Nonlinear Systems Analysis

Lecture Note 12

Section 4.4

Comparison Functions (Lyapunov Stability)

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Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- From autononous to non-autonomous

- The sol of the nonautonomous syst

  $\dot{x} = f(t, x)$, starting at $x(t_0) = x_0$,

  depends on both t and t_0.

- Should refine the definitions

  to let stability hold

  uniformly in the initial time t_0.

- Need some special comparison functions.

Definition 4.2: Class K Functions

- $\alpha : [0, a) \to [0, \infty)$ is a continuous function.
- IF $\alpha(\cdot)$ is strictly increasing and $\alpha(0) = 0$,

  THEN it is said to belong to class $\mathcal{K}$

- IF $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$,

  THEN it is said to belong to class $\mathcal{K}_\infty$

Definition 4.3: Class KL Functions

- $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$

  is a continuous function.

- IF

  (1) for each fixed s,

  the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$

  w.r.t. r and,

  (2) for each fixed r,

  the mapping $\beta(r, s)$ is decreasing w.r.t. s

  and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$

- THEN it is said to belong to class $\mathcal{KL}$

Example 4.16

- $\alpha(r) = \tan^{-1}(r)$

- $\alpha(r) = r^c, \quad c > 0$

- $\alpha(r) = \min\{r, r^2\}$

Example 4.16

- $\beta(r,s) = r^c e^{-s}$, $\quad c > 0$

- $\beta(r,s) = r/(ksr+1)$, $\quad k > 0$

Lemma 4.2

- α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$,

  α_3 and α_4 be class $\mathcal{K}_\infty$ functions

  and β be a class $\mathcal{KL}$ function.

- Denote the inverse of α_i by α_i^{-1}.

- THEN

  - α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$
  - α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.
  - $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$
  - $\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$
  - $\sigma(r,s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$.

Lemma 4.3

- Let $V : D \to R$ be a continuous P.D. function defined on a domain $D \subset R^n$ that contains the origin.

- Let $B_r \subset D$ for some $r > 0$.

- Then, there exist class $\mathcal{K}$ functions α_1, α_2, defined on $[0, r]$, such that

$$\alpha_1(||x||) \;\leq\; V(x) \;\leq\; \alpha_2(||x||)$$

for all $x \in B_r$.

Lemma 4.3

- If $D = R^n$, α_1, α_2 will be defined on $[0, \infty)$ and the foregoing inequality will hold $\forall x \in R^n$.

- Moreover, if $V(x)$ is radially unbounded, then α_1, α_2 can be chosen to belong to class $\mathcal{K}_\infty$.

- If $V(x) = x^T P x$,

$$\lambda_{\text{min}}(P)||x||_2^2 \;\leq\; x^T P x \;\leq\; \lambda_{\text{max}}(P)||x||_2^2$$

Lemma 4.4

- Consider the scalar autonomous D.E.

$$\dot{y} \;=\; -\,\alpha(y), \qquad y(t_0) = y_0$$

  where $\alpha(\cdot)$ is a local Lipschitz class $\mathcal{K}$ func

  defined on $[0, a)$.

- For all $0 \leq y_0 < a$,

  it has a unique solution $y(t)$ $\forall t \geq t_0$.

- Moreover, $y(t) \;=\; \sigma(y_0, t - t_0)$

  where σ is a class $\mathcal{KL}$ function

  defined on $[0, a) \times [0, \infty)$.

Lemma 4.4: Examples

- If $\dot{y} = -ky$, $k > 0$,

- If $\dot{y} = -ky^2$, $k > 0$,

- For the proof of Thm 4.1:
- Want to choose β, δ

  such that $B_\delta \subset \Omega_\beta \subset B_r$
- So, for a P.D. function $V(x)$,
- Because

- Then

- Also, want to show that

  when $\dot{V}(x)$ is N.D., $x(t) \to 0$ as $t \to \infty$.

- Using Lemma 4.3,

  there is a class $\mathcal{K}$ function α_3

  such that $\dot{V}(x) \leq -\alpha_3(||x||)$

- Hence, $\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$

- Comparison lemma (Lemma 3.4) shows that

  $V(x(t))$ is bounded by the solution of

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \qquad y(0) = V(x(0))$$

- Lemma 4.2 shows that

  $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function.

- Lemma 4.4 shows that

  the solution is $y(t) = \beta(y(0), t)$,

  where β is a class $\mathcal{KL}$ function.

- Consequently, $V(x(t))$ satisfies

  $V(x(t)) \leq \beta(V(x(0)), t)$,

  which shows that $V(x(t)) \to 0$ as $t \to \infty$.

- $V(x(t)) \leq V(x(0))$ implies

  $\alpha_1(||x(t)||) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(||x(0)||)$

- Hence, $||x(t)|| \leq \alpha_1^{-1}(\alpha_2(||x(0)||)$,

  where $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function

- Similarly, $V(x(t)) \leq \beta(V(x(0)), t)$ implies

  $\alpha_1(||x(t)||) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(||x(0)||), t)$

- Therefore, $||x(t)|| \leq \alpha_1^{-1}(\beta(\alpha_2(||x(0)||), t))$,

  where $\alpha_1^{-1}(\beta(\alpha_2(r), t))$ is a class $\mathcal{KL}$ func.