Nonlinear Systems Analysis

Lecture Note 14

Section 4.6

LTV Systems & Linearization (Lyapunov Stability)

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Outline

- Consider the linear time-varying systems:

$$\dot{x} = A(t)x \qquad (4.29)$$

- $x = 0$ is an equilibrium point

- The stability behavior of the origin
  as an equilibrium point
  can be completely characterized
  in terms of the state transition matrix
  of the system.

- Form linear system theory,
  we know that the solution is given by

$$x(t) \quad = \quad \Phi(t, t_0)\, x(t_0)$$

where $\Phi(t, t_0)$ is
the state transition matrix.

Theorem 4.11: G.U.A.S.

- The equilibrium point $x = 0$ of (4.29) is (globally) uniformly asymptotically stable
- if and only if the state transition matrix satisfies the inequality

$$||\Phi(t, t_0)|| \leq k\, e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

for some positive constants k and λ.

Theorem 4.11: G.U.A.S.: Proof

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- Theorem 4.11 shows that,
  for linear systems,
  U.A.S. of the origin = E.S..

- Note that,
  for linear time-varying systems,
  U.A.S. cannot be characterized
  by the location of the eigenvalues of A.

- Thm 4.11 is not helpful as a stability test
  because it needs to solve the state eqn.

- However, it guarantees
  the existence of a Lyapunov function.
  See Example 4.21, for example.

- Let $x = 0$ be the E.S. E.P. of

$$\dot{x} = A(t)\,x(t), \qquad (4.29)$$

- Suppose $A(t)$ is continuous and bounded.

- Let $Q(t)$ be a cont., bdd., P.D., symm. matrix.

$$0 \le c_3 I \le Q(t) \le c_4 I, \qquad \forall t \ge 0$$

- THEN, there is a cont. diff., bdd., P.D., symm. matrix $P(t)$ satisfying

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \quad (4.28)$$

- Hence, $V(t,x) = x^T P(t) x$ is a Lyapunov function of the system that satisfies the conditions of Thm 4.10.

Theorem 4.12: E.S.: Proof

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Linearization to Non-autonomous System

- Consider the nonlinear nonautonomous sys

$$\dot{x} = f(t, x)$$

  where $f : [0, \infty] \times D \to R^n$ is cont. diff.
  and $D = \{x \in R^n \mid ||x||_2 < r\}$.

- Suppose the origin $x = 0$ is an E.P.
  for the systems at $t = 0$;
  that is, $f(t, 0) = 0$ for all $t \geq 0$.

- Furthermore, suppose

  the Jacobian matrix $[\partial f / \partial x]$ is bdd. and

  Lipschitz on D, uniformly in t; thus,

$$\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\|_2 \leq L_1 ||x_1 - x_2||_2,$$

  $\forall x_1, x_2 \in D$, $\forall t \geq 0$ for all $1 \leq i \leq n$.

- By the mean value theorem,

$$f_i(t, x) = f_i(t, 0) + \frac{\partial f_i}{\partial x}(t, z_i)x$$

  where z_i is a point on the line segment connecting x to the origin.

- Since $f(t, 0) = 0$,

  we can write $f_i(t, x)$ as

$$\begin{aligned} f_i(t, x) &= \frac{\partial f_i}{\partial x}(t, z_i)x \\ &= \frac{\partial f_i}{\partial x}(t, 0)x + \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x \end{aligned}$$

- Hence, $f(t, x) = A(t)x + g(t, x)$

  where

  . $A(t) = \frac{\partial f}{\partial x}(t, 0)$ and

  . $g_i(t, x) = \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right] x$

- The function $g(t, x)$ satisfies

$$\begin{aligned} \|g(t,x)\|_2 &\leq \left(\sum_{i=1}^{n} \left\| \frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0) \right\|_2^2 \right)^{1/2} \|x\|_2 \\ &\leq L \, \|x\|_2^2 \end{aligned}$$

  where $L = \sqrt{n} L_1$.

- Therefore, in a small nbhd of the origin, we may approximate the nonlinear system by its linearization about the origin.

- The next theorem states Lyapunov's indirect method for showing E.S. of the origin in the nonautonomous case.

- Let $x = 0$ be an E.P. for the NL sys

$$\dot{x} = f(t, x)$$

where $f : [0, \infty] \times D \to R^n$ is cont. diff.,

$D = \{x \in R^n \mid ||x||_2 < r\}$,

and the Jacobian matrix $[\partial f / \partial x]$ is bdd.

and Lipschitz on D, uniformly in t.

- Let

$$A(t) = \frac{\partial f}{\partial x}(t, x) \Big|_{x=0}$$

- Then, the origin is an E.S. E.P.

for the nonlinear system

if it is an E.S. E.P. for the linear system

$$\dot{x} = A(t)x$$

Theorem 4.13: E.S. of Non-Auto Syst.: Proof

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