Nonlinear Systems Analysis

Lecture 7

3.1: Existence & Uniqueness

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Outline

- 3.1: Existence & Uniqueness
  - Introduction
  - Local Existence and Uniqueness
  - Lipschitz Property & Continuity
  - Global Existence and Uniqueness

- Fundamental properties of solutions of ODEs:

  existence,

  uniqueness,

  continuous dependence on initial conditions,

  and continuous dependence on parameters.

- Starting an experiment at t_0,

  we expect that

  the system will move and

  its states will be defined at $t > t_0$.

- With a deterministic system,

  we expect that

  we can repeat the experiment exactly,

  i.e. get same motion and same state

  at $t > t_0$.

- To obtain this prediction,

  the initial-value problem

  $\dot{x} = f(t, x), \qquad x(t_0) = x_0$

  must have a unique solution.

- The existence and uniqueness
  can be ensured
  by imposing some constraints on $f(t, x)$.

- The key constraint is
  the Lipschitz condition:

  for all (t, x) and (t, y)
  in some neighborhood of (t_0, x_0).

- An essential factor
  in the validity of any math model is
  the continuous dependence of its solutions
  on the data of the problem.

- The data are
  the initial state x_0,
  the initial time t_0, and
  the $f(t, x)$.

- Arbitrarily small errors in the data will not
  result in large errors in the solutions.

- Sensitivity equations

  to describe the effect

  of small parameter variations

  on the performance of the system.

- Comparison principle

  to bound the solution

  of a scalar differential inequality

  by the solution of

- To study the sufficient conditions

  for the existence and uniqueness

  of the solution

  of the initial-value problem (3.1).

- A solution of (3.1) over interval $[t_0, t_1]$ is

  a continuous function $x : [t_0, t_1] \to R^n$

  such that

  $\dot{x}(t)$ is defined and

  $\dot{x} = f(t, x(t))$ for all $t \in [t_0, t_1]$.

- If $f(t, x)$ is continuous in t and x, then the solution $x(t)$ will be continuously differentiable.

- If $f(t, x)$ is continuous in x, but only piecewise continuous in t, then a solution $x(t)$ could only be piecewise continuously differentiable.

- A ball: $B_r(x_0) = \left\{ x \in R^n \;\middle|\; ||x - x_0|| \le r \right\}$

- Theorem 3.1

  (Local Existence and Uniqueness)

  Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition

$$||f(t, x) - f(t, y)|| \le L||x - y||$$

  $\forall x, y \in B_r(x_0), \quad \forall t \in [t_0, t_1]$.

  Then, there exists some $\delta > 0$ such that $\dot{x} = f(t, x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

- Proof:
- First, $x(t)$ satisfies both the following eqns:

- View its RHS as a mapping of the continuous function $x : [t_0, t_1] \to R^n$,
- Denote it by $(Tx)(t)$,
- Write it as $x(t) = (Tx)(t)$
- Note that $(Tx)(t)$ is continuous in t.
- A solution of it is a fixed point of the mapping T that maps x into Tx.
- Existence of a fixed pint can be established by using the contraction mapping theorem.

- We need to define a Banach space χ
  and a closed set $S \subset \chi$
  such that T maps S into S
  and is a contraction over S.
- Let $\chi =$
  (set of all cont. fun.)
  with norm $||x||_C =$
  and $S =$
- We restrict the choice of δ
  to satisfy $\delta \leq \; t_1 - t_0$
  so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$.

- Notice that $||x(t)||$ dentoes a norm on R^n,
  while $||x||_C$ denotes a norm on χ.
- Also, B is a ball in R^n,
  while S is a ball in χ.
- By definition, T maps χ into χ.

- To show that T maps S into S, write

  $(Tx)(t) - x_0$

- By piecewise continuity of f,

  we know that $f(t, x_0)$ is bounded on $[t_0, t_1]$.

  Let $h = \max_{t \in [t_0, t_1]} ||f(t, x_0)||$.

- Using the Lipschitz condition and

  the fact that for each $x \in S$,

  $||x(t) - x_0|| \;\; \leq \;\; r, \quad \forall t \in [t_0, t_0 + \delta]$,

  we obtain

  $||(Tx)(t) - x_0|| \;\; \leq$

- To show that

  T is a contraction mapping over S:

- Let $x, y \in S$, consider

  $||(Tx)(t) - (Ty)(t)|| =$

- Therefore, for $\delta \leq \frac{\rho}{L}$,

  $||Tx - Ty||_C \quad \leq$

- Choosing $\rho < 1$ and $\delta \leq \rho/L$ ensures that

  T is a contraction mapping over S.

- By the contraction mapping theorem,

  if δ is chosen to satisfy

  $\delta \quad \leq$

  then (C.2) will havea unique solution in S.

- Our final goal is to establish

  uniqueness of the solution

  among all continuous functions $x(t)$,

  that is, uniqueness in χ.

- It turns out that

  any solution of (C.2) in χ will lie in S.

- Note that

  since $x(t_0) = x_0$ is inside the ball B,

  any continuous solution $x(t)$ must

  lie inside B for some interval of time.

- Suppose that

  $x(t)$ leaves the ball B and

  let $t_0 + \mu$ be the first time

  $x(t)$ intersects the boundary of B.

  Then, $||x(t_0 + \mu) - x_0|| = r$.

- On the other hand, for all $t \leq t_0 + \mu$,

  $||x(t) - x_0|| \quad \leq$

- Hence,

  the solution $x(t)$ cannot leave the set B

  within the time interval $[t_0, t_0 + \delta]$,

  which implies that

  any solution in χ lies in S.

- Consequently,

  uniqueness of the solution in S

  implies uniqueness in χ.

- QED

- A function is Lipschitz in x

  with a Lipschitz constant: L

- A function $f(x)$ is said to be

  local Lipschitz on a domain

  (open and connected set) $D \subset R^n$

  if each point of D has a neighborhood D_0

  such that

  f satisfies the Lipschitz condition (3.2)

  for all points in D_0

  with some Lipschitz constant L_0.

- A local Lipschitz function on a domain D
  is not necessarily Lipschitz on D,
  since the Lipschitz condition may not hold
  uniformly (with the same constant L)
  for all points in D.

- A local Lipschitz function on a domain D
  is Lipschitz on every compact
  (cloesed and bounded) subset of D.

- A function $f(x)$ is said to be
  globally Lipschitz
  if it is Lipschitz on R^n.

- Lemma 3.1 shows
  how a Lipschitz constant can be calculated
  using knowledge of $[\partial f / \partial x]$.

- Lemma 3.1

- Let $f : [a, b] \times D \to R^m$
  be continuous for some domain $D \subset R^n$.

- Suppose that
  $[\partial f / \partial x]$ exists and
  is continuous on $[a, b] \times D$.

- For a convex subset $W \subset D$,

  if there is a constant $L \geq 0$ such that

  on $[a, b] \times W$,

  then

  for all $t \in [a, b]$, $x, y \in W$.

- Lemma 3.2

  If $f(t, x)$ and $[\partial f / \partial x](t, x)$ are continuous

  on $[a, b] \times D$, for some domain $D \subset R^n$,

  then f is local Lipschitz in x on $[a, b] \times D$.

- Lemma 3.3

  If $f(t, x)$ and $[\partial f / \partial x](t, x)$ are continuous

  on $[a, b] \times R^n$,

  then f is globally Lipschitz in x on $[a, b] \times R^n$

  iff $[\partial f / \partial x]$ is uniformly bounded

  on $[a, b] \times R^n$.

- If $f(x)$ is Lipschitz on W,

  then it is uniformly continuous on W

  (Exercise 3.20).

  The converse is not true.

- The Lipschitz property is stronger

  than continuity.

- Lemma 3.2 shows that

  the Lipschitz property is weaker

  than continuous differentiability.

- Example

$$f(x) = \begin{bmatrix} \sin 3x_1 + \cos 3x_2 \\ 2\cos x_1 - \sin x_2 \end{bmatrix}$$

- Example 3.1

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

- Example 3.2

$$f(x) = \begin{bmatrix} x_2 \\ -\text{sat}(x_1 + x_2) \end{bmatrix}$$

- Theorem 3.2

  (Global Existence and Uniqueness)

  Suppose that

  $f(t, x)$ is piecewise continuous in t

  and satisfies

  $$||f(t,x) - f(t,y)|| \;\; \leq \;\; L \, ||x - y||$$

  $\forall x, y \in R^n, \;\; \forall t \in [t_0, t_1]$.

  Then, the state equation $\dot{x} = f(t, x)$,

  with $x(t_0) = x_0$,

  has a unique solution over $[t_0, t_1]$.

- Local Lipschitz property of a function is basically a smoothness requirement.
  It is implied by continuous differentiability.
  Except for discontinuous nonlinearities,
  it is reasonable to expect
  models of physical sysems
  to have locally Lipschitz RHS functions.

- Global Lipshitz property is restrictive.

- The following Theorem 3.3 shows that
  global existence and uniqueness only needs
  the local Lipschitz property of f
  at the expense of having to know
  more about the solution of the system.

- Theorem 3.3

  (Global Existence and Uniqueness)

- Let $f(t, x)$ be piecewise continuous in t

  and local Lipschitz in x

  for all $t \geq t_0$ and all x in a domain $D \subset R^n$.

- Let W be a compact subset of D, $x_0 \in W$,

  and suppose it is known that

  every solution of $\dot{x} = f(t, x), \;\; x(t_0) = x_0$

  lies entirely in W.

- Then, there is a unique solution

  that is defined for all $t \geq t_0$.