Nonlinear Systems Analysis

Lecture Note 16

Section 4.8
(Ultimate) Boundedness
(Lyapunov Stability)

Feng-Li Lian

NTU-EE

Sep05 – Jan06

Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Lyapunov analysis can be used to show the boundedness of the solution of the state equation, even when there is no E.P. at the origin.

- If $f(t,0)=0, \forall t \geq 0$, the origin is an E.P. for $\dot{x}=f(t,x)$ at $t=0$

- For example: consider the scalar eqn.:

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no E.P.s and whose solution is given by

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau d\tau$$

- The solution satisfies the bound

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\ &= e^{-(t-t_0)}a + \delta[1 - e^{-(t-t_0)}] \\ &= \delta + (a-\delta)e^{-(t-t_0)} \\ &\leq a, \quad \forall t \geq t_0 \end{aligned}$$

- Which shows that the solution is bouded for all $t \geq t_0$, uniformly in t_0, that is, with a bound independent of t_0.

- While this bound is valid for all $t \geq t_0$,

  it becomes a conservative estimate

  of the solution as time progresses,

  because it does not take into consideration

  the exponentially decaying term.

- If we pick any number b

  such that $\delta < b < a$,

  it can be easily seen that

$$|x(t)| \leq b, \quad \forall t \geq t_0 + \ln(\frac{a-\delta}{b-\delta})$$

- The bound b,

  which again is independent of t_0,

  gives a better estimate of the sol.

  after a transient period has passes.

- In this case,

  the sol. is said to be

  uniformly ultimately bounded and

  b is called the ultimate bound.

- Showing that the sol. of $\dot{x} = -x + \delta \sin t$ has the uniform boundedness and ultimate boundedness properties can be done via Lyapunov analysis w/o using the explicit sol. of the state eqn.

- Starting with $V(x) = x^2/2$, then,

$$\dot{V} \;=$$

- The RHS is not N.D. because, near the origin, $\delta|x|$ dominates $-x^2$.

- However, outside the set $\{|x| \le \delta\}$, i.e., $\{|x| > \delta\}$, $\dot{V}$ is negative.

- Choose $c > \delta^2/2$.

- Since $\dot{V}$ is negative on the boundary $V = c$,

  sols. starting in the set $\{V(x) \leq c\}$

  will remain therein for all future time.

- Hence, the sol. are uniformly bounded.

- Moreover, if we pick any number ϵ

  such that $(\delta^2/2) < \epsilon < c$,

  then $\dot{V}$ will be negative in $\{\epsilon \leq V \leq c\}$.

- Which shows that, in this set,

  V will decrease monotonically

  until the sol. enters the set $\{V \leq \epsilon\}$.

- Form the time on,

  because $\dot{V}$ is negative

  on the boundary $V = \epsilon$, the sol. cannot

  leave the set $\{V \leq \epsilon\}$.

- Thus, we can conclude that

  the sol. is uniformly ultimately bounded

  with the ultimate bound $|x| \leq \sqrt{2\epsilon}$.

$V(x) = x^2/2 \leq \epsilon$

- Consider the following system:

$$\dot{x} = f(t, x) \quad (4.32)$$

  where $f : [0, \infty) \times D \to R^n$

  is piecewise continuous in t and

  locally Lipschitz in x on $[0, \infty) \times D$,

  and $D \subset R^n$ is a domain

  containing the origin.

- Definition 4.6:

  The solutions of $\dot{x} = f(t, x)$ are

- uniformly bounded (UB)

  if there exists a positive constant c,

  independent of $t_0 \geq 0$, and

  for every $a \in (0, c)$,

  there is $\beta = \beta(a) > 0$,

  independent of t_0, such that

$$||x(t_0)|| \leq a \Rightarrow ||x(t)|| \leq \beta, \quad \forall t \geq t_0 \quad (4.33)$$

- globally uniformly bounded (GUB)

  if (4.33) holds for arbitrarily large a.

- uniformly ultimately bounded (UUB)

  with ultimate bound b

  if existing $b > 0$ and $c > 0$, indep. of $t_0 \geq 0$,

  and for every $a \in (0, c)$,

  there is $T = T(a, b) \geq 0$, indep. of t_0,

  such that $||x(t_0)|| \leq a \Rightarrow ||x(t)|| \leq b, \quad \forall t \geq t_0 + T \quad (4.34)$

- globally uniformly ultimately bounded (GUUB)

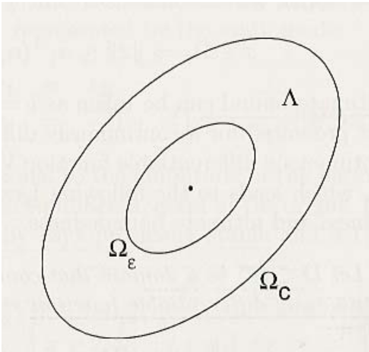
  if (4.34) holds for arbitrarily large a.

- For autonomous systems,

  we may drop the word "uniformly"

  since the solution depends only on $t - t_0$.

- Consider a cont. diff., P.D. fun. $V(x)$ and suppose that $\{V(x) \leq c\}$ is compact, for some $c > 0$.



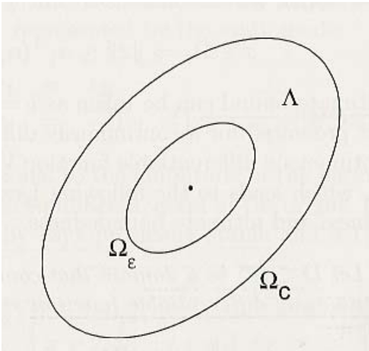
- Let $\Lambda = \{\epsilon \leq V(x) \leq c\}$ for some positive constant $\epsilon < c$.

- Suppose the derivative of V along the traj. of $\dot{x} = f(t, x)$ satisfies

$$\dot{V}(t, x) \leq -W_3(x), \; \forall x \in \Lambda, \; \forall t \geq t_0 \quad (4.35)$$

where $W_3(x)$ is a cont. P.D. function.

- Inequality (4.35) implies that $\Omega_c = \{V(x) \leq c\}$ and $\Omega_\epsilon = \{V(x) \leq \epsilon\}$ are positively invariant since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\epsilon$, the derivative $\dot{V}$ is negative.



- Since $\dot{V}$ is negative in Λ, a trajectory starting in Λ must move in a direction of decreasing $V(x(t))$.

- In fact, while in Λ,

  V satisfies (4.22), (4.24) of Thm 4.9.

$$W_1(x) \le V(t,x) \le W_2(x), \quad (4.22)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad (4.24)$$

- Therefore, the trajectory behaves

  as if the origin was U.A.S.

  and satisfies an inequality of the form

$$||x(t)|| \le \beta(||x(t_0)||, t - t_0)$$

  for some class $\mathcal{KL}$ function β.

- $V(x(t))$ will continue decreasing

  until the traj. enters Ω_ϵ in finite time

  and stays therein for all future time.

- The fact that

  the trajectory enters Ω_ϵ in finite time

  can be shown as follows:

- Because $W_3(x)$ is continuous

  and Λ is compact,

  let $k = \min_{x \in \Lambda} W_3(x) > 0$.

- Since $W_3(x)$ is P.D., k is positive.

  Hence, $W_3(x) \geq k, \ \forall x \in \Lambda$ (4.36)

$\dot{V}(t,x) \leq -W_3(x)$, (4.35)

$W_3(x) \geq k$, (4.36)

$\Lambda = \{\epsilon \leq V(x) \leq c\}$

- Inequalities (4.35) and (4.36) imply that

$$\dot{V}(t,x) \leq -k, \ \forall x \in \Lambda, \ \forall t \geq t_0$$

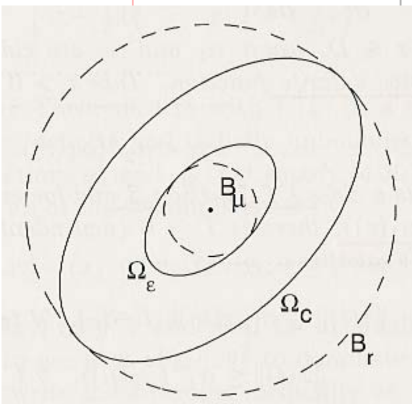
- Therefore,

$$\begin{aligned} V(x(t)) &\leq V(x(t_0)) - k(t-t_0) \\ &\leq c - k(t-t_0) \end{aligned}$$

- Which shows that $V(x(t))$ reduces to ϵ within the time interval $[t_0, t_0 + (c-\epsilon)/k]$.

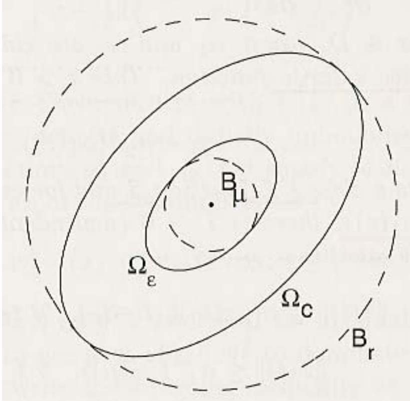
- In many problems, $\dot{V} \leq -W_3$ is obtained by using norm inequalities.

- In such cases, it is more likely that we arrive at



$$\dot{V}(t,x) \leq -W_3(x), \quad \forall \mu \leq ||x|| \leq r, \ \forall t \geq t_0 \quad (4.37)$$

- If r is sufficiently larger than μ, we can choose c and ϵ such that Λ is nonempty and contained in $\{\mu \le ||x|| \le r\}$.



- In particular, let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

$$W_1(x) \le V(t,x) \le W_2(x), \quad (4.22)$$

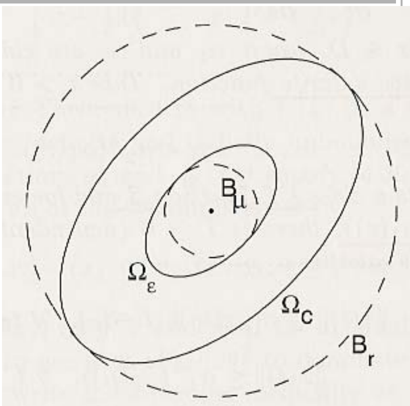
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad (4.24)$$

$$\Lambda = \{\epsilon \le V(x) \le c\}$$

- From the left inequality of (4.38), we have

$$V(x) \le c \Rightarrow \alpha_1(||x||) \le c$$

$$\Leftrightarrow ||x|| \le \alpha_1^{-1}(c)$$



- Therefore, taking $c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$.

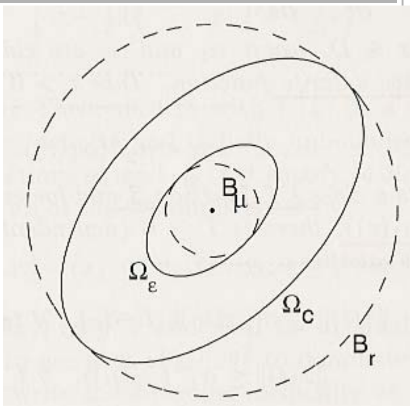
$$W_1(x) \le V(t,x) \le W_2(x), \quad (4.22)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad (4.24)$$

$$\Lambda = \{\epsilon \le V(x) \le c\}$$

- From the right inequality of (4.38),

$$||x|| \le \mu \Rightarrow V(x) \le \alpha_2(\mu)$$



- Consequently, taking $\epsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\epsilon$.

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

- To obtain $\epsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$.

- All trajectories starting in Ω_c enter Ω_ϵ within a finite time T.

- To calculate the ultimate bound on $x(t)$, use the left inequality of (4.38) to write

$$V(x) \le \epsilon \Rightarrow \alpha_1(||x||) \le \epsilon \Leftrightarrow ||x|| \le \alpha_1^{-1}(\epsilon)$$

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

- Recalling that $\epsilon = \alpha_2(\mu)$,

$$x \in \Omega_\epsilon \Rightarrow ||x|| \le \alpha_1^{-1}(\alpha_2(\mu))$$

- Hence, the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.

Lyapunov Analysis - 11

- The ideas just presented $\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$ (4.38)
  for a cont. diff. functon $V(x)$
  can be extended to $V(t,x)$,
  as long as $V(t,x)$ satisfies inequality (4.38),
  which leads to the Lyapunov-like theorem
  for showing uniform boundedness
  and ultimate boundedness.

Theorem 4.18: G.U.U.B.

- Let $D \subset R^n$ be a domain
  that contains the origin and
  $V : [0,\infty) \times D \to$
  be a cont. diff. func. such that

$$\alpha_1(||x||) \leq V(t,x) \leq \alpha_2(||x||) \quad (4.39)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \ \forall ||x|| \geq \mu > 0 \quad (4.40)$$

$\forall t \geq 0$ and $\forall x \in D$,
where α_1 and α_2 are class $\mathcal{K}$ functions
and $W_3(x)$ is a cont. P.D. functon.

- Take $r > 0$ such that $B_r \subset D$ and
  suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$ (4.41)

- Then, there exists a class $\mathcal{KL}$ function β
  and for every initial state $x(t_0)$,
  satisfying $||x(t_0)|| \le \alpha_2^{-1}(\alpha_1(r))$,
  there is $T \ge 0$ (dependent on $x(t_0)$ and μ)
  such that the soluton of $\dot{x} = f(t, x)$ satisfies

$$||x(t)|| \le \beta(||x(t_0)||, t - t_0), \ \forall t_0 \le t \le t_0 + T \quad (4.42)$$

$$||x(t)|| \le \alpha_1^{-1}(\alpha_2(\mu)), \ \forall t \ge t_0 + T \quad (4.43)$$

- Hence, $x(t)$ is U.B. (4.42) & U.U.B. (4.43).

- Moreover, if $D = R^n$ and
  α_1 belongs to class $\mathcal{K}_\infty$,
  then (4.42) and (4.43) hold
  for any initial state $x(t_0)$,
  with no restriction on how large μ is.

- Proof: See Appendix C. 9.

- Inequalities (4.42) and (4.43) show that

  $x(t)$ is uniformly bounded

  for all $t \geq t_0$ and

  uniformly ultimately bounded

  with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$.

- The ultimate bound is

  a class $\mathcal{K}$ function of μ;

  hence, the smaller the value of μ,

  the smaller the ultimate bound.

- As $\mu \rightarrow 0$,

  the ultimate bound approaches zero.

- The main application of Thm 4.18 arises

  in studying the stability of perturbed syst.

- Consider a mass-spring system
  with a hardening spring,
  linear viscous damping,
  and a periodic external force
  can be represented
  by the Duffings equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos wt$$

- Taking $x_1 = y, x_2 = \dot{y}$ and
  assuming certain numerical values
  for the various constants,
  the system is represented by the state model

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M\cos wt\end{aligned}$$

  where $M \geq 0$ is proportional to
  the amplitude of the periodic external force.

Example 4.24

- When $M = 0$,

  the system has an E.P. at the origin.

Example 4.24

- When $M > 0$, we apply Thm 4.18

  with $V(x)$ as a candidate func.

$$\alpha_1(||x||) \le V(t,x) \le \alpha_2(||x||) \quad (4.39)$$

- $V(x)$ is P.D. and R.U.;

  hence, by Lemma 4.3,

  there exist class $\mathcal{K}_\infty$ functions α_1 and α_2

  that satisfy (4.39) globally.

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall ||x|| \ge \mu > 0 \quad (4.40)$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M \cos wt \end{aligned}$$

$$V(x) = \frac{3}{2}x_1^2 + x_2^2 + x_1 x_2 + \frac{1}{2}x_1^4$$

Example 4.24

Example 4.24

Example 4.24