Nonlinear Systems Analysis

Lecture Note 14

Section 4.6 LTV Systems & Linearization (Lyapunov Stability)

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Outline Feng-Li Lian © 2005 NTUEE-NSA-Ch4.6-2

- Introduction (L9)
- Autonomous Systems (4.1 L9)
 - · Basic stability definitions
 - · Lyapunov's stability theorems
 - Variable gradient method
 - Region of attraction
 - Instability
- The Invariance Principle (4.2, L10)
 - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
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- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

• Consider the linear time-varying systems:

$$\dot{x} = A(t)x \qquad (4.29)$$

- x = 0 is an equilibrium point
- The stability behavior of the origin as an equilibrium point can be completely characterized in terms of the state transition matrix of the system.

U.A.S. of LTV Systems

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Form linear system theory,
 we know that the solution is given by

$$x(t) = \Phi(t, t_0) x(t_0)$$

where $\Phi(t,t_0)$ is

the state transition matrix.

- The equilibrium point x = 0 of (4.29) is (globally) uniformly asymptotically stable
- if and only if the state transition matrix satisfies the inequality

$$||\Phi(t, t_0)|| \le k e^{-\lambda(t-t_0)}, \quad \forall t \ge t_0 \ge 0$$

for some positive constants k and λ .

Theorem 4.11: G.U.A.S.: Proof

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Theorem 4.11: G.U.A.S.

 Theorem 4.11 shows that, for linear systems,
 U.A.S. of the origin = E.S..

- Note that,
 for linear time-varying systems,
 U.A.S. cannot be characterized
 by the location of the eigenvalues of A.
- Thm 4.11 is not helpful as a stability test because it needs to solve the state eqn.
- However, it quarantees
 the existence of a Lyapunov function.
 See Example 4.21, for example.

Theorem 4.12: E.S.

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• Let x = 0 be the E.S. E.P. of

$$\dot{x} = A(t) x(t), \quad (4.29)$$

- Suppose A(t) is continuous and bounded.
- Let Q(t) be a cont., bdd., P.D., symm. matrix.

$$0 \le c_3 I \le Q(t) \le c_4 I, \quad \forall t \ge 0$$

• THEN, there is a cont. diff., bdd., P.D., symm. matrix P(t) satisfying

$$-\dot{P}(t) = P(t)A(t) + A^{T}(t)P(t) + Q(t) \quad (4.28)$$

• Hence, $V(t,x) = x^T P(t)x$ is a Lyapunov function of the system that satisfies the conditions of Thm 4.10.

Theorem 4.12: E.S.: Proof

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Linearization to Non-autonomous System

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Consider the nonlinear nonautonomous sys

$$\dot{x} = f(t, x)$$

where $f:[0,\infty]\times D\to R^n$ is cont. diff. and $D=\{x\in R^n\mid ||x||_2< r\}.$

• Suppose the origin x = 0 is an E.P. for the systems at t = 0; that is, f(t, 0) = 0 for all $t \ge 0$.

• Furthermore, suppose the Jacobian matrix $[\partial f/\partial x]$ is bdd. and Lipschitz on D, uniformly in t; thus,

$$\left\| \frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2) \right\|_2 \le L_1 ||x_1 - x_2||_2,$$

 $\forall x_1, x_2 \in D$, $\forall t \geq 0$ for all $1 \leq i \leq n$.

By the mean value theorem,

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t, \mathbf{z_i})x$$

where z_i is a point on the line segment connecting x to the origin.

Linearization to Non-autonomous System

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• Since f(t,0) = 0, we can write $f_i(t,x)$ as

$$f_{i}(t,x) = \frac{\partial f_{i}}{\partial x}(t,z_{i})x$$

$$= \frac{\partial f_{i}}{\partial x}(t,0)x + \left[\frac{\partial f_{i}}{\partial x}(t,z_{i}) - \frac{\partial f_{i}}{\partial x}(t,0)\right]x$$

• Hence, f(t,x) = A(t)x + g(t,x)

where

- . $A(t) = rac{\partial f}{\partial x}(t,0)$ and
- . $g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) \frac{\partial f_i}{\partial x}(t,0)\right] x$

• The function g(t,x) satisfies

where $L = \sqrt{n}L_1$.

$$||g(t,x)||_{2} \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial f_{i}}{\partial x}(t,z_{i}) - \frac{\partial f_{i}}{\partial x}(t,0) \right\|_{2}^{2} \right)^{1/2} ||x||_{2}$$

$$\leq L ||x||_{2}^{2}$$

Linearization to Non-autonomous System

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- Therefore, in a small nbhd of the origin, we may approximate the nonlinear system by its linearization about the origin.
- The next theorem states
 Lyapunov's indirect method
 for showing E.S. of the origin
 in the nonautonomous case.

• Let x = 0 be an E.P. for the NL sys

$$\dot{x} = f(t, x)$$

where $f:[0,\infty]\times D\to R^n$ is cont. diff., $D=\{x\in R^n\mid ||x||_2< r\},$ and the Jacobian matrix $[\partial f/\partial x]$ is bdd. and Lipschitz on D, uniformly in t.

• Let

$$A(t) = \frac{\partial f}{\partial x}(t, x) \Big|_{x=0}$$

Theorem 4.13: E.S. of Non-Auto Syst.

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Then, the origin is an E.S. E.P.
for the nonlinear system
if it is an E.S. E.P. for the linear system

$$\dot{x} = A(t)x$$

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