Nonlinear Systems Analysis

Lecture 10

Section 4.2
Invariance Principle (Lyapunov Stability)

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Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- The pendulum equation with friction:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2$$

- The energy Lyapunov function fails to satisfy the asymptotic cond. of Thm 4.1 because $\dot{V}(x) = -b x_2^2$ is only negative semidefinite.

- But, $\dot{V}(x)$ is negative everywhere, except on the line $x_2 = 0$, where $\dot{V}(x) = 0$.

- For the system to maintain $\dot{V}(x) = 0$, the trajectory of the system must be confined to the line $x_2 = 0$.

- THEN,

- Hence, on $-\pi < x_1 < \pi$ of the $x_2 = 0$ line,

  the system can maintain $\dot{V}(x) = 0$

  only at the origin $x = 0$.

- So, $V(x(t))$ must decrease toward 0

  and, consequently, $x(t) \to 0$ as $t \to \infty$,

  which is consistent with the fact that,

  due to friction,

  energy cannot remain constant

  while the system is in motion.

- LaSalle's invariance principle:

- In a domain about the origin

- IF we can find a Lyapunov function

  whose derivative along the trajectories of

  the system is negative semidefinite, and

- IF we can establish that

  except at the origin,

  no trajectory can stay identically at points

  where $\dot{V}(x) = 0$,

- THEN, the origin is asymptotically stable.

Positive Limit Set & Invariant Set – 1

- Let $x(t)$ be a solution of $\dot{x} = f(x)$ (4.1).

- IF there is a sequence $\{t_n\}$,
  with $t_n \to \infty$ as $n \to \infty$,
  such that $x(t_n) \;\to\; p$ as $n \to \infty$,
  THEN, the point p is said to be
  a positive limit point of $x(t)$.

- The set of all positive limit points of $x(t)$
  is called the positive limit set of $x(t)$.

- IF $x(0) \in M \;\Rightarrow\; x(t) \in M, \forall t \in R$,
  THEN, the set M is said to be
  an invariant set with respect to (4.1).

Positive Limit Set & Invariant Set – 2

- A set M is said to be
  a positively invariant set if

$$x(0) \in M \Rightarrow x(t) \in M, \ \forall t \geq 0$$

- We also say that
  $x(t)$ approaches a set M as $t \to \infty$,
  if for each $\epsilon > 0$ there is $T > 0$
  such that

$$\text{dist}(x(t), M) \;<\; \epsilon, \quad \forall t > T$$

where $\text{dist}(p, M) = \inf_{x \in M} ||p - x||$.

- The equilibrium point and the limit cycle
  are invariant sets,
  since any solution starting in either set
  remains in the set for all $t \in R$.

- $x(t)$ approaches M as $t \to \infty$
  does not imply that $\lim_{t\to\infty} x(t)$ exists.

- Lemma 4.1:
  a fundamental property of limit sets

- IF a solution $x(t)$ of (4.1)
  is bounded and
  belongs to D for $t \geq 0$,

- THEN its positive limit set L^+ is
  a nenempty, compact, invariant set.

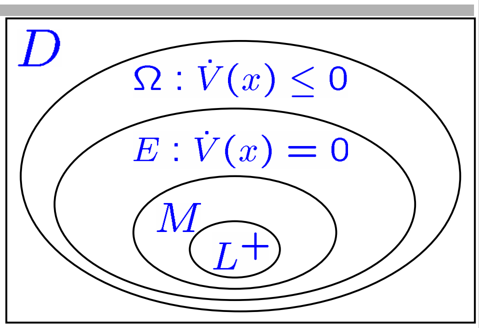
- Moreover, $x(t)$ approaches L^+ as $t \to \infty$.

- Proof: See Appendix C.3.

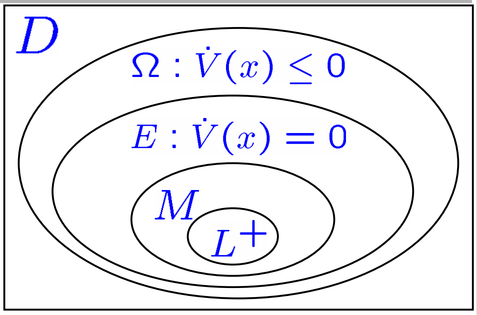
Theorem 4.4: LaSalle's Theorem

- Let $\Omega \subset D$ be a compact set that is positively invariant w.r.t. (4.1).
- Let $V : D \to R$ be a cont. diff. func. such that $\dot{V}(x) \leq 0$ in Ω.
- Let E be the set of all points in Ω where $\dot{V}(x) = 0$.
- Let M be the largest invariant set in E.
- Then every solution starting in Ω approaches M as $t \to \infty$.

Theorem 4.4: LaSalle's Theorem: Proof



Theorem 4.4: LaSalle's Theorem: Proof



Corollaries 4.1 & 4.2:

- We are interested in showing that $x(t) \to 0$ as $t \to \infty$.

- So, to show that

  the largest invariant set in E is the origin,

  OR

  no solution can stay identically in E,
  other than the trivial solution $x(t) = 0$.

Corollary 4.1: Barbashin's Theorem

- Let $x = 0$ be an E.P. for (4.1).
- Let $V : D \to R$ be a
  continuously differentiable
  positive definite function
  on a domain D containing the origin $x = 0$,
  such that $\dot{V}(x) \leq 0$ in D.
- Let $S = \left\{ x \in D \;\middle|\; \dot{V}(x) = 0 \right\}$ and
  suppose that
  no solution can stay identically in S,
  other than the trivial solution $x(t) \equiv 0$.
- THEN, the origin is asymptotically stable.

Corollary 4.2: Krasovskii's Theorem

- Let $x = 0$ be an E.P. for (4.1).
- Let $V : R^n \to R$ be a
  a continuously differentiable,
  radially unbounded,
  positive definite function
  such that $\dot{V}(x) \leq 0$ for all $x \in R^n$.
- Let $S = \left\{ x \in R^n \;\middle|\; \dot{V}(x) = 0 \right\}$ and
  suppose that
  no solution can stay identically in S,
  other than the trivial solution $x(t) \equiv 0$.
- THEN, the origin is globally asymptotically stable.

Corollaries 4.1 & 4.2:

- When $\dot{V}(x)$ is negative definite,

  $\Rightarrow \quad S = \{0\}$.

- Then, Corollaries 4.1 and 4.2 coincide with Theorems 4.1 and 4.2, respectively.

Example 4.8: Generalized Pendulum Example

- Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

where $h_1(\cdot)$ and $h_2(\cdot)$ are locally Lipschitz and satisfy

$$h_i(0) = 0,$$

$$y h_i(y) > 0, \ \forall y \neq 0 \text{ and } y \in (-a, a)$$

- Consider again the system of Example 4.8,

  but this time let $a = \infty$ and assume that

  $h_1(\cdot)$ satisfies the additional condition:

$$\int_0^z h_1(y)dy \quad \to \quad \infty \qquad \text{as} \quad |z| \to \infty$$

- The Lyapunov function

$$V(x) \quad = \quad \int_0^{x_1} h_1(y)dy \; + \; \frac{1}{2}x_2^2$$

  is

- Relax the negative definiteness requirement of Lyapunov theorem

- Estimate the region of attraction
  - Not only for $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$
  - Can be compact positively invariant set

- Used in system with an equilibrium set, rather than an isolated equilibrium point.
  - adaptive control example 4.10, sec 1.2.6

- $V(x)$ does not have to be positive definite.
  - neural network example 4.11, sec 1.2.5