Nonlinear Systems Analysis

Lecture 8

3.2: Dependence on Data
3.3: Sensitivity Analysis
3.4: Comparison Principle

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Outline

- 3.2: Continuous Dependence on Data
  - Gronwall-Bellman Inequality
  - Closeness of Solutions
  - Continuous Dependence on Initial State and Parameters
- 3.3: Sensitivity Analysis
  - Differentiability of Solutions
  - Sensitivity Equations
- 3.4: Comparison Principle

- Here, we discuss the dependence of the solution of (3.1) on the initial state x_0, and the RHS function $f(t, x)$.

- Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$.

- Dependence on x_0:

- $B_\delta(y_0) = \left\{ x \in R^n \;\middle|\; ||x - y_0|| \; < \; \delta \right\}$

- Given $\epsilon > 0$, there is $\delta > 0$ such that for all z_0 in $B_\delta(y_0)$, $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $||z(t) - y(t)|| < \epsilon$ for all $t \in [t_0, t_1]$.

- Dependence on $f(t,x)$:

- One way to look at

$$\dot{x} = f(t,x)$$
$$\dot{x}_m = f_m(t,x)$$

IF $f_m(t,x) \to f(t,x)$, as $m \to \infty$

THEN $x_m \to x$

- The other way:

- Assume that f depends continuously on a set of constant parameters; that is, $f = f(t,x,\lambda)$, where $\lambda \in R^p$.

- Let $x(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ defined on $[t_0,t_1]$, with $x(t_0,\lambda_0) = x_0$.

- And $x(t,\lambda)$ be a solution of $\dot{x} = f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $x(t_0,\lambda) = x_0$.

- The solution is said to

  depend continuously on λ

  if for any $\epsilon > 0$, there is $\delta > 0$

  such that

  for all λ in $B_\delta(\lambda_0)$,

  $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$

  defined on $[t_0, t_1]$, with $x(t_0, \lambda) = x_0$, and

  satisfies $||x(t, \lambda) - x(t, \lambda_0)|| \ < \ \epsilon$

  for all $t \in [t_0, t_1]$.

- Lemma A.1: (Gronwall-Bellman Inequality)

- Let $\lambda : [a, b] \to R$ be continuous and

  $\mu : [a, b] \to R$ be cont. and nonnegative.

- IF a continuous function

  $y : [a, b] \to R$ satisfies

$$y(t) \ \leq \ \lambda(t) \ + \ \int_a^t \mu(s) y(s) ds$$

  for $a \leq t \leq b$,

  THEN on the same interval

$$y(t) \ \leq \ \lambda(t) \ + \ \int_a^t \lambda(s) \ \mu(s) \ \exp\left(\int_s^t \ \mu(\tau) \ d\tau\right) \ ds$$

- In particular,
  if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda \exp[\int_a^t \mu(\tau)\, d\tau]$$

- If, in addition,
  $\mu(t) \equiv \mu \geq 0$ is a constant, then

$$y(t) \leq \lambda \exp[\mu(t-a)]$$

- Theorem 3.4:
- Let $f(t,x)$ be
  piecewise continuous in t and
  Lipschitz in x on $[t_0, t_1] \times W$
  with a Lipschitz constant L,
  where $W \subset R^n$ is an open connected set.

- Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y), \quad y(t_0) = y_0$$

and $$\dot{z} = f(t,z) + g(t,z), \quad z(t_0) = z_0$$

such that $y(t), z(t) \in W$ for all $t \in [t_0, t_1]$.

- Suppose that

$$||g(t,x)|| \leq \mu, \quad \forall (t,x) \in [t_0, t_1] \times W$$

for some $\mu > 0$.

- THEN, $\forall t \in [t_0, t_1]$

$$||y(t) - z(t)|| \leq ||y_0 - z_0|| \exp^{[L\,(t-t_0)]} + \frac{\mu}{L} \left\{ \exp^{[L\,(t-t_0)]} - 1 \right\}$$

- Proof:
- The solutions $y(t)$ and $z(t)$ are given by

- Substracting the two equations
  and taking norms yield

- By the Gronwall-Bellman inequality
  (Lemma A.1)

  $||y(t) - z(t)|| \leq$

- Integrating the RHS by parts, we obtain

- Theorem 3.5:
- Let $f(t, x, \lambda)$ be

  continuous in (t, x, λ) and

  locally Lipschitz in x (uniformly in t and λ)

  on $[t_0, t_1] \times D \times \{||\lambda - \lambda_0|| \leq c\}$

  where $D \subset R^n$ is an open connected set.

- Let $y(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$

  with $y(t_0, \lambda_0) = y_0 \in D$.

- Suppose $y(t, \lambda_0)$ is defined and

  belongs to D for all $t \in [t_0, t_1]$.

- Then, given $\epsilon > 0$, there is $\delta > 0$

  such that IF

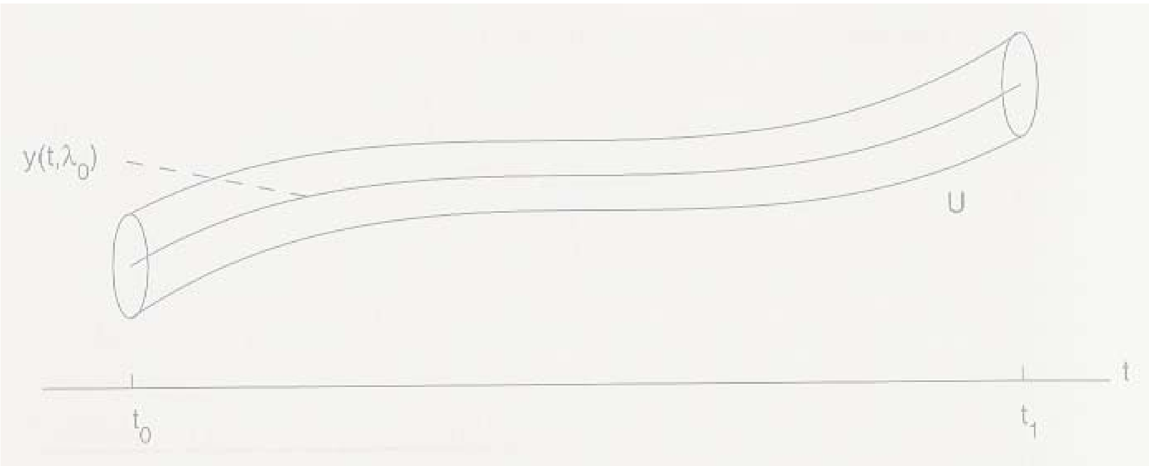
  THEN there is a unique solution $z(t, \lambda)$

  of $\dot{x} = f(t, x, \lambda)$

  defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$,

  and $z(t, \lambda)$ satisfies

- Proof Concept:



- Suppose that

  $f(t, x, \lambda)$ is continuous in (t, x, λ) and

  has continuous first partial derivatives

  w.r.t. x and λ

  for all $(t, x, \lambda) \in [t_0, t_1] \times R^n \times R^p$.

- Let λ_0 be a nominal value of λ, and

  suppose that

  the nominal state equation $\dot{x} = f(t, x, \lambda_0)$,

  with $x(t_0) = x_0$

  has a unique solution $x(t, \lambda_0)$ over $[t_0, t_1]$.

- From Thm 3.5,

  for all λ sufficiently close to λ_0,

  that is, $||\lambda - \lambda_0||$ sufficiently small,

  $\dot{x} = f(t, x, \lambda)$, with $x(t_0) = x_0$

  has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$

  that is close to the nominal solution $x(t, \lambda_0)$.

- The continuous differentiability of f

  w.r.t. x, λ

  implies the additional property that

  the solution $x(t, \lambda)$ is differentiable

  w.r.t. λ near λ_0.

- To see that, write

$x(t,\lambda) =$

- Take partial derivatives wrt λ yields

$x_\lambda(t,\lambda) =$

- Differentiating wrt t,
  it can be seen that $x_\lambda(t,\lambda)$ satisfies

$\frac{\partial}{\partial t} x_\lambda(t,\lambda) =$

$A(t,\lambda) =$

$B(t,\lambda) =$

- For λ sufficiently close to λ_0,

  the matrices $A(t, \lambda)$ and $B(t, \lambda)$ are

  defined on $[t_0, t_1]$.

  Hence, $x_\lambda(t, \lambda)$ is defined

  on the same interval.

- At $\lambda = \lambda_0$,

  the RHS of (3.4) depends only on

  the nominal solution $x(t, \lambda_0)$.

- Let $S(t) = x_\lambda(t, \lambda_0)$;

  then $S(t)$ is the unique solution of

- $S(t)$ is called the sensitivity function, and

  (3.5) is called the sensitivity equation.

- Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions.

- For small $||\lambda - \lambda_0||$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$:

- Procedure for calculating $S(t)$:
  - Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$
  - Evaluate the Jacobian matrices

$$A(t, \lambda_0) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right|_{x=x(t,\lambda_0), \lambda=\lambda_0}$$

$$B(t, \lambda_0) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right|_{x=x(t,\lambda_0), \lambda=\lambda_0}$$

  - Solve the sensitivity equation (3.5) for $S(t)$.

- Alternative approach for calculating $S(t)$:

$$\dot{x} = f(t, x, \lambda_0), \quad x(t_0) = x_0,$$

$$\dot{S} = \left[\frac{\partial f(t,x,\lambda)}{\partial x}\right]_{\lambda=\lambda_0} S + \left[\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right]_{\lambda=\lambda_0}$$

$$S(t_0) = 0$$

which is solved numerically.

- Sometimes we only want to compute the bounds of $x(t)$ without solving it.

- The Gronwall-Bellman Inequality is a tool. Another tool is the comparison lemma.

- Consider a differential inequality

  $\dot{v}(t) \leq f(t, v(t))$

  and a differential equation

  $\dot{u}(t) = f(t, u(t))$.

- And two facts:
  - If $v(t)$ is differentiable at t,
    then $D^{+}v(t) = \dot{v}(t)$.
  - If $\frac{1}{h}|v(t+h) - v(t)| \leq g(t,h)$, $\forall h \in (0,b]$
    and $\lim_{h \to 0^+} g(t,h) = g_0(t)$
    then $D^{+}v(t) \leq g_0(t)$.

upper RH derivative:

$$D^{+}v(t) = \lim_{h \to 0^+} \sup \frac{v(t+h) - v(t)}{h}$$

The limit $h \to 0^+$ means that$h$ approaches zero from above.

- Lemma 3.4: (Comparison Lemma)
- Consider $\dot{u} = f(t,u)$, $u(t_0) = u_0$
  where $f(t,u)$ is
  continuous in t and
  locally Lipschitz in u,
  for all $u \in J \subset R$.
- Let $[t_0, T)$ (T could be infinity)
  be the maximal interval of existence
  of the solution $u(t)$,
  and suppose $u(t) \in J$ for all $t \in [t_0, T)$.

- Let $v(t)$ be a continuous function whose upper RH derivative $D^+v(t)$ satisfies the differential inequality $D^+v(t) \leq f(t, v(t))$, $v(t_0) \leq u_0$ with $v(t) \in J$ for all $t \in [t_0, T)$.

- Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

- Example 3.8:

- Consider the scalar D.E.

$$\dot{x} = f(x) = -(1 + x^2)x, \quad x(0) = a$$

has a unique solution on $[0, t_1)$, for some $t_1 > 0$, because $f(x)$ is local Lipschitz.

- Let

- $v(t)$ is differentiable and

  its derivative is given by

- Hence,

  $v(t)$ satisfies the differential inequality

- Let $u(t)$ be the solution of the D.E.

- Then, by the comparison lemma,

  the solution $x(t)$ is defined for all $t \geq 0$

  and satisfies