Nonlinear Systems Analysis

Lecture Note 11

Section 4.3
Linear Systems & Linearization (Lyapunov Stability)

Feng-Li Lian
NTU-EE

Sep05 – Jan06

Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Consider the linear time-invariant system

$$\dot{x} = A\,x \qquad (4.9)$$

  has an E.P. at the origin.

- The E.P. is isolated IFF $\det(A) \neq 0$.

- If $\det(A) = 0$,
  - The matrix A has a nontrivial null space.
  - Every point in the null space of A is an E.P. for the system (4.9).
  - The system has an equilibrium subspace.

- Notice that a linear system cannot have multiple isolated equilibrium points.

- Stability properties of the origin can be characterized by the locations of the eigenvalues of A.

- From linear system theory

(a) the solution of (4.9) for a given $x(0)$

is given by

(b) for any matrix A

there is nonsingular matrix P

that transforms A into its Jordan form;

where J_i is a Jordan block

associated with the eigenvalue λ_i of A.

- A Jordan block of order one

takes the form $J_i =$

- A Jordan block of order $m > 1$

takes the form

$$J_i = \begin{bmatrix} \; & \; \\ \; & \; \end{bmatrix}_{m \times m}$$

- Therefore,

$\exp(At) \;=$

- From Linear Algebra,

  IF an $n \times n$ matrix A has
  a repeated eigenvlaue λ_i
  of algebraic multiplicity q_i,

  THEN Jordan blocks associated with λ_i
  have order one

  IFF rank $(A - \lambda_i I) = n - q_i$.

Theorem 4.5

- The E.P. $x = 0$ of $\dot{x} = Ax$ is stable

  IFF

  (1) all eigenvalues of A satisfy

  Re $\lambda_i \leq 0$ and

  (2) for every eigenvalue with Re $\lambda_i = 0$

  and algebraic multiplicity $q_i \geq 2$,

  rank $(A - \lambda_i I) = n - q_i$,

  where n is the dimension of x.

- The E.P. $x = 0$ is

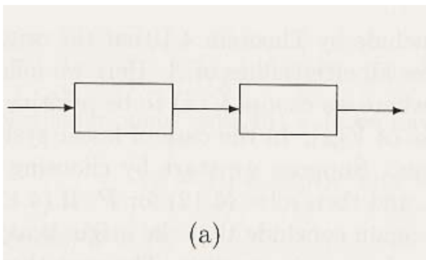
  (globally) asymptotically stable

  IFF all eigenvalues of A satisfy Re $\lambda_i < 0$.

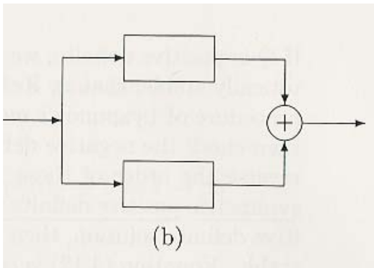
Theorem 4.5: Proof

Example 4.12

- Consider a series connection and a parallel connection of two identical systems.



(a)



(b)

- Each system is represented by

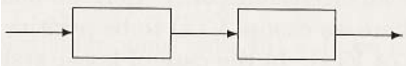
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \;\; 0] x$$

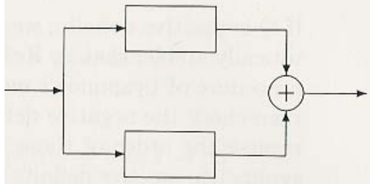
where u and y are the input and output, respectively.

Example 4.12

- Let A_s and A_p be the matrices of the series and parallel connections, when modeled without driving inputs.



(a)



(b)

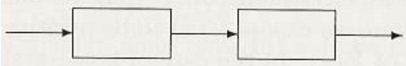
- Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

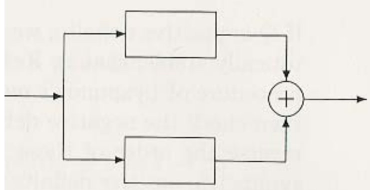
$$A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

Example 4.12

- The matrices A_p and A_s have the same e-values on the I-axis, $\pm j$, with algebraic multiplicity $q_i = 2$.



(a)



(b)

- Also, rank $(A_p - jI) = 2 = n - q_i$, while rank $(A_s - jI) = 3 \neq n - q_i$.

- Thus, by Theorem 4.5,

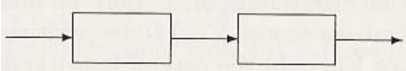
  the origin of the parallel connection is stable,

  the origin of the series connection is unstable.

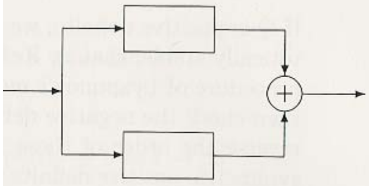
Example 4.12

- To physically see the difference between the two cases, notice that in the parallel connection, nonzero initial conditions produce sinusoidal oscillations of freq 1 rad/sec, which are bounded functions of time.

- The sum of these sinusoidal signals remains bounded.



(a)

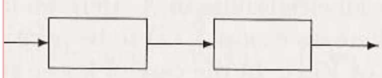


(b)

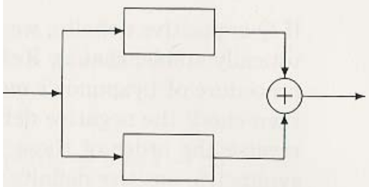
Example 4.12

- On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of freq 1 rad/sec, which acts as a driving input for the second component.

- Since the second component has an undamped natural freq of 1 rad/sec, the driving input causes "resonance" and the response grows unbounded.



(a)



(b)

- IF all eigenvalues of A satisfy Re $\lambda_i < 0$,

  THEN

  A is called a Hurwitz (stability) matrix

- The origin of $\dot{x} = A\,x$

  is asymptotically stable

  IFF A is Hurwitz.

- Lyapunovs method can be used

  to investigate

  the asymptotic stability of the origin of

  $\dot{x} = A\,x$.

- Consider a quadratic Lyapunov function

  candidate

$$V(x) = x^T P\,x$$

  where P is a real symmetric P.D. matrix.

- The derivative of V

  along the trajectories of $\dot{x} = A\,x$

  is given by

  $\dot{V}(x) =$

Theorem 4.6

- A matrix A is Hurwitz;

  that is, Re $\lambda_i < 0$ for all e-values of A,

  IFF

  for any given P.D. symmetric matrix Q

  there exists a P.D. symmetric matrix P and

  that satisfies the Lyapunov equation (4.12).

- Moreover,

  IF A is Hurwitz,

  THEN P is the unique solution of (4.12).

Theorem 4.6: Proof

- Sufficiency follows from Theorem 4.1

  with the Lyapunov function $V(x) = x^T P x$,

  as we have already shown.

- To prove necessity, assume that

  all eigenvalues of A satisfy Re $\lambda_i < 0$ and

  consider the matrix P, defined by

- Next, need to show that

  the matrix P is symmetric and P.D.

- Symmetric is from the form of P;

  P.D. will be proved in the following.

- Supposing it is not so,

  there is a vector $x \neq 0$

  such that $x^T P_x = 0$.

- Next, show that

  P is the unique solution of (4.12).

- Now, substituting (4.13) in the LHS

  of (4.12) yields

Theorem 4.6 & Proof

- To show that it is the unique solution,

  suppose there is another solution $\tilde{P} \neq P$.

- Then,

Example 4.13

- Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{12}$.

- The Lyapunov equation (4.12)

can be rewritten as

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Example 4.13

- The unique solution of this equation

  is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

- The matrix P is P.D.

  since its leading principal minors

  (1.5 and 1.25) are positive.

- Hence, all eigenvalues of A are

  in the open left-half complex plane.

Property after Linearization

- Let us go back to the nonlinear system

$$\dot{x} = f(x)$$

  where $f : D \to R^n$ is

  a continuously differentiable map

  from a domain $D \subset R^n$ into R^n.

- Suppose the origin $x = 0$ is in D and

  an E.P. for the system;

  that is, $f(0) = 0$.

- By the mean value theorem,

$f_i(x) \;=$

- The function $g_i(x)$ satisfies

- By continuity of $[\partial f/\partial x]$,

- In a small neighborhood of the origin
  we can approximate the nonlinear system
  by its linearization about the origin

$$\dot{x} = Ax, \text{ where } A = \frac{\partial f}{\partial x}(0)$$

Theorem 4.7: Lyapunov Indirect Method

- LET $x = 0$ be an E.P. for the NL syst

$$\dot{x} = f(x) \quad (\text{ or } = A\,x + g(x)\,)$$

  where $f : D \to R^n$ is

  continuously differentiable

  and D is a neighborhood of the origin.

- LET

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=0}$$

Theorem 4.7

- THEN,

  1. If Re $\lambda_i < 0$ for all eigenvalues of A,

     the origin is asymptotically stable.

  2. If Re $\lambda_i > 0$ for one or more of the eigenvalues of A,

     the origin is unstable.

Theorem 4.7: Proof Concept

- 1st part: (asymptotically stable)

  - use $V(x) = x^T P x$

  - Thm 4.6

  - Thm 4.1

- 2nd part: (unstable)

  (1) no eig(A) on the I-axis

  - assme in open RHP and open LHP

  - use Thm 4.6, Thm 4.3

  (2) some eig(A) on the I-axis

  - shift the I-axis, then work like the above

  - use Thm 4.6, Thm 4.3

Examples 4.14

- Consider the scalar system $\dot{x} = a\,x^3$

- The linearized system is:

- If $a = 0$,

  the system is linear, and

  is stable by Thm 4.5.

Examples 4.14

- If $a < 0$,

  choose $V(x) = x^4$,

- If $a > 0$,

  choose $V(x) = x^4$,

Examples 4.15: Pendulum Equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}$$

E.P. $= (0,0)$ and $(\pi, 0)$.