Nonlinear Systems Analysis

Lecture 9

Section 4.1
Autonomous Systems (Lyapunov Stability)

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Outline

- Introduction (L9)
- Autonomous Systems (4.1 L9)
  - Basic stability definitions
  - Lyapunov's stability theorems
  - Variable gradient method
  - Region of attraction
  - Instability
- The Invariance Principle (4.2, L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L11)
- Comparison Functions (4.4, L12)
- Non-autonomous Systems (4.5, L13)
- Linear Time-Varying Systems & Linearization (4.6, L14)
- Converse Theorems (4.7, L15)
- Boundedness & Ultimate Boundedness (4.8, L16)
- Input-to-State Stability (4.9, L17)

- Stability theory plays a central role
  in systems theory and engineering.
  In this book, we will discuss
  stability of equilibrium points (Chap 4),
  input-output stability (Chap 5), and
  stability of periodic orbits (Chap 8).

- Stability of equilibrium points is usually
  characterized in the sense of Lyapunov,
  a Russian mathematician and engineer.

- An equilibrium point is stable
  if all solutions
  starting at nearby points stay nearby;
  otherwise, it is unstable.
  It is asymptotically stable
  if all solutions starting at nearby points
  not only stay nearby,
  but also tend to the equilibrium points
  as time approaches infinity.

Introduction

- **Section 4.1:**
  Basic theorems of Lyapunov's method for autonomous systems
- **Section 4.2:**
  An extension of the basic theory, LaSalle.
- **Section 4.3:**
  Stability of E.P. of $\dot{x}(t) = Ax(t)$:
  by the location of the eigenvalues of A.
- **Section 4.4:**
  Class $\mathcal{K}$ and class $\mathcal{KL}$ functions
- **Section 4.5:**
  Uniform stability,
  uniform asymptotic stability, and exponential stability for nonautonomous systems
- **Section 4.6:**
  Linear time-varying systems and linearization
- **Section 4.7:**
  Converse theorems
- **Section 4.8:**
  Boundedness and utlimate boundedness
- **Section 4.9:**
  Input-to-state stability

Autonomous Systems

- Consider the autonomous system

$$\dot{x} = f(x) \quad (4.1)$$

  where $f : D \to R^n$ is
  a locally Lipschitz map
  from a domain $D \subset R^n$ into R^n.

- Suppose $\bar{x} \in D$ is an equilibrium point of (4.1); that is, $f(\bar{x}) = 0$.
  Our goal is to characterize and study the stability of $\bar{x}$.

- For convenience,

  we state all definitions and theorems

  for the case

  when the equilibrium point is

  at the origin of R^n; that is, $\bar{x} = 0$.

- Suppose $\bar{x} \neq 0$ and

  consider the change of variables $y =$

  Then $\dot{y} =$

- In the new variable y,

  the system has equilibrium at the origin.

  Therefore, without loss of generality (wlog),

  we will always assume that

  $f(x)$ satisfies $f(0) = 0$

  and study the stability of the origin $x = 0$.

- Definition 4.1

  The equilibrium point $x = 0$ of (4.1) is

- stable: For each $\epsilon > 0$, if there is $\delta = \delta(\epsilon) > 0$ such that

$$||x(0)|| < \delta \quad \Rightarrow \quad ||x(t)|| < \epsilon, \forall t \geq 0$$

- unstable: If it is not stable.

- asymptotically stable: If it is stable and δ can be chosen such that

$$||x(0)|| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

- The pendulum example.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}$$

  has two equilibrium points

  at $(x_1 = 0, x_2 = 0)$ and $(x_1 = \pi, x_2 = 0)$.

- Consider two cases:
  - $b = 0$
  - $b > 0$

- Let $b = 0$, (neglecting friction),
  trajectories in the neighborhood
  of the first equilibrium pt are closed orbits.

- Therefore, by starting sufficiently close to
  the equilibrium point,
  trajectories can be guaranteed to stay
  within any specified ball
  centered at the equilibrium point.

- Hence, the $\epsilon - \delta$ requirement
  for stability is satisfied.

- However, the equilibrium point
  is not asymptotically stable
  since trajectories starting
  off the equilibrium point
  do not tend to it eventually.
  Instead, they remain in their closed orbits.

- Let $b > 0$, (friction is considered)
  the equilibrium point at the origin
  becomes a stable focus.

- Inspection of the phase portrait
  of a stable focus shows that
  the $\epsilon - \delta$ requirement
  for stability is satisfied.

- In addition, trajectories starting
  close to the equilibrium point
  tend to it as t tends to ∞.

- So, it is AS.

- The second equilibrium point
  at $x_1 = \pi$ is a saddle point.

- Clearly the $\epsilon - \delta$ requirement
  cannot be satisfied
  since, for any $\epsilon > 0$,
  there is always a trajectory
  that will leave the ball $\{x \in R^n \mid ||x - \bar{x}|| \leq \epsilon\}$
  even when $x(0)$ is arbitrarily close to
  the equilibrium point $\bar{x}$.

- Actually finding all solutions

  ⇒ May be difficult or even impossible.

  ⇒ Try energy concepts first.

- Define the energy of the pendulum $E(x)$ as potential energy + kinetic energy, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

$$E(x) \quad =$$

- When friction is neglected ($b = 0$), the system is conservative; that is, there is no dissipation of energy.

- Hence, $E =$ constant during the motion of the system or, in other words,

$$\frac{dE(x)}{dt} \quad =$$

- Since $E(x) = c$ forms a closed contour

  around $x = 0$ for small c,

  we can again arrive at the conclusion that

  $x = 0$ is a stable equilibrium point.

- When friction is accounted for ($b > 0$),

  energy will dissipate

  during the motion of the system,

  that is,

  along the trajectories of the system,

$$\frac{dE(x)}{dt} =$$

- Due to friction,
  E cannot remain constant indefinitely
  while the system is in motion.

- Hence, it keeps decreasing
  until it eventually reaches zero,
  showing that
  the trajectory tends to $x = 0$
  as t tends to ∞.

- Thus, by examining the derivative of E
  along the trajectories of the system,
  it is possible to determine
  the stability of the equilibrium point.

- In 1892, Lyapunov showed that
  certain other functions could be used
  instead of energy to determine stability
  of an equilibrium point.

- Let $V : D \rightarrow R$ be
  a continuously differentiable function
  defined in a domain $D \subset R^n$
  that contains the origin.

- The derivative of V
  along the trajectories of (4.1) is

$\dot{V}(x) \quad =$

- Theorem 4.1:

  Let $x = 0$ be an equilbrium point for (4.1)
  and $D \subset R^n$ be a domain containing $x = 0$.
  Let $V : D \rightarrow R$ be
  a continuously differentiable function
  such that

  $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$ (4.2)

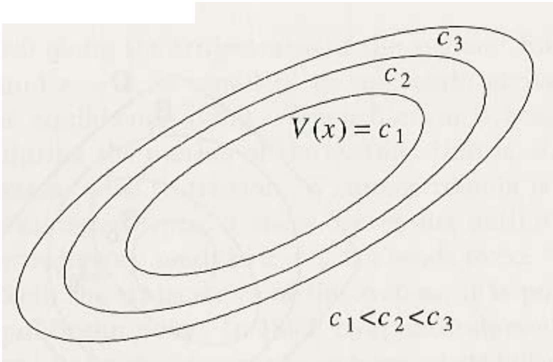
  $\dot{V}(x) \leq 0$ in D (4.3)

  Then, $x = 0$ is stable.
  Moreover, if $\dot{V}(x) < 0$ in $D - \{0\}$ (4.4)
  then $x = 0$ is asymptotically stable.

Lyapunov's Stability Theorem: Proof

- A continuously differentiable function $V(x)$ satisfying (4.2) and (4.3) is called a Lyapunov function.

- The surface $V(x) = c$, for some $c > 0$, is called a Lyapunov surface or a level surface.



- The condition $\dot{V} \leq 0$ implies that when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ and can never come out again.

- When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c.

- A function $V(x)$ satisfying condition (4.2)

  that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$,

  is said to be positive definite.

- If it satisfies the weaker condition

  $V(x) \geq 0$ for $x \neq 0$,

  it is said to be positive semidefinite.

- A function $V(x)$ is said to be

  negative definite or negative semidefinite

  if $-V(x)$ is positive definite or

  positive semidefinite, respectively.

- If $V(x)$ does not have a definite sign

  as per one of these four cases,

  it is said to be indefinite.

- Rephrase Lyapunov's theorem:
- The origin is stable

  if there is a continuously differentiable

  positive definite function $V(x)$

  so that $\dot{V}(x)$ is negative semidefinite.

- The origin is asymptotically stable

  if it is stable and

  $\dot{V}(x)$ is negative definite.

- A class of scalar functions for which

  sign definiteness can be easily checked

  is the class of functions of

  the quadratic form

  $V(x) \quad =$

- In this case,

  $V(x)$ is positive definite (positive semidefinite)

  IFF all the eigenvalues of P are positive (nonnegative),

  IFF all the leading principal minors of P are positive

  (all principal minors of P are nonnegative).

- The matrix P is positive definite

  (positive semidefinite)

  and write $P > 0$ ($P \geq 0$).

- Consider

$$V(x) = ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2$$

- The leading principal minors of P are

Example 4.1 – 2

Example 4.2: Odd Function – 1

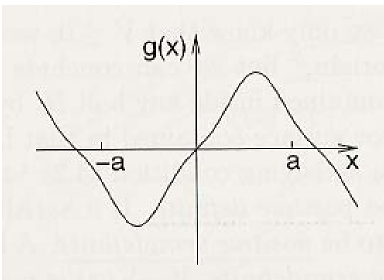
- Consider the differential equation

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0;$$

$$xg(x) > 0, \quad \forall x \neq 0 \text{ and } x \in (-a, a)$$



Example 4.3: Pendulum without Friction – 1

- Consider the pendulum eqn w/o friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1$$

and let us study the stability of the equilibrium point at the origin.

- A natural Lyapunov function candidate is the energy function

$$V(x) =$$

Example 4.4: Pendulum with Friction – 1

- Consider the pendulum eqn with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2$$

- Again, let us try

  $V(x) = a(1 - \cos x_1) + (1/2)x_2^2$

  as a Lyapunov function candidate.

$$\dot{V}(x) =$$

- Try another Lyapunov function candidate

$$V(x) \;=\; a(1 - \cos x_1) \;+$$

Variable Gradient Method: SKIP

- A procedure

  that searchs for a Lyapunov function

  in a backward manner.

- That is, investigate an expression

  for the derivative $\dot{V}(x)$ and

  go back to choose the parameters of $V(x)$

  so as to make $\dot{V}(x)$ negative definite.

- Region of attraction

  Region of asymptotic stability

  Domain of attraction

  Basin

- When the origin $x = 0$ is

  asymptotically stable,

  how far from the origin

  the trajectory can be and

  still converge to the origin

  as t approaches ∞.

- Let $\phi(t; x)$ be the solution of (4.1)
  that starts at initial state x at time $t = 0$.

- Then, the region of attraction is defined
  as the set of all points x
  such that $\phi(t; x)$ is defined for all $t \geq 0$
  and $\lim_{t\to\infty} \phi(t; x) = 0$.

- Finding the exact region of attraction
  analytically might be difficult
  or even impossible.

- However, Lyapunov functions can be used
  to estimate the sets
  contained in the region of attraction.

- From the proof of Theorem 4.1,

  if there is a Lyapunov function

  that satisfies the conditions of

  asymptotic stability over a domain D and,

  if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$

  is bounded and contained in D,

  then every trajectory starting in Ω_c

  remains in Ω_c and

  approaches the origin as $t \to \infty$.

- Thus, Ω_c is an estimate of

  the region of attraction.

- The estimate may be conservative,

  that is, it may be much smaller than

  the actual region of attraction.

- In Section 8.2, we will solve examples

  on estimating the region of attraction and

  see some ideas to enlarge the estimates.

Region of Attraction is R^n – 1

- But, Under what conditions
  will the region of attraction be
  the whole space R^n?

- For any initial state x,
  the trajectory $\phi(t; x)$ approaches the origin
  as $t \to \infty$,
  no matter how large $||x||$ is.

- If an asymptotically stable E.P.
  at the origin has this property,
  it is said to be
  globally asymptotically stable.

Region of Attraction is R^n – 2

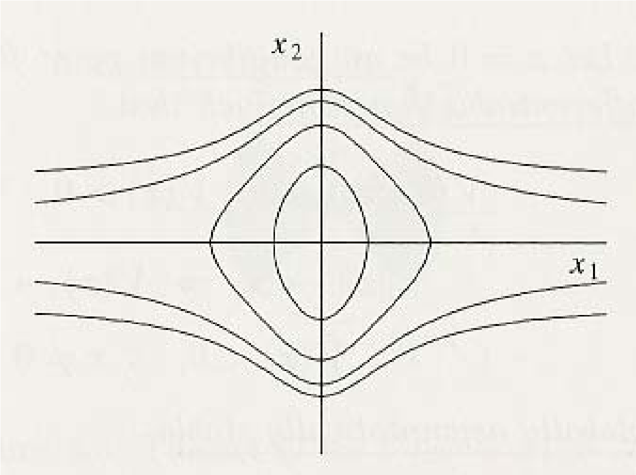
- From the proof of Theorem 4.1,
  for the global asymptotic stability,
  if $x \in R^n$ can be included
  in the interior of a bounded set Ω_c
  That is, $D = R^n$;
  but, is that enough?

- The problem is that
  for large c, the set Ω_c need not be bounded.

- For example, consider the function

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

- Fig. 4.4 shows the surfaces $V(x) = c$ for various positive values of c.



- An extra condition that ensures that

  Ω_c is bounded for all values of $c > 0$ is

$$V(x) \to \infty \text{ as } ||x|| \to \infty$$

- A function satisfying this condition

  is said to be radially unbounded.

- Theorem 4.2:

  Barbashin-Krasovskii Theorem:

- Let $x = 0$ be an E.P. for (4.1).

- Let $V : R^n \to R$

  be a continuously differentiable function

  such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \quad (4.5)$$

$$||x|| \to \infty \quad \Rightarrow \quad V(x) \to \infty \quad (4.6)$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0 \quad (4.7)$$

  then $x = 0$ is

  globally asymptotically stable.

Globally Asymptotically Stable: Proof

Example 4.6 – 1

- Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 \end{aligned}$$

where

$$\begin{aligned} a &> 0 \\ h(\cdot) &: \quad \text{locally Lipschitz} \\ h(0) &= 0 \\ yh(y) &> 0, \quad \forall y \neq 0 \end{aligned}$$

- If $x = 0$ is a G.A.S. E.P. of a system, then it must be the unique E.P. of the system.

- For if there were another E.P. $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$, $\forall t \geq 0$; hence, it would not approach the origin, which contradicts the claim that the origin is G.A.S.

- Therefore, G.A.S. is not studied for multiple equilibria systems like the pendulum equation.

Instability Theorem – 1

- Let $V : D \to R$ be
  a continuously differentiable function
  on $D \subset R^n$ that contains $x = 0$.

- Suppose $V(0) = 0$ and
  there is a point x_0 arbitrarily close to $x = 0$
  such that $V(x_0) > 0$.

- Choose $r > 0$, such that the ball
  $B_r = \{x \in R^n \mid ||x|| \leq r\}$ contained in D,
  and let $U = \{x \in B_r \mid V(x) > 0\}$ (4.8)

Instability Theorem – 2

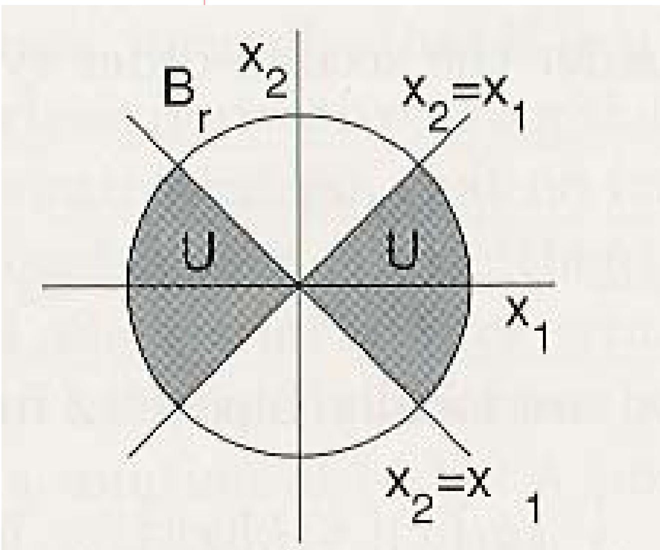
- The set U is a nonempty set
  contained in B_r.

- Its boundary is
  the surface $V(x) = 0$ & the sphere $||x|| = r$.

- Since $V(0) = 0$,
  $x = 0$ lies on the boundary of U inside B_r.

- Notice that
  U may contain more than one component.

- For example, Fig. 4.5 shows that the set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.



- The set U can be always constructed provided that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to $x = 0$.

- Theorem 4.3: Chetaev's Theorem
- Let $x = 0$ be an E.P. for (4.1).
- Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$, and $V(x_0) > 0$ for some x_0 with arbitrarily small $||x_0||$.
- Define a set U as in (4.8) and suppose that $\dot{V}(x) > 0$ in U.
- THEN, $x = 0$ is an unstable E.P.

Example 4.7 – 1

- Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $g_{1,2}(\cdot)$ are locally Lipschitz functions that satisfy the inequalities

$$\begin{aligned} |g_1(x)| &\leq k||x||_2^2, \\ |g_2(x)| &\leq k||x||_2^2 \end{aligned}$$

in a neighborhood D of the origin.

Example 4.7 – 2