

Nonlinear Systems Analysis

Lecture Note

Section 8.3

Invariance-like Theorems (Advanced Stability Analysis)

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Outline

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- The Center Manifold Theorem (8.1)
- Region of Attraction (8.2)
- **Invariance-Like Theorems (8.3)**
- Stability of Periodic Systems (8.4)

- For **autonomous systems**
LaSalle's invariance theorem shows that

$$\text{trajectory} \longrightarrow E = \{\dot{V}(x) = 0\}$$

- For **non-autonomous systems**

$$\{\dot{V}(t, x)\}$$

- If it can be shown that

$$\dot{V}(t, x) \leq -W(x) \leq 0$$

$$E = \{W(x) = 0\}$$

$$\text{trajectory} \longrightarrow E \quad \text{as } t \rightarrow \infty$$

Barbalat's Lemma

- **Lemma 8.2**
- Let $\phi : R \rightarrow R$ be a **unif. cont. func.** on $[0, \infty)$.

- Suppose that

$$\lim_{t \rightarrow \infty} \int_0^t \phi(\tau) d\tau$$

exists and is finite.

- Then,

$$\phi(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

- **Proof:**
- If is **not true**,
then there is a positive constant k_1
such that for every $T > 0$,
we can find $T_1 \geq T$ with $|\phi(T_1)| \geq k_1$.
- Since $\phi(t)$ is **unif. cont.**,
there is a positive constant k_2
such that $|\phi(t + \tau) - \phi(t)| < k_1/2$
for all $t \geq 0$ and all $0 \leq \tau \leq k_2$.

- Hence,

$$\begin{aligned}
 |\phi(t)| &= |\phi(t) - \phi(T_1) + \phi(T_1)| \\
 &\geq |\phi(T_1)| - |\phi(t) - \phi(T_1)| \\
 &> k_1 - \frac{1}{2}k_1 \\
 &= \frac{1}{2}k_1, \forall t \in [T_1, T_1 + k_2]
 \end{aligned}$$

- Since $\phi(t)$ retains the **same sign**
for $T_1 \leq t \leq T_1 + k_2$, then

$$\left| \int_{T_1}^{T_1+k_2} \phi(t) dt \right| = \int_{T_1}^{T_1+k_2} |\phi(t)| dt > \frac{1}{2}k_1 k_2$$

- Thus $\int_0^t \phi(\tau) d\tau$ **cannot** converge to
a **finite** limit as $t \rightarrow \infty$,
a **contradiction**.

Theorem 8.4

- **Theorem 8.4**
- Let $D \subset R^n$ be a domain containing $x = 0$ and suppose $f(t, x)$ is piecewise cont. in t and locally Lipschitz in x , uniformly in t , on $[0, \infty) \times D$.
- Furthermore, $\forall t \geq 0$, suppose $f(t, 0)$ is unif. bdd.

Theorem 8.4

- Let $V : [0, \infty) \times D \rightarrow R$ be a cont. diff. func such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x)$$

$\forall t \geq 0, \forall x \in D$,
where $W_1(x), W_2(x)$ are cont. P.D. func. and $W(x)$ is cont. P.S.D. func. on D .
- Choose $r > 0$ such that $B_r \subset D$ and let $\rho < \min_{\|x\|=r} W_1(x)$.

- Then, with $x(t_0) \in \{x \in B_r \mid W_2(x) \leq \rho\}$ all sol. of $\dot{x} = f(t, x)$ are **bdd** and satisfy

$$W(x(t)) \rightarrow 0 \text{ as } t \rightarrow \infty$$

- Moreover, if all the assumptions hold **globally** and $W_1(x)$ is **radially unbounded**, the statement is true for **all** $x(t_0) \in R^n$.

Theorem 8.4: Part of Proof

- **Part of Proof:**
- Since $V(t, x(t))$ is **monotonically nonincreasing** and **bounded from below by zero**, it **converges** as $t \rightarrow \infty$.

- So, for $W(x)$,

$$\begin{aligned} \int_{t_0}^t W(x(\tau)) d\tau &\leq - \int_{t_0}^t \dot{V}(\tau, x(\tau)) d\tau \\ &= V(t_0, x(t_0)) - V(t, x(t)) \end{aligned}$$

- Therefore, $\lim_{t \rightarrow \infty} \int_{t_0}^t W(x(\tau)) d\tau$ **exists** and is **finite**.

- Since $x(t)$ is **bdd.**, $\forall t \geq t_0$
 $\dot{x} = f(t, x(t))$ is **bdd., uniformly in t .**
 - Hence, $x(t)$ is **unif. cont. in t** on $[t_0, \infty)$.
 - So, because $W(x)$ is **unif. cont. in x**
on the **compact set B_r ,**
consequently,
 $W(x(t))$ is **unif. cont. in t** on $[t_0, \infty)$.
 - Therefore, by Lemma 8.2,
 $W(x(t)) \rightarrow 0$ as $t \rightarrow \infty$.
 - The limit $W(x(t)) \rightarrow 0$ implies that
 $x(t)$ **approaches E** as $t \rightarrow \infty$,
where
 - Therefore, the **positive limit set** of $x(t)$
is a subset of E .
- $$E = \{x \in D \mid W(x) = 0\}$$

Theorem 8.5: U.A.S.

- **Theorem 8.5**
- Let $D \subset \mathbb{R}^n$ be a domain containing
 $x = 0$
and suppose $f(t, x)$ is **piecewise cont.**
in t
and **locally Lipschitz in x**
for all $t \geq 0$ and $x \in D$.
- Let $x = 0$ be an **E.P.** for $\dot{x} = f(t, x)$
at $t = 0$.

- Let $V : [0, \infty) \times D \rightarrow R$ be a **cont. diff.** func such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

$$V(t + \delta, \phi(t + \delta; t, x)) - V(t, x) \leq -\lambda V(t, x), \quad 0 < \lambda < 1$$

$\forall t \geq 0, \forall x \in D$, for some $\delta > 0$,

where

$W_1(x), W_2(x)$ are **cont. P.D.** func.

on D

and $\phi(\tau; t, x)$ is the sol. of the system starts at (t, x) .

- Then, the origin is **U.A.S.**

- If all assumptions hold **globally** and $W_1(x)$ is **radially unbounded**, then the origin is **G.U.A.S.**

- If

$$W_1(x) \geq k_1 \|x\|^c, \quad W_2(x) \leq k_2 \|x\|^c,$$

where $k_1, k_2, c > 0$,

then the origin is **E.S.**

- **Example 8.11**
- Consider the LTV system

$$\dot{x} = A(t)x$$

where $A(t)$ is cont. for all $t \geq 0$.

- Suppose that there is a **cont. diff. symm.** $P(t)$ that satisfies

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0$$

as well as matrix diff. eq.

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t)$$

where $C(t)$ is cont. in t .

- The derivative of the quadratic func.

$$V(t, x) = x^T P(t)x$$

along the traj. of the system is

$$\dot{V} = -x^T C^T(t)C(t)x \leq 0$$

- Let the sol. be $\phi(\tau; t, x) = \Phi(\tau, t)x$,
where $\Phi(\tau, t)$ is state transition matrix.

- Therefore,

$$\begin{aligned} V(t + \delta, \phi(t + \delta; t, x)) - V(t, x) &= \int_t^{t+\delta} \dot{V}(\tau, \phi(\tau; t, x)) d\tau \\ &= -x^T \int_t^{t+\delta} \Phi^T(\tau, t) C^T C \Phi(\tau, t) d\tau \\ &= -x^T W(t, t + \delta) x \end{aligned}$$

$$\text{where } W(t, t + \delta) = \int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau$$

- Suppose there is **positive** a constant $k \leq c_2$ such that

$$W(t, t + \delta) \geq kI, \forall t \geq 0$$

then

$$V(t + \delta, \phi(t + \delta; t, x)) - V(t, x) \leq -k\|x\|_2^2 \leq -\frac{k}{c_2}V(t, x)$$

- Thus, all assumptions of Thm 8.5 are satisfied globally with

$$W_i(x) = c_i\|x\|_2^2, i = 1, 2, \lambda = \frac{k}{c_2} < 1$$

- Then, $x = 0$ is **G.E.S.**
- Note that $W(t, t + \delta)$ is the **observability Gramian** of $(A(t), C(t))$ and $W(t, t + \delta) \geq kI$ is implied by **uniform observability** of $(A(t), C(t))$.

Example 8.12: Adaptive Control

- **Example 8.12** (from Sec 1.2.6)
- Model reference adaptive control:

$$\text{plant model: } \dot{y}_p = a_p y_p + k_p u$$

$$\text{reference model: } \dot{y}_m = a_m y_m + k_m r$$

- If $\gamma > 0$ is the **adaptation gain**, $e_o = y_p - y_m$ is the **output error**, and ϕ_1, ϕ_2 are the **parameter errors**.

- The closed-loop eq.

$$\dot{e}_o = a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)]$$

$$\dot{\phi}_1 = -\gamma e_o r(t)$$

$$\dot{\phi}_2 = -\gamma e_o [e_o + y_m(t)]$$

- Assume that $k_p > 0$, $a_m < 0$
- $r(t)$ is **piecewise cont.** and **bdd.**
- Using

$$V = \frac{1}{2} \left[\frac{e_o^2}{k_p} + \frac{1}{\gamma} (\phi_1^2 + \phi_2^2) \right]$$

as a Lyapunov function candidate,

- We obtain

$$\begin{aligned} \dot{V} &= \frac{a_m}{k_p} e_o^2 + e_o(\phi_1 r + \phi_2 e_o + \phi_2 y_m) - \phi_1 e_o r - \phi_2 e_o(e_o + y_m) \\ &= \frac{a_m}{k_p} e_o^2 \leq 0 \end{aligned}$$

- By applying Thm 8.4, we conclude that
for any $c > 0$ and
for all initial states in $\{V \leq c\}$,
all state variables are bounded
for all $t \geq t_0$ and $\lim_{t \rightarrow \infty} e_o(t) = 0$.
- This shows that $y_p \rightarrow y_m$,
but it says **nothing** about $\phi_1, \phi_2 \rightarrow 0$!
- In fact, **they may not converge to zero.**
- If r, y_m are **nonzero constant** signals,
the closed-loop system will have
an **equilibrium subspace**
 $\{e_o = 0, \phi_2 = (a_m/k_p)\phi_1\}$.

- Hence, we need to apply Thm 8.5 to derive the conditions of $\phi_1, \phi_2 \rightarrow 0$. That is, the conditions which the origin $(e_o = 0, \phi_1 = 0, \phi_2 = 0)$ is **U.A.S.**

- Reformulate the system as:

$$\dot{x} = \begin{bmatrix} a_m & k_p r(t) & k_p y_p(t) \\ -\gamma r(t) & 0 & 0 \\ -\gamma y_p(t) & 0 & 0 \end{bmatrix} x,$$

where $x = \begin{bmatrix} e_o \\ \phi_1 \\ \phi_2 \end{bmatrix}$

- Suppose $\lim_{t \rightarrow \infty} [r(t) - r_{ss}(t)] = 0$,
Then, $\lim_{t \rightarrow \infty} [y_m(t) - y_{ss}(t)] = 0$,
Together with $\lim_{t \rightarrow \infty} e_o(t) = 0$,
the above linear system can be represented by

$$\dot{x} = [A(t) + B(t)]x$$

where

$$A(t) = \begin{bmatrix} a_m & k_p r_{ss}(t) & k_p y_{ss}(t) \\ -\gamma r_{ss}(t) & 0 & 0 \\ -\gamma y_{ss}(t) & 0 & 0 \end{bmatrix}$$

and $\lim_{t \rightarrow \infty} B(t) = 0$

- Because $\lim_{t \rightarrow \infty} B(t) = 0$,
if $\dot{x} = A(t)x$ is **U.A.S.**,
then $\dot{x} = [A(t) + B(t)]x$ is **U.A.S.**.

- Using V as a Lypunov function candidate, we obtain

$$\dot{V} = \frac{a_m}{k_p} e_o^2 = -x^T C^T C x,$$

where
$$C = \sqrt{\frac{-a_m}{k_p}} [1 \ 0 \ 0]$$

- From Example 8.11, if $(A(t), C)$ is **uniformly observable**, then the origin will be **U.A.S.**
- And, **uniform observability** of $(A(t), C)$ implies **uniform observability** of $(A(t) - K(t)C, C)$ for any **piecewise cont., bdd** $K(t)$.

- Take

$$K(t) = \sqrt{\frac{k_p}{-a_m}} [a_m \ -\gamma r_{ss}(t) \ -\gamma t_{ss}(t)]^T$$

and obtain

$$A(t) - K(t)C = \begin{bmatrix} 0 & k_p r_{ss}(t) & k_p y_{ss}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \sqrt{\frac{-a_m}{k_p}} [1 \ 0 \ 0]$$

- Hence, by investigating **observability** of this pair for a given reference signal, we can determine whether the **conditions of Thm 8.5** are satisfied.

- For example,
if r is a **nonzero constant** signal,
it can be easily seen that
the pair is **not observable**.
- On the other hand,
if $r(t) = a \sin wt$ with positive a, w ,
we have
 $r_{ss}(t) = r(t)$ and $y_{ss} = aM \sin(wt + \delta)$,
where M, δ are determined
by the transfer func. of the ref. model.

- It can be verified that
the pair is **uniformly observable**;
hence, the origin ($e_o = 0, \phi_1 = 0, \phi_2 = 0$)
is **U.A.S.** and
the parameter errors $\phi_1(t), \phi_2(t) \rightarrow 0$
as $t \rightarrow \infty$.
- Note that $r(t) = a \sin wt$ is said to be
persistently exciting,
while a **constant reference** is
not persistently exciting.

- Consider **zero-input LTV system**

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

$$y(t) = C(t)x(t)$$

and let $\phi_i(t, t_0, x_0^i)$ the associated sol.
or,

$$x(t) = \begin{bmatrix} \phi_1(t, t_0, x_0^1) \\ \vdots \\ \phi_n(t, t_0, x_0^n) \end{bmatrix} \in \mathbf{R}^{n \times 1}$$

- Note that

$$x(t) \text{ or } [\phi_i(t, t_0, x_0^i)] = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is
the state transition matrix
from t_0 to t .

- Let

$$y^i(t) = C(t)\phi_i(t, t_0, x_0^i)$$

So, over $[t_0, t_1]$

the pair $(C(t), A(t))$ is **observable**
iff $y^i(\cdot)$ are **linear indep. vector func.**

- Note that $y(t) = C(t)\Phi(t, t_0)x(t_0)$

- That is,
the columns of $C(t)\Phi(t, t_0)$ are
linear indep.

- That is, there are distinct points t_1, \dots, t_p such that

$$\text{rank} \begin{bmatrix} C(t_1)\Phi(t_1, t_0) \\ C(t_2)\Phi(t_2, t_0) \\ \vdots \\ C(t_p)\Phi(t_p, t_0) \end{bmatrix} = n$$

- For LTI systems, because

$$\begin{aligned} \Phi(t_i, t_0) &= \exp(A(t_i - t_0)) \\ &= I + A(t_i - t_0) + \frac{A^2}{2!}(t_i - t_0)^2 + \dots \end{aligned}$$

the **Observability Matrix** becomes

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$