Nonlinear Systems Analysis

Lecture Note 13

Section 4.5
Non-autonomous Systems (Lyapunov Stability)

Feng-Li Lian
NTU-EE

Sep05 – Jan06

Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Consider the nonautonomous system:

$$\dot{x} = f(t, x) \qquad (4.15)$$

where $f : [0, \infty] \times D \to R^n$

is piecewise continuous in t and

locally Lipschitz in x on $[0, \infty] \times D$,

and $D \subset R^n$ is a domain

that contains the origin $x = 0$.

- If $f(t, 0) = 0, \forall t \geq 0$,

the origin is an E.P. for (4.15) at $t = 0$

- An equilibrium point at the origin could be

a translation of a nonzero E.P.

OR,

a translation of a nonzero sol. of the syst.

- To see the latter point,

suppose $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$.

- The change of variables

- So, we can determine the stability behavior of the solution $\bar{y}(\tau)$ of the original system by examining the stability behavior of the origin (as an E.P. for the transformed system).

- Notice that IF $\bar{y}(\tau)$ is not constant, the transformed system will be nonautonomous even when the original system is antonomous, that is, even when $g(\tau, y) = g(y)$.

- Studying the stability behavior of solutions in the sense of Lyapunov can be done only in the context of studying the stability behavior of the equilibria of nonautonomous systems.

- The notions of stability and asymptotic stability of E.P. of nonautonomous systems are basically the same as those introduced in Definition 4.1 for autonomous systems.

- While the sol. of an autonomous system depends only on $(t - t_0)$, the sol. of a nonautonomous system may depend on both t and t_0.

- So, in general, the stability behavior of E.P. will depend on t_0.

- The origin $x = 0$ is a stable E.P. for (4.15)

  if, for each $\varepsilon > 0$, and any $t_0 \geq 0$

  there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon, \quad \forall t \geq t_0$$

- The existence of δ for every t_0

  does not necessarily guarantee that

  there is one constant δ,

  dependent only on ε,

  that would work for all t_0,

  as illustrated by the next example.

Example 4.17: Stability Case

- The linear first-order system

$$\dot{x} = (6t \sin t - 2t)x$$

  has the solution

$$\begin{aligned} x(t) &= x(t_0) \exp\left[\int_{t_0}^{t} (6\tau \sin\tau - 2\tau) d\tau\right] \\ &= x(t_0) \exp\left[6 \sin t - 6t \cos t - t^2 - 6 \sin t_0 + 6t_0 \cos t_0 + {t_0}^2\right] \end{aligned}$$

- Hence,

$$|x(t)| < \Big|x(t_0)\Big| \, c(t_0), \quad \forall t \geq t_0$$

- For any $\varepsilon > 0$,

  the choice $\delta = \varepsilon / c(t_0)$ shows that

  the origin is stable.

- Suppose t_0 takes on the successive values

  $t_0 = 2n\pi$, for $n = 0, 1, 2, ...$, and

  $x(t)$ is evaluated π seconds later

  in each case.

- Then,

$$x(t_0 + \pi) = x(t_0) \exp \Big[(4n + 1)(6 - \pi)\pi \Big]$$

  which implies that, for $x(t_0) \neq 0$,

$$\frac{x(t_0 + \pi)}{x(t_0)} \to \infty \text{ as } n \to \infty$$

- Thus, given $\varepsilon > 0$,

  there is no δ independent of t_0

  that would satisfy the stability requirement

  uniformly in t_0.

- The linear first-order system

$$\dot{x} = -\frac{x}{1+t}$$

  has the solution

$$x(t) = x(t_0)\ \exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau} d\tau\right)$$

$$= x(t_0)\,\frac{1+t_0}{1+t}$$

- Since $|x(t)| \le |x(t_0)|$, $\forall t \geq t_0$,

  the origin is clearly stable.

- Actually, given any $\varepsilon > 0$,

  we can choose δ independent of t_0.

- It is also clear that

$$x(t) \to 0 \quad \text{as } t \to \infty$$

- Consequently, assording to Definition 4.1,

  the origin is asymptotically stable.

- But, the convergence of x(t) to the origin

  is not uniform wrt the initial time t_0.

- It is equivalent to saying that,

  given any $\varepsilon > 0$, there's $T = T(\varepsilon, t_0) > 0$

  such that $|x(t)| < \varepsilon$, for all $t \geq t_0 + T$.

- Alghough this is ture for every t_0,

  the constant T cannot be chosen

  independent of t_0.

Definition 4.4 of Stability, US, Instability, AS, UAS, GUAS

- The equilibrium point $x = 0$ of (4.15) is
- stable if, for each $\varepsilon > 0$,

  there is $\delta = \delta(\varepsilon, t_0) > 0$

  such that $||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$,

  $\forall t \geq t_0 \geq 0 \quad (4.16)$
- uniformly stable (US) if, for each $\varepsilon > 0$,

  there is $\delta = \delta(\varepsilon) > 0$, independent of t_0,

  such that (4.16) is satisfied.
- unstable if it is not stable.

Definition 4.4 of Stability, US, Instability, AS, UAS, GUAS

- asymptotically stable (AS)

  if it is stable and there is $c = c(t_0) > 0$

  such that $x(t) \to 0$ as $t \to \infty$,

  for all $||x(t_0)|| < c$.
- uniformly asymptotically stable (UAS)

  if it is US and there is $c > 0$, indep. of t_0,

  such that for all $||x(t_0)|| < c$,

  $x(t) \to 0$ as $t \to \infty$, uniformly in t_0;

  i.e., for each $\eta > 0$, there is $T = T(\eta) > 0$

  such that $||x(t)|| < \eta$, $\forall t \geq t_0 + T(\eta)$, $\forall ||x(t_0)|| < c$

- globally uniformly asymptotically stable (GUAS)

  if it is US,

  $\delta(\varepsilon)$ can be chosen to satisfy

  $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, and,

  for each pair of positive numbers η & c,

  there is $T = T(\eta, c) > 0$ such that

$$||x(t)|| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \quad \forall ||x(t_0)|| < c$$

- The E.P. $x = 0$ of (4.15) is
- uniformly stable (US)

  IFF there exist a class $\mathcal{K}$ function α and

  a positive constant c,

  independent of t_0, such that

$$\forall t \geq t_0 \geq 0, \quad \forall ||x(t_0)|| < c, \qquad ||x(t)|| \leq \alpha\left(||x(t_0)||\right) \quad (4.19)$$

- uniformly asymptotically stable (UAS)

  IFF there exist a class $\mathcal{KL}$ function β and

  a positive constant c,

  independent of t_0, such that

$$\forall t \geq t_0 \geq 0, \quad \forall ||x(t_0)|| < c, \quad ||x(t)|| \ \leq \ \beta\Big(||x(t_0)||, t - t_0\Big) \qquad (4.20)$$

- globally uniformly asymptotically stable (G UAS)

  IFF inequality (4.20) is satisfied

  for any initial state $x(t_0)$.

- The E.P. $x = 0$ of (4.15) is

- exponentially stable (ES)

  if there exist positive constants c, k, & λ

  such that

$$||x(t)|| \quad \leq \quad k \quad ||x(t_0)|| \quad e^{-\lambda(t-t_0)}, \qquad \forall ||x(t_0)|| < c$$

- globally exponentially stable (GES)

  if the above inequality is satisfied

  for any initial state $x(t_0)$.

Theorem 4.8: U.S.

- Let $x = 0$ be an E.P. for (4.15) and

  $D \subset R^n$ be a domain containing $x = 0$.

- Let $V : [0, \infty] \times D \to R$ be

  a continuously differentiable func such that

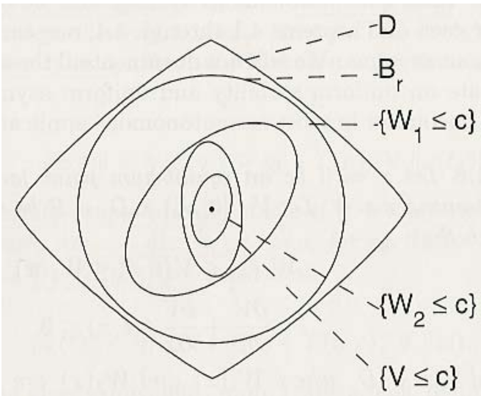
  $W_1(x) \le V(t, x) \le W_2(x)$

  $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le 0, \qquad \forall t \ge 0 \text{ and } \forall x \in D$

  where $W_1(x)$ and $W_2(x)$ are continuous P.D. func on D.

- Then, $x = 0$ is uniformly stable.

Theorem 4.8: U.S.: Proof



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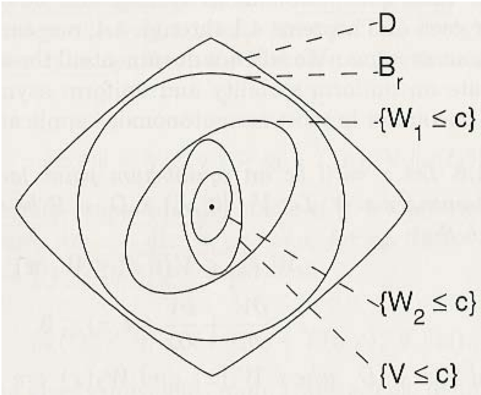

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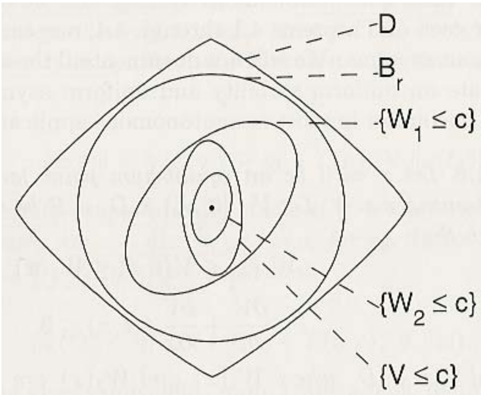

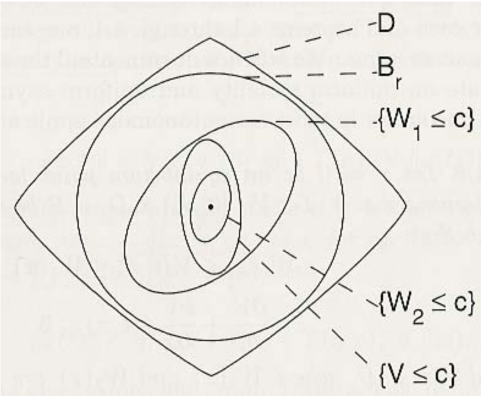

- Suppose the assumptions of Theorem 4.8

  are satisfied with strengthened inequality:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x)$$

  $\forall t \ge 0$ and $\forall x \in D$,

  where $W_3(x)$: a cont. P.D. func. on D

- Then, $x = 0$ is

  uniformly asymptotically stable.

- Moreover, if r and c are chosen such that

  $B_r = \{||x|| \leq r\} \subset D$ & $c < \min_{||x||=r} W_1(x)$,

  then every trajectory starting

  in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

  $$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall t \geq t_0 \geq 0$$

  for some class $\mathcal{KL}$ function β.

- Finally,

  if $D = R^n$ and $W_1(x)$ is radially unbounded,

  then $x = 0$ is

  globally uniformly asymptotically stable.

Theorem 4.9: U.A.S. & G.U.A.S.: Proof

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PD, ND & US, UAS, GUAS

- A function $V(t, x)$ is said to be
- positive semidefinite
  if $V(t, x) \geq 0$,
- positive definite
  if $V(t, x) \geq W_1(x)$
  for some positive definite function $W_1(x)$,
- radially unbounded
  if $W_1(x)$ is so,
- decrescent
  if $V(t, x) \leq W_2(x)$.
- negative definite (semidefinite)
  if $-V(t, x)$ is positive definite (semidefinite).

positive semidefinite

if $V(t, x) \geq 0$,

positive definite

if $V(t, x) \geq W_1(x)$

for some positive definite

radially unbounded

if $W_1(x)$ is so,

- A function $V(t, x)$ is said to be
- positive semidefinite — if $V(t, x) \geq 0$
- positive definite — if $V(t, x) \geq W_1(x)$ for some PD function $W_1(x)$
- radially unbounded — if $W_1(x)$ is so
- decrescent — if $V(t, x) \leq W_2(x)$
- negative definite (semidefinite) — if $-V(t, x)$ is PD (PSD)

- Thms 4.8 and 4.9 say that the origin is
- uniformly stable

  if there is a continuously differentiable, PD, decresscent function $V(t, x)$, whose derivative along the trajectories of the system is NSD
- uniformly asymptototically stable

  if the derivative is ND
- globally uniformly asymptotically stable

  if the conditions for UAS hold globally with a radially unbounded $V(t, x)$

Theorem 4.10: E.S. & G.E.S.

- Let $x = 0$ be an E.P. for (4.15) and $D \subset R^n$ be a domain containing $x = 0$.
- Let $V : [0, \infty] \times D \to R$ be a continuously differentiable function s.t.

$$k_1||x||^a \leq V(t,x) \leq k_2||x||^a \quad (4.25)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3||x||^a \quad (4.26)$$

$\forall t \geq 0$ and $\forall x \in D$,

where k_1, k_2, k_3, and a are + constants.

- Then, $x = 0$ is exponentially stable.
- If the assumptions hold globally, then $x = 0$ is globally exponentially stable.

Theorem 4.10: E.S. & G.E.S.: Proof

Example 4.19: G.U.A.S.

- Consider the scalar system

$$\dot{x} = -[1 + g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$.

Example 4.19: G.U.A.S.

Example 4.20: G.E.S.

- Consider the system

$$\dot{x}_1 = -x_1 - g(t)x_2$$
$$\dot{x}_2 = x_1 - x_2$$

where $g(t)$ is cont. diff. and satisfies

$$0 \leq g(t) \leq k \text{ and } \dot{g}(t) \leq g(t), \ \forall t \geq 0$$

Example 4.20: G.E.S.

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- The linear time-varying system

$$\dot{x} = A(t)x$$

has an E.P. at $x = 0$.

- Let $A(t)$ be continuous for all $t \geq 0$.

- Suppose there is a

cont. diff., sym., bdd, PD matrix $P(t)$;

that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \ \forall t \geq 0$$

which satisfies the matrix diff. eqn (4.28)

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

where $Q(t)$ is cont., sym., and PD;

that is,

$$Q(t) \geq c_3 I > 0, \ \forall t \geq 0$$

Example 4.21: G.E.S. of LTV System

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