Nonlinear Systems Analysis

Lecture Note 15

Section 4.7

Converse Theorems (Lyapunov Stability)

Feng-Li Lian

NTU-EE

Sep05 – Jan06

Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Two Questions:
  - Is there a function that satisfies the conditions of the Thms? (Thm 4.9, 4.10, e.x.)
  - How can we search for such a function?

- In many cases, Lyapunov theory provides an affirmative answer to the first question.
- The answer takes the form of a converse Lyapunov theorem, which is the inverse of one of Lyapunov's theorems.
- Most of these converse theorems are proven by actually constructing auxiliary functions that satisfy the conditions of the respective theorems.

- But, the construction almost always assumes the knowledge of the sol. of the diff. eqn.

- In this section, we give three converse Lyapunov theorems.

- The first one is a converse Lyapunov thm when the origin is exponentially stable and,

- The second, when it is uniformly asymptotically stable.

- The third thm applies to autonomous syst. and defines the converse Lyapunov func. for the whole region of attraction of an asymptotically stable equilibrium point.

- Let $x = 0$ be an EP for the NL system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is cont. diff.,

$D = \{x \in R^n \mid ||x|| < r\}$,

and the Jacobian matrix $[\partial f / \partial x]$ is

bdd on D, uniformly in t.

- Let k, λ, and r_0 be positive const.

with $r_0 < r/k$.

- Let $D_0 = \{x \in R^n \mid ||x|| < r_0\}$.

- Assume that the traj. of the syst. satisfy

$$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}, \quad \forall x(t_0) \in D_0, \ \forall t \geq t_0 \geq 0$$

- Then, there is a function

$V : [0, \infty) \times D_0 \to R$

that satisfies the inequalities

$$c_1||x||^2 \leq V(t, x) \leq c_2||x||^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3||x||^2$$

$$\left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x||$$

for some positive const. c_1, c_2, c_3, and c_4.

- Moreover, if $r = \infty$ and the origin is G.E.S., then $V(t, x)$ is defined and satisfies the aforementioned inequalities on R^n.

- Furthermore, if the system is autonomous, V can be chosen independent of t.

- Due to the equivalence of norms, it is sufficient to prove the thm for the 2-norm.

- Let $\phi(\tau; t, x)$ denote the sol. of the syst. that starts at (t, x); that is, $\phi(t; t, x) = x$.

- For all $x \in D_0$, $\phi(\tau; t, x) \in D$ for all $\tau \geq t$.

- Let

$$V(t, x) = \int_t^{t+\delta} \phi^T(\tau; t, x)\phi(\tau; t, x) d\tau$$

where δ is a positive constant to be chosen.

- Due to the exponentially decaying bound on the trajectories, we have

$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}$, $\forall x(t_0) \in D_0, \ \forall t \geq t_0 \geq 0$

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau$$
$$= \int_t^{t+\delta} ||\phi(\tau;t,x)||_2^2 d\tau$$
$$\leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau-t)} d\tau ||x||_2^2$$
$$= \frac{k^2}{2\lambda}(1-e^{-2\lambda\delta})||x||_2^2$$

- On the other hand, the Jacobian matrix $[\partial f/\partial x]$ is bdd on D.

- Let

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|_2 \leq L, \ \forall x \in D$$

- Then, $||f(t,x)||_2 \leq L||x||_2$ and $\phi(\tau;t,x)$ satisfies the lower bound

$$||\phi(\tau;t,x)||_2^2 \geq ||x||_2^2 e^{-2L(\tau-t)}$$

- Hence,

$$V(t,x) \geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau ||x||_2^2$$
$$= \frac{1}{2L}(1-e^{-2L\delta})||x||_2^2$$

- Thus, $V(t,x)$ satisfies the first inequality of the theorem with

$$c_1 = \frac{1-e^{-2L\delta}}{2L} \text{ and } c_2 = \frac{k^2(1-e^{-2\lambda\delta})}{2\lambda}$$

- To calculate the derivative of V along the trajectories of the system, define the sensitivity functions

$$\phi_t(\tau;t,x) = \frac{\partial}{\partial t}\phi(\tau;t,x)$$

$$\phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x)$$

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau$$

- Then,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$$

$$= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \phi^T(t;t,x)\phi(t;t,x)$$

$$+ \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_t(\tau;t,x)d\tau$$

$$+ \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)d\tau f(t,x)$$

$$= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - ||x||_2^2$$

$$+ \int_t^{t+\delta} 2\phi^T(\tau;t,x)\left[\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x)\right] d\tau$$

- It is not difficult to show that (Ex 3.30)

$$\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t, x) \equiv 0, \ \forall \tau \geq t$$

- Therefore,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - ||x||_2^2 \\ &\leq -(1 - k^2 e^{-2\lambda\delta})||x||_2^2 \end{aligned}$$

- By choosing $\delta = \ln(2k^2)/(2\lambda)$,
  the second inequality of the thm.
  is satisfied with $c_3 = 1/2$.

- To show the last inequality, let us note that
  $\phi_x(\tau; t, x)$ satisfies the sensitivity eqn.

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau; t, x))\phi_x, \ \phi_x(t; t, x) = I$$

- Since $||\frac{\partial f}{\partial x}(t, x)||_2 \leq L$ on D,
  ϕ_x satisfies the bound

$$||\phi_x(\tau; t, x)||_2 \leq e^{L(\tau - t)}$$

- Therefore,

$$\left\|\frac{\partial V}{\partial x}\right\|_2 = \left\|\int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)d\tau\right\|_2$$

$$\leq \int_t^{t+\delta} 2\left\|\phi(\tau;t,x)\right\|_2 \left\|\phi_x(\tau;t,x)\right\|_2 d\tau$$

$$\leq \int_t^{t+\delta} 2ke^{-\lambda(\tau-t)}e^{L(\tau-t)}d\tau||x||_2$$

$$= \frac{2k}{\lambda - L}[1 - e^{-(\lambda-L)\delta}]||x||_2$$

$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}$, $\forall x(t_0) \in D_0,\ \forall t \geq t_0 \geq 0$

- The last inequality of the thm. is satisfies

with $c_4 = \dfrac{2k}{(\lambda - L)}[1 - e^{-(\lambda-L)\delta}]$

- If all the assumptions hold globally,

  then r_0 can be chosen arbitrarily large.

- If the system is autonomous,

  then $\phi(\tau;t,x)$ depends only on $(\tau - t)$; i.e.,

$$\phi(\tau;t,x) = \psi(\tau - t;x)$$

- Then,

$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau$

$$V(t,x) = \int_t^{t+\delta} \psi^T(\tau - t;x)\psi(\tau - t;x)d\tau$$

$$= \int_0^{\delta} \psi^T(s;x)\psi(s;x)ds$$

which is independent of t.

- QED

Theorem 4.15: E.S. of NL & L Systems

- Let $x = 0$ be an E.P. for the NL syst.

$$\dot{x} = f(t, x)$$

  where $f : [0, \infty) \times D \to R^n$ is cont. diff.,
  $D = \{x \in R^n \mid ||x||_2 < r\}$, and
  the Jacobian matrix $[\partial f / \partial x]$ is bdd and
  Lipschitz on D, uniformly in t.

- Let $A(t) = \frac{\partial f}{\partial x}(t, x)\Big|_{x=0}$

- Then,
  $x = 0$ is an E.S. E.P. for the NL syst.
  iff it is an E.S. E.P for the L syst.

$$\dot{x} = A(t)x$$

Theorem 4.15: E.S. of NL & L Systems: Proof

- The "if" part follows from Thm 4.13.

- To prove the "only if" part,
  write the linear system as

$$\dot{x} = f(t, x) - [f(t, x) - A(t)x] = f(t, x) - g(t, x)$$

- Recalling the argument preceding
  Thm 4.13, we know that

$$||g(t, x)||_2 \le L||x||_2^2, \ \forall x \in D, \ \forall t \ge 0$$

- Since $x = 0$ is an E.S. E.P. of
  the NL syst.,
  there are positive const $k, \lambda,$ and c
  such that

$$||x(t)||_2 \leq k||x(t_0)||_2 e^{-\lambda(t-t_0)},$$

$$\forall t \geq t_0 \geq 0, \forall ||x(t_0)||_2 < c$$

- Choosing $r_0 < \min\{c, r/k\}$,
  all the conditions of Thm 4.14 are satisfied.

- Let $V(t,x)$ be the function
  provided by Thm 4.14 and
  use it as a Lyapunov function candidate
  for the L syst.

- Then,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) - \frac{\partial V}{\partial x} g(t,x) \\
&\leq -c_3||x||_2^2 + c_4 L ||x||_2^3 \\
&< -(c_3 - c_4 L \rho)||x||_2^2, \qquad \forall ||x||_2 < \rho
\end{aligned}$$

- The choice $\rho < \min\{r_0, c_3/(c_4 L)\}$
  ensures that $\dot{V}(t,x)$ is N.D. in $||x||_2 < \rho$.

- Consequently, all the conditions of
  Thm 4.10 are satisfied in $||x||_2 < \rho$,
  and we conclude that
  the origin is an E.S. E.P. for the L. syst. QED

Corollary 4.3

- Let $x = 0$ be an E.P. of the NL syst.

$$\dot{x} = f(x)$$

  where $f(x)$ is cont. diff.

  in some nbhd of $x = 0$.

- Let $A = \left[\frac{\partial f}{\partial x}\right](0)$

- Then,

  $x = 0$ is an E.S. E.P. for the NL system

  iff A is Hurwitz.

Theorem 4.16: U.A.S.

- Let $x = 0$ be an E.P. for the NL syst.

$$\dot{x} = f(t, x)$$

  where $f : [0, \infty) \times D \to R^n$ is cont. diff.,
  $D = \{x \in R^n \mid ||x||_2 < r\}$, and
  the Jacobian matrix $[\partial f / \partial x]$ is
  bdd on D, uniformly in t.

- Let β be a class $\mathcal{KL}$ function and
  r_0 be a positive constant
  such that $\beta(r_0, 0) < r$.

- Let $D_0 = \{x \in R^n \mid ||x|| < r_0\}$.

- Assume that the traj. of the syst. satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0),$$

$$\forall x(t_0) \in D_0, \quad \forall t \ge t_0 \ge 0$$

- Then, there is a cont. diff. function
  $V : [0, \infty) \times D_0 \to R$
  that satisfies the inequalities

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|)$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le \alpha_4(\|x\|)$$

where $\alpha_1, \alpha_2, \alpha_3$, and α_4 are
class $\mathcal{K}$ functions defined on $[0, r_0]$.

- If the system is autonomous,
  V can be chosen independent of t.

- Proof: See Appendix C.7.

- Let $x = 0$ be an AS EP for the NL syst

$$\dot{x} = f(x)$$

  where $f : D \to R^n$ is locally Lipschitz and $D \subset R^n$ is a domain that contains $x = 0$.

- Let $R_A \subset D$ be the region of attraction of $x = 0$.

- Then, there is a smooth, PD function $V(x)$ and a cont., PD function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A$$

and for any $c > 0$,

$\{V(x) \leq c\}$ is a compact subset of R_A.

- When $R_A = R^n$,

  $V(x)$ is radially unbounded.

- Proof: See Appendix C.8.

- An interesting feeture of Thm 4.17 is that
  any bounded subset S of
  the region of attraction
  can be included in a compact set
  of the form $\{V(x) \le c\}$
  for some constant $c > 0$.

- This feature is useful because quite often
  we have to limit our analysis to
  a positively invariant, compact set
  of the form $\{V(x) \le c\}$.

- With the property $S \subset \{V(x) \le c\}$,
  our analysis will be valid
  for the whole set S.

- On the other hand,
  if all we know is the existence
  of a Lyapunov function $V_1(x)$ on S,
  we will have to choose a constant c_1
  such that $\{V_1(x) \le c_1\}$ is compact and
  included in S;
  then our analysis will be limited to
  $\{V_1(x) \le c_1\}$, which is only a subset of S.