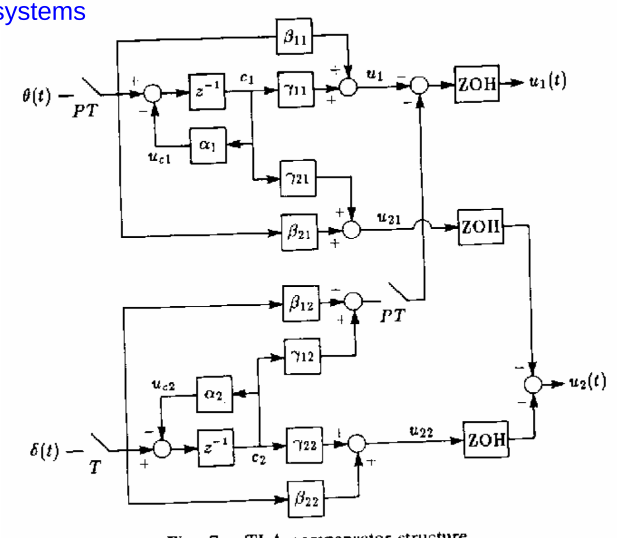


Fig. 7. TLA compensator structure.

M.C. Berg, N. Amit, J.D. Powell, "Multirate digital control system design", IEEE-TAC 33(12): 1139-1150, Dec 1988