Nonlinear Systems Analysis

Lecture Note 17

Section 4.9

Input-to-State Stability (Lyapunov Stability)

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Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Consider the system

$$\dot{x} = f(t, x, u) \quad (4.44)$$

where $f : [0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and locally Lipschitz in x and u.

- The input $u(t)$ is a piecewise continuous, bdd function of t for all $t \geq 0$.

- Suppose the unforced system

$$\dot{x} = f(t, x, 0) \quad (4.45)$$

has a G.U.A.S. E.P. at $x = 0$.

- What can we say about the behavior of the system (4.44) in the presence of a bounded input $u(t)$?

- For the L.T.I. system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A,

the solution is

$$x(t) = e^{(t-t_0)A}x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}Bu(\tau)d\tau$$

- And use the bound $||e^{(t-t_0)A}|| \leq ke^{-\lambda(t-t_0)}$

to estimate the solution by

$$\begin{aligned} ||x(t)|| &\leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}||B||\ ||u(\tau)||d\tau \\ &\leq ke^{-\lambda(t-t_0)}||x(t_0)|| + \frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \end{aligned}$$

- This estimate shows that
  the zero-input response
  decays to zero exponentially fast,
  while the zero-state response is bounded
  for every bounded input.

- In fact, the estimate shows more than
  a bounded-input-bounded-state (BIBO)
  property.

- It shows that
  the bound on the zero-state response
  is proportional to
  the bound on the input.

- For a general nonlinear system,

  it should not be surprising that

  these properties may not hold even when

  the origin of the unforced syst. is G.U.A.S.

- e.g., consider the scalar system

$$\dot{x} = -3x + (1 + 2x^2)u$$

  which has a G.E.S. origin when $u = 0$.

- When $x(0) = 2$ and $u(t) \equiv 1$,

  the solution $x(t) = \frac{(3-e^t)}{(3-2e^t)}$

  is unbounded;

  it even has a finite escape time.

- Let us view the system

  $\dot{x} = f(t, x, u)$ as

  a perturbation of the unforced syst

  $\dot{x} = f(t, x, 0)$.

- Supose we have a Lyapunov func $V(t,x)$ for the unforced system and let us calculate the derivative of V in the presence of u.

- Due to the boundedness of u, it is plausible that in some cases it should be possible to show that $\dot{V}$ is negative outside a ball of radius μ, where μ depends on $\sup ||u||$.

- This would be expected, for example, when the functon $f(t,x,u)$ satisfies the Lipschitz condition

$$||f(t,x,u) - f(t,x,0)|| \leq L||u||, \qquad (4.46)$$

- Showing that $\dot{V}$ is negative outside a ball of radius μ would enable us to apply Thm 4.18 to show that $x(t)$ satisfies (4.42), (4.43).

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \ \forall t_0 \leq t \leq t_0 + T \quad (4.42)$$

$$||x(t)|| \leq \alpha_1^{-1}(\alpha_2(\mu)), \ \forall t \geq t_0 + T \quad (4.43)$$

- These inequalities show that

  $||x(t)||$ is bdd by a class $\mathcal{KL}$ function

  $\beta(||x(t_0)||, t - t_0)$ over $[t_0, t_0 + T]$ and

  by a class $\mathcal{K}$ function $\alpha_1^{-1}(\alpha_2(\mu))$

  for $t \geq t_0 + T$.

- Consequently,

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu))$$

  is valid for all $t \geq t_0$.

- The system $\dot{x} = f(t, x, u)$ is said to be

  input-to-state stable

  if there exist a class $\mathcal{KL}$ function β

  and a class $\mathcal{K}$ function γ

  such that for any initial state $x(t_0)$

  and any bdd input $u(t)$,

  the sol. $x(t)$ exists for all $t \geq t_0$ and

  satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} ||u(\tau)||\right), \quad (4.47)$$

- Inequality (4.47) guarantees that
  for any bdd input $u(t)$,
  the state $x(t)$ will be bounded.

- Furthermore, as t increases,
  the state $x(t)$ will be ultimately bounded
  by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} ||u(t)||$.

- Since, with $u(t) \equiv 0$,
  (4.47) reduces to

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$$

input-to-state stability implies that
the origin of the unforced system (4.45)
is G.U.A.S.

$$\dot{x} = f(t, x, u) \quad (4.44)$$

$$\dot{x} = f(t, x, 0) \quad (4.45)$$

- The notion of input-to-state stability is defined for the global case where the initial state and the input can be arbitrarily large.

- A local version of this notion is presented in Ex 4.60.

- Let $V : [0, \infty) \times R^n \to R$ be a cont. diff. func. such that $\forall (t, x, u) \in [0, \infty) \times R^n \times R^m$

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||) \quad (4.48)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W_3(x), \quad \forall ||x|| \geq \rho(||u||) > 0, \quad (4.49)$$

where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and $W_3(x)$ is a cont. P.D. func. on R^n.

- Then, the system (4.44) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Lemma 4.6: Converse Theorem for G.E.S.

- Suppose $f(t, x, u)$ is cont. diff. and globally Lipshitz in (x, u), uniformly in t.

$$\dot{x} = f(t, x, u) \quad (4.44)$$

$$\dot{x} = f(t, x, 0) \quad (4.45)$$

- If the unforced syst (4.45), i.e., $u \equiv 0$ has a GES EP at the origin $x = 0$, then the system (4.44) is ISS.

Discussion of Lemma 4.6

- Lemma 4.6 requires a globally Lipschitz function f and G.E.S. of $x = 0$ of the unforced system to conclude input-to-state stability.

- It is easy to construct examples where the lemma does not hold in absence of one of these 2 conditions.

- The system $\dot{x} = -3x + (1 + x^2)u$,
  which we discussed earlier in the Sec,
  doesn't satisfy the global Lipschitz cond.

- The system $\dot{x} = -\frac{x}{1+x^2} + u =^{def} f(x,u)$
  has a globally Lipschitz f
  since the partial derivatives of f
  w.r.t. x & u are globally bounded.

- The origin of $\dot{x} = -\frac{x}{1+x^2}$ is G.A.S.,
  as it can be seen by the Lyapunov function $V(x) = x^2/2$,
  whose derivative $\dot{V}(x) = -\frac{x^2}{1+x^2}$
  is N.D. for all x.

- It is locally E.S.
  because the linearization at the origin
  is $\dot{x} = -x$.

- However, it is not G.E.S.

Discussion of Lemma 4.6

- It is easiest seen through the fact that the system is not I.S.S..
- Notice that with $u(t) \equiv 1$, $f(x, u) \geq 1/2$. Hence, $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$, which shows that the sol. is unbounded.
- In the absence of G.E.S. or globally Lipschitz functions, we may still be able to show ISS by applying Thm 4.19.

Example 4.25

- The system $\dot{x} = f(x, u) = -x^3 + u$ has a GAS origin when $u = 0$.
- Taking $V = \frac{1}{2}x^2$

Example 4.26

- The system

$$\dot{x} = f(x, u) = -x - 2x^3 + (1 + x^2)u^2$$

has a GES origin when $u = 0$,

but Lemma 4.6 does not apply

since f is not globally Lipschitz.

- Taking $V = \frac{1}{2}x^2$

Examples 4.25 & 4.26

- Note that, in examples 4.25 & 4.26,

  $V(x) = x^2/2$ satisfies

  (4.48) of Thm 4.19

  with $\alpha_1(r) = \alpha_2(r) = r^2/2$.

- Hence, $\alpha_1^{-1}(\alpha_2(r)) = r$

  and $\gamma(r)$ reduces to $\rho(r)$.

$$\alpha_1(||x||) \le V(t, x) \le \alpha_2(||x||) \quad (4.48)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \le -W_3(x), \quad \forall ||x|| \ge \rho(||u||) > 0, \quad (4.49)$$

- Applications of **I.S.S.** to

  stability analysis of **cascade systems**

- Consider

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (4.52)$$

where

$f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and

$f_2 : [0, \infty) \times R^{n_2} \to R^{n_2}$

are **piecewise cont. in** t

and **locally Lipschitz in** $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Suppose both

$$\dot{x}_1 = f_1(t, x_1, 0)$$

$$\dot{x}_2 = f_2(t, x_2)$$

have **G.U.A.S. E.P.**

at their respective **origins**.

- Under **what condition**

  will **the origin** $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  of the **cascade** system

  posses the same property?

- Under the stated assumptions,

  and with x_2 as input

  if $\dot{x}_1 = f_1(t, x_1, x_2)$ is ISS

  and $x_2 = 0$ of $\dot{x}_2 = f_2(t, x_2)$ is GUAS,

  then $x = 0$ of the cascade system:

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (4.52)$$

  is GUAS.

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (4.52)$$

- Let $t_0 \geq 0$ be the initial time.

- The sol. of (4.51) & (4.52) satisfy

$$||x_1(t)|| \leq \beta_1\left(||x_1(s)||, t-s\right) + \gamma_1\left(\sup_{s \leq \tau \leq t} ||x_2(\tau)||\right) \quad (4.53)$$

$$||x_2(t)|| \leq \beta_2\left(||x_2(s)||, t-s\right) \quad (4.54)$$

  globally, where $t \geq s \geq t_0$,

  β_1, β_2 are class $\mathcal{KL}$ functions

  and γ_1 is a class $\mathcal{K}$ function.

- **Apply (4.53) with $s = (t + t_0)/2$**

$$||x_1(t)|| \;\le\; \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{\frac{t+t_0}{2}\le\tau\le t} ||x_2(\tau)||\right) \qquad \textbf{(4.55)}$$

- **To estimate $x_1(\frac{t+t_0}{2})$,**

  apply (4.53) with $s = t_0$ and

  t replaced by $\frac{t+t_0}{2}$ to obtain

$$\left\|x_1\left(\frac{t+t_0}{2}\right)\right\| \;\le\; \beta_1\left(||x_1(t_0)||, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{t_0\le\tau\le\frac{t+t_0}{2}} ||x_2(\tau)||\right) \qquad \textbf{(4.56)}$$

- **Using (4.54), we obtain**

$$\sup_{t_0\le\tau\le\frac{t+t_0}{2}} ||x_2(\tau)|| \;\le\; \beta_2(||x_2(t_0)||, 0) \qquad \textbf{(4.57)}$$

$$\sup_{\frac{t+t_0}{2}\le\tau\le t} ||x_2(\tau)|| \;\le\; \beta_2(||x_2(t_0)||, \frac{t-t_0}{2}) \qquad \textbf{(4.58)}$$

- Substituting (4.56) through (4.58)

  into (4.55) and using the inequalities

$$||x_1(t_0)|| \leq ||x(t_0)||,$$

$$||x_2(t_0)|| \leq ||x(t_0)||, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$||x(t)|| \leq ||x_1(t)|| + ||x_2(t)||$$

  yield

$$||x(t)|| \leq \beta\left(||x(t_0)||, t - t_0\right)$$

  where

$$\beta\left(r, s\right) = \beta_1\left(\underline{\beta_1\left(r, s/2\right) + \gamma_1\left(\beta_2(r, 0)\right)},\ s/2 \right) + \gamma_1\left(\beta_2\left(r, s/2\right)\right) + \beta_2\left(r, s\right)$$

- So, β is a class $\mathcal{KL}$ func for all $r \geq 0$.

  Hence, $x = 0$ is GUAS