Nonlinear Systems Analysis

Lecture 5

2.4: Limit Cycles
2.6: Existence of Periodic Orbits

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Outline

- 2.4: Limit Cycles
  - Van der Pol Equation
- 2.6: Existence of Periodic Orbits
  - Poincare-Bendixson Criterion
  - Bendixson Criterion
  - Index Method

- A system oscillates

  when it has a nontrivial periodic solution:

- The image of a periodic solution

  in the phase portrait

  is a closed trajectory,

  which is usually called

  a periodic orbit or a closed orbit.

- In 2nd-order linear system: Oscillation

  - with eigenvalues

  - $x = 0$ is a

  - the solution:

$$z_1(t) =$$

$$z_2(t) =$$

  where

  - the harmonic oscillator

- Two fundamental problems
  with the linear oscillator:

  1. robustness:
     perturbation will destroy the oscillation.
     the linear oscillator is
     not structurally stable.

  2. the amplitude of oscillation is
     dependent on the initial conditions.

- It is possible to build
  physical nonlinear oscillators
  such that

  1. the nonlinear oscillator is
     structurally stable.

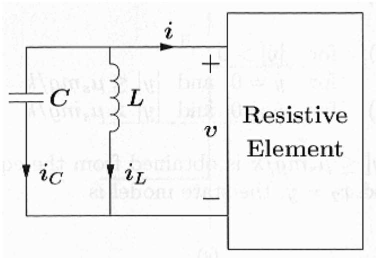
  2. the amplitude of oscillation
     (at steady state)
     is independent of initial conditions.

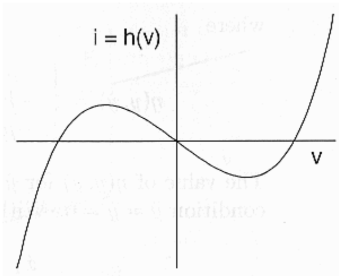
- The negative-resistance oscillator:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \epsilon h'(x_1) x_2$$

the system has only one EP
at $x_1 = x_2 = 0$.

1.2.4: Page 12





$$h(0) = 0$$
$$h'(0) < 0$$
$$h(v) \to \infty \text{ as } v \to \infty$$
$$h(v) \to -\infty \text{ as } v \to -\infty$$

- The Jacobian matrix:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\epsilon h'(0) \end{bmatrix}$$

- Since $h'(0) < 0$,
  the origin is
  either an unstable node
  or unstable focus,
  depending on the value of $\epsilon h'(0)$.

- All trajectories starting near the origin
  would diverge away from it
  and head toward infinity.

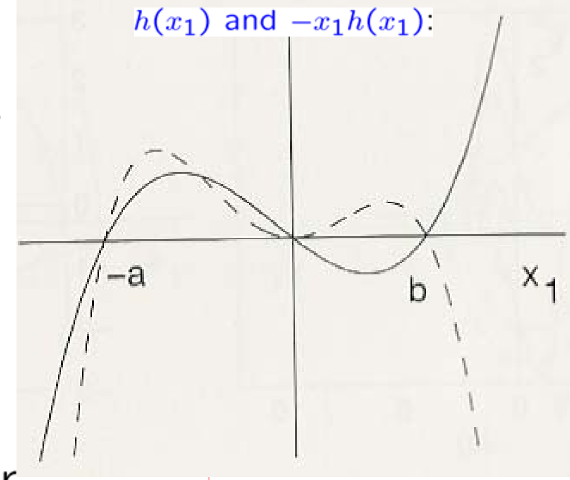
- The resistive element is "active",
  and supplies energy.

- The total energy stored in the capacitor and inductor at any time t is given by:

- The rate of change of energy is given by:

$$\dot{E} =$$

- Near the origin, the trajectory gains energy since for small $|x_1|$, $x_1 h(x_1)$ is negative.



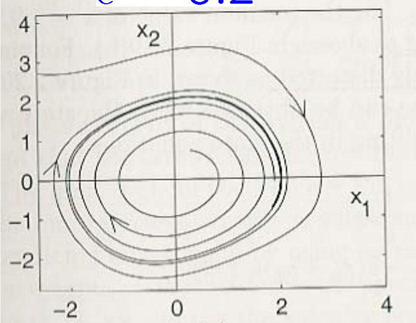
- Also, the trajectory gains energy within the strip $-a < x_1 \leq b$, and loses energy outside the strip.
- A stationary oscillation will occur if, along a trajectory, the net exchange of energy over one cycle is zero.

- **Example 2.6** Van der Pol equation:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2\end{aligned}$$

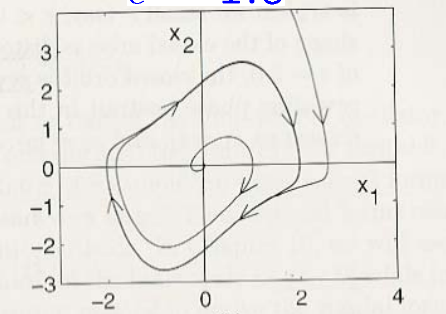
- For $\epsilon = 0.2, 1.0, 5.0$ are shown in Figs.

$\epsilon = 0.2$



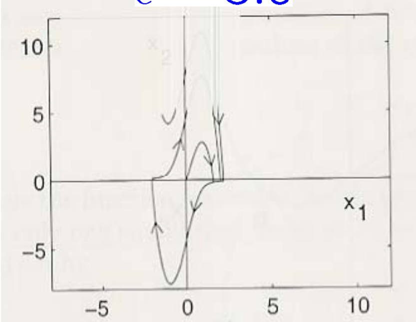
(a)

$\epsilon = 1.0$



(b)

$\epsilon = 5.0$



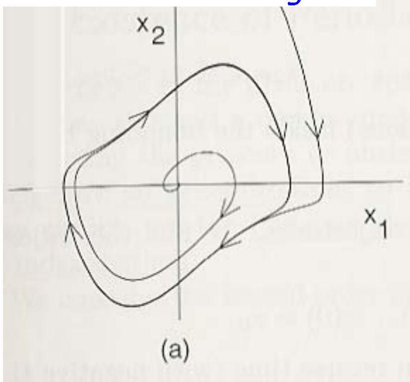
(a)

2.4: Stable & Unstable Limit Cycles

- An isolated periodic orbit is called a limit cycle.

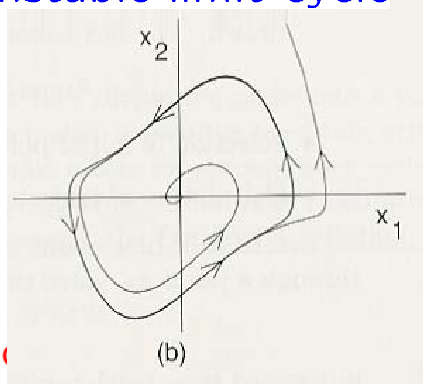
- stable and unstable limit cycles:

stable limit cycle



stable:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

unstable limit cycle



unstable:
$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 - \epsilon(1 - x_1^2)x_2$$

- Two special forms:
  - $\epsilon \to 0$: the averaging method
  - $\epsilon \to \infty$: the singular perturbation meth

2.6: Existence of Periodic Orbits – 1

- Periodic orbits in the plane are special that they divide the plane into a region inside the orbit and a region outside it.

- This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems.

- The most celebrated of these criteria are
  the Poincaré-Bendixson theorem,
  the Bendixson criterion, and
  the index method.

- Theorem (Poincaré-Bendixson):

  Let γ^+ be a bounded positive semiorbit of

  $\dot{x} = f(x)$,

  i.e., $\gamma^+(y) = \left\{ \phi(t, y) \mid 0 \leq t < \infty \right\}$

  and L^+ be its positive limit set.

  If L^+ contains no EP,

  then it is a periodic orbit.

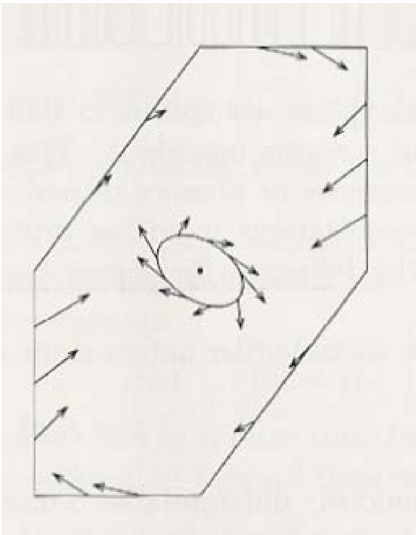
- Lemma 2.1, Presence of Limit Cycles (Poincaré-Bendixson Criterion):
  Consider $\dot{x} = f(x)$ and
  let M be a closed bounded subset
  of the plane,
  such that

  - M (1) contains no EP, OR
    (2) contains only one EP
    such that the Jcaobian matrix $[\partial f / \partial x]$
    at this point has eigenvalues
    with positive real parts.
    (Hence, the EP is unstable focus or node.)

  - Every trajectory starting in M
    stays in M for all future time.

- Then, M contains
  a periodic orbit of $\dot{x} = f(x)$.

- Intuition:
  Bounded trajectories in the plane
  will have to approach
  periodic orbits or equilibrium points
  as time tends to infinity.



- If M contains no EP,
  then it must contain a periodic orbit.

- If M contains only one EP
  that satisfies the stated conditions,
  then in the vicinity of that point
  all trajecotries will be moving away from it.

- Therefore, we can choose
  a simple closed curve around the EP
  such that the vector field on the curve
  points outward.

- Consider a simple closed curve
  defined by $V(x) = c$,
  where $V(x)$ is continuously differentiable.

- The vector field $f(x)$
  at a point x on the curve
  points inward
  if the inner product of $f(x)$ and
  the gradient vector $\nabla V(x)$ is ______________;
  that is,

  $f(x) \cdot \nabla V(x) \ =$

- The vector field $f(x)$ points outward
  if $f(x) \cdot \nabla V(x)$________0.

- It is tangent to the curve
  if $f(x) \cdot \nabla V(x)$________0.

- Trajectories can leave a set
  only if the vector field points outward
  at some points on its boundary.

- For a set of the form $M = \{V(x) \leq c\}$,
  for some $c > 0$,
  trajectories are trapped inside M
  if $f(x) \cdot \nabla V(x)$________0
  on the boundary $V(x) = c$.

- For annular region of the form
  $M = \{W(x) \geq c_1$ and $V(x) \leq c_2\}$,
  for some $c_1 > 0, c_2 > 0$
  trajectories are trapped inside M
  if $f(x) \cdot \nabla V(x)$________0 on $V(x) = c_2$
  and $f(x) \cdot \nabla W(x)$________0 on $W(x) = c_1$.

- **Example 2.7:**

- Consider the harmonic oscillator:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

  the annual region $M = \{c_1 \leq V(x) \leq c_2\}$,
  where $V(x) = x_1^2 + x_2^2$ and $c_2 > c_1 > 0$.

- The set M is

- Trajectories are trapped inside M since $f(x) \cdot \nabla V(x)$ _______ 0 everywhere.

- By PBC, there is a periodic orbit in M.

- PBC assures the existence of a periodic orbit, but not its uniqueness.

- Harmonic oscillator has a continuum of periodic orbits in M.

- **Example 2.8**: The system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

has a unique EP at (0,0).

- the Jacobian matrix:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} =$$

- Let $M = \{V(x) \le c\}$,
  where $V(x) = x_1^2 + x_2^2$ and $c > 0$.

- M is

- On the surface $V(x) = c$, we have:

  $f(x) \cdot \nabla V(x)$

  $=$

- **Example 2.9**:

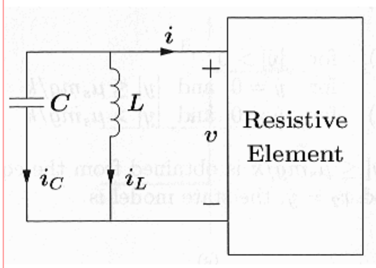
  The negative-resistance oscillator:

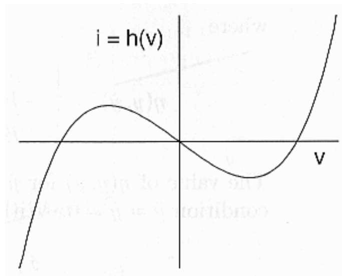
$$\ddot{v} + \epsilon h'(v)\dot{v} + v = 0$$

  where ϵ is a positive constant

  h satisfies the conditions:

$$h(0) = 0, \qquad h'(0) < 0,$$
$$\lim_{v \to \infty} h(v) = \infty, \quad \lim_{v \to -\infty} h(v) = -\infty$$





- To simplify the analysis,

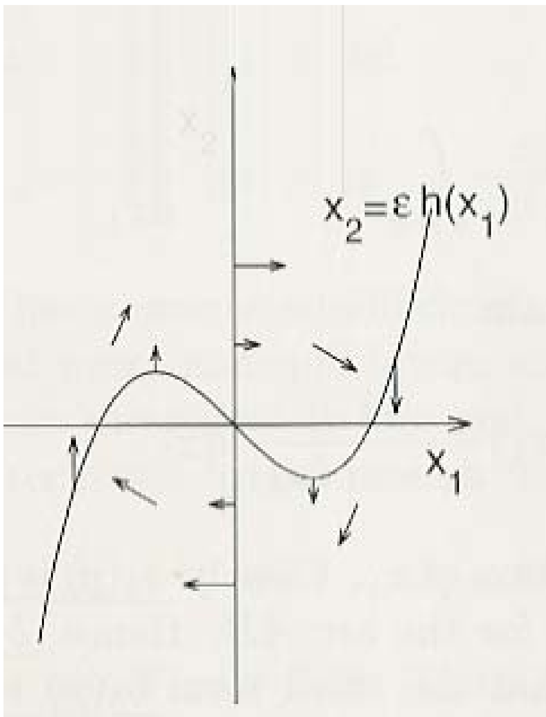
  we impose the additional requirements:

$$h(v) = -h(-v),$$
$$h(v) < 0 \text{ for } 0 < v < a,$$
$$h(v) > 0 \text{ for } v > a$$

$$\begin{aligned} h(0) &= 0 \\ h'(0) &< 0 \\ h(v) \to \infty &\quad \text{as} \quad v \to \infty \\ h(v) \to -\infty &\quad \text{as} \quad v \to -\infty \end{aligned}$$

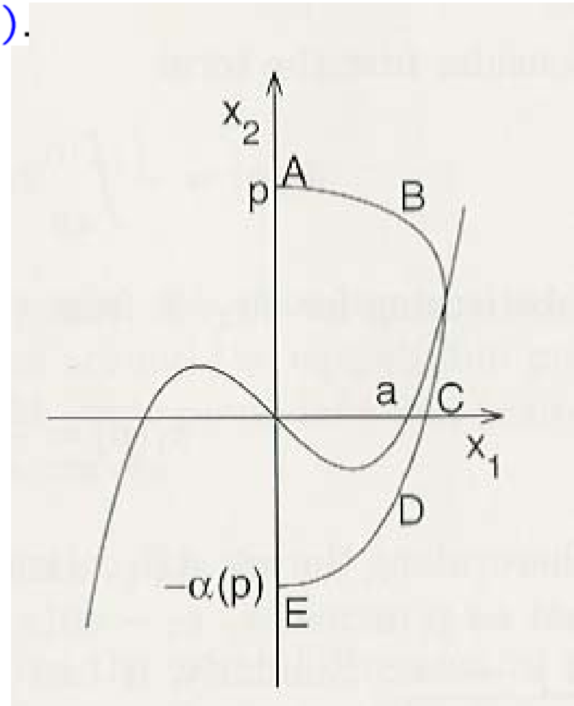
- Choose the state variables as:

- The state model as:



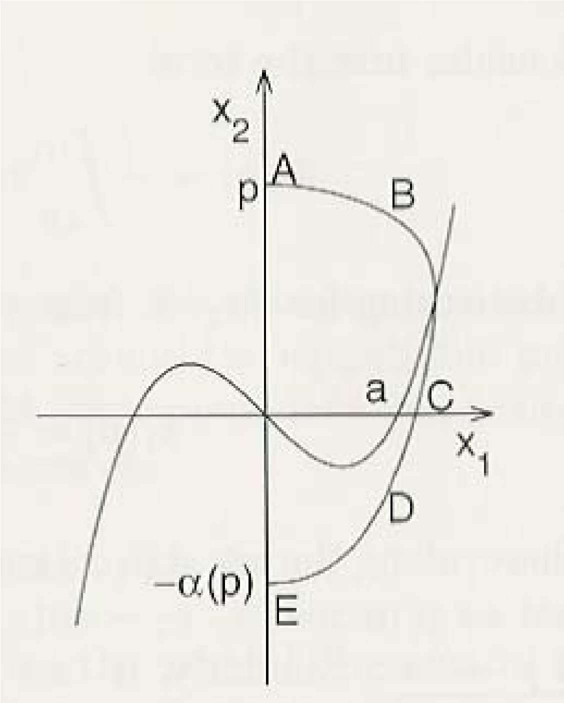
- A solution

  from $A = (0, p)$ to $E = (0, -\alpha(p))$.



- Consider the function $V(x)$:

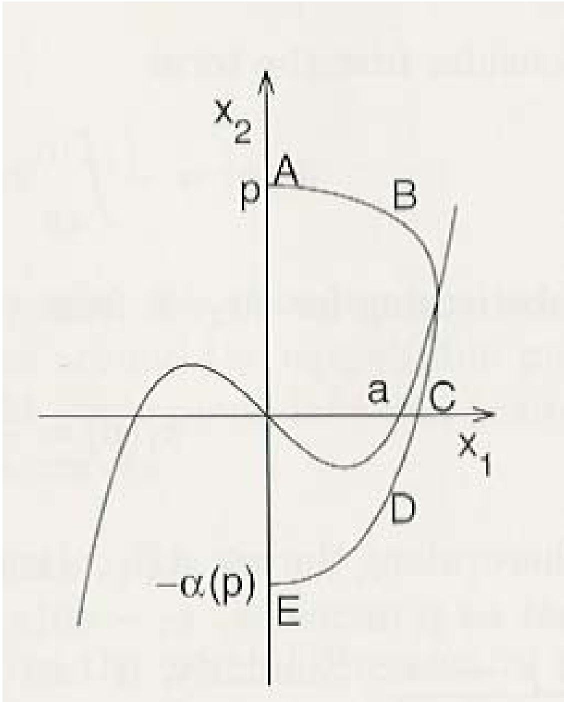
- The time derivative of $V(x)$



2.6: Negative-Resistance Oscillator – 5

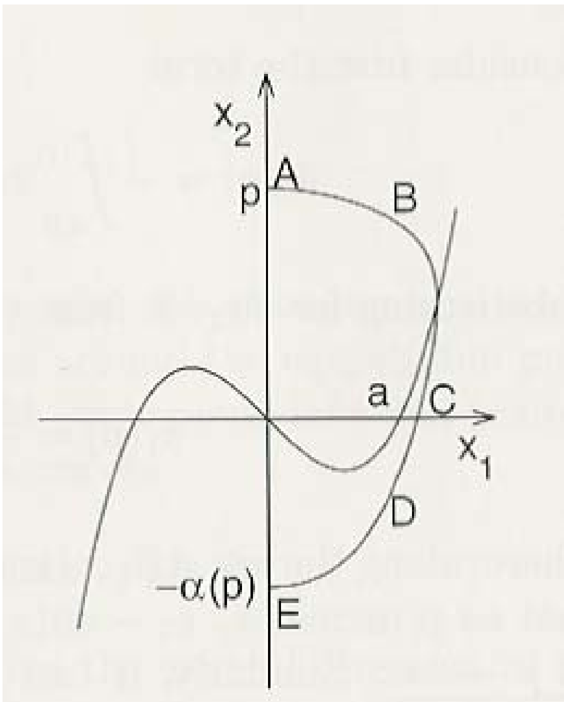
- Now,

$\delta(p) = V(E) - V(A) =$

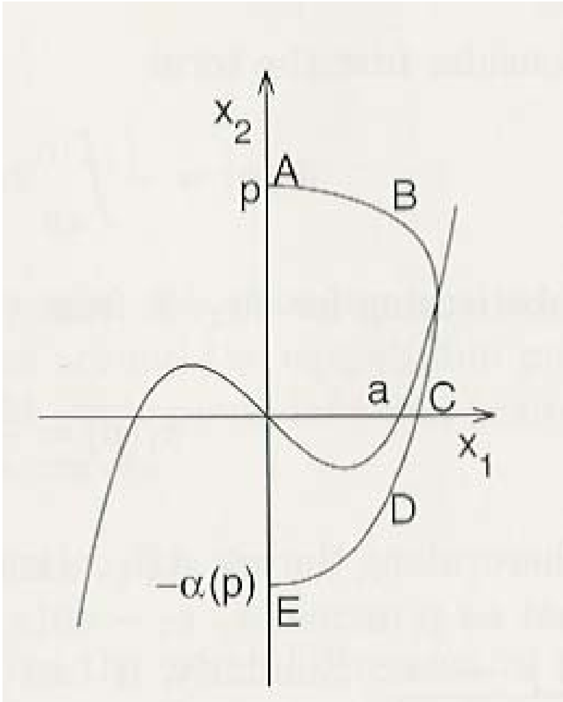


2.6: Negative-Resistance Oscillator – 6

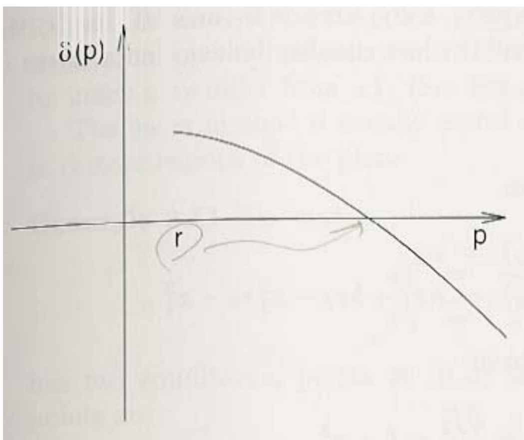
- Consider the first term $\delta_1(p)$:



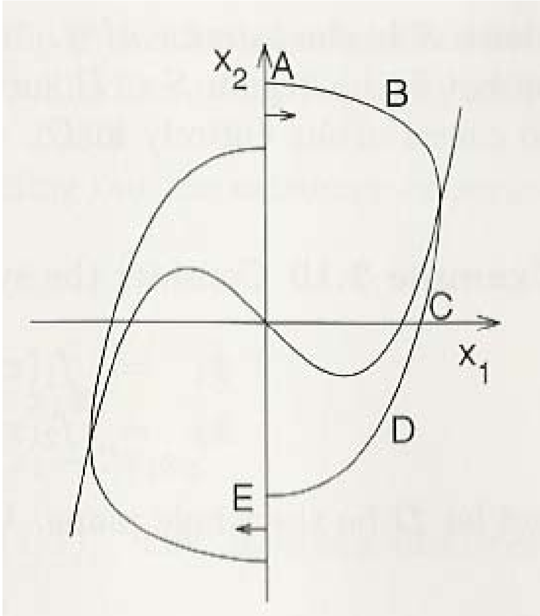
- Consider the second term $\delta_2(p)$:



- In summary:



- Since $h(\cdot)$ is an odd function, due to its symmetry, if (x_1, x_2) is a solution, then so is $(-x_1, -x_2)$. See Fig.



- Let M be the region enclosed by this closed curve.

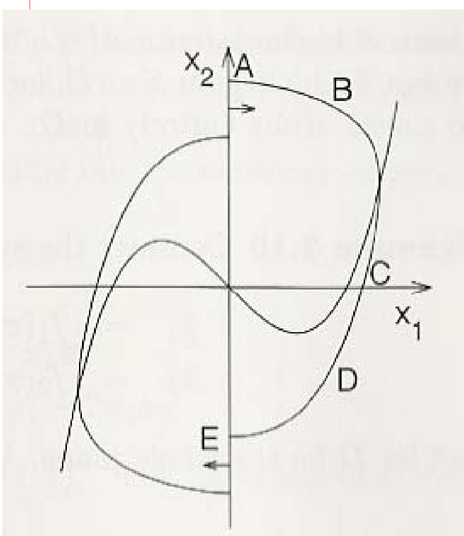
- Then very trajectory starting in M at $t = 0$ will remain inside for all $t \geq 0$.

- M is closed, bounded, and has a unique EP at the origin.

- The Jacobian matrix at the origin:

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\epsilon h'(0) \end{bmatrix}$$

has eigenvalues wiht positive real parts since $h'(0) < 0$.
By PBC, there is a closed orbit in M.



- This closed orbit is unique iff $\alpha(p) = p$.
Only one value of p, see Fig.

- Every nonequilibrium solution spirals toward the unique closed orbit.

- To rule out the existence of periodic orbits:

- Lemma 2.2, Absence of Limit Cycles (Bendixson Criterion)
  If, on a simply connected region D of the plane,
  the express $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$
  is not identially zero and
  does not change sign,
  then $\dot{x} = f(x)$ has no periodic orbits
  lying entirely in D.

- Proof:

- **Example 2.10**: Consider the system:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) = ax_1 + bx_2 - x_1^2 x_2 - x_1^3 \end{aligned}$$

and let D be the whole plane.

- Consider $\dot{x} = f(x)$
- Let C be a simple closed curve not passing through any of its EP.
- Consider the orientation of the vector field $f(x)$ at a point $p \in C$.

- Letting p traverse C in the counterclockwise direction, the vector $f(x)$ rotates continuously and, upon returning to the original position, must have rotated an angle $2\pi k$, $k \in Z$, where the angle is measured counterclockwise.

- The integer k is called the index of the closed curve C.

- If C is chosen to encircle a single isolated EP $\bar{x}$, then k is called the index of $\bar{x}$.

- Lemma 2.3:

(a) The index of a node, a focus, or a center is $+1$.

(b) The index of a (hyberbolic) saddle is -1.

(c) The index of a closed orbit is $+1$.

(d) The index of a closed curve not encircling any EP is 0.

(e) The index of a closed curve is equal to the sum of the indices of the EP within it.

- Colollary 2.1:

  Inside any periodic orbit γ,

  there must be at least one EP

  Suppose the EPs inside γ are hyperbolic,

  then if N is the number of nodes and foci

  and S is the number of saddles,

  it must be that $N - S = 1$.

- An EP is hyperbolic

  if the Jacobian at that point has

  no eigenvalues on the imaginary axis.

- If the EP is not hyperbolic,

  then its index may differ from ± 1.

- The index method is usually useful

  in ruling out the existence of periodic orbits

  in certain regions of the plane.

2.6: Poincaré Index of an EP – 6

- **Example 2.11**: The system:

$$\dot{x}_1 = -x_1 + x_1 x_2$$
$$\dot{x}_2 = x_1 + x_2 - 2x_1 x_2$$

has two EPs at (0,0) and (1,1).

The Jacobian matrices at these points are

$$\left[\frac{\partial f}{\partial x}\right] =$$

$$\left[\frac{\partial f}{\partial x}\right]_{(0,0)} =$$

$$\left[\frac{\partial f}{\partial x}\right]_{(1,1)} =$$

Phase Portrait of Van der Pol Oscillator

Filename: expVanderPol.m

```
%%% Nonlinear Systems Analysis, NTU-EE, 91 Fall
%%% Phase Portrait of Van der Pol oscillator
%%% Date: Sep. 26, 2002
%%% By Feng-Li Lian, fengli@ntu.edu.tw

close all; clear all

xinit = [
 [(0:0.5:3) (0:0.5:3) -(0:0.5:3) -(0:0.5:3)]'
 [(0:0.5:3) -(0:0.5:3)  (0:0.5:3) -(0:0.5:3)]' ];

figure(11)
  plot( xinit(:,1), xinit(:,2), 'rx' ); hold on

for i = 1:28
  [ t, x ] = ode45( @funVanderPol, [ 0 20 ], xinit(i,:));

  figure(11)
  plot( x(:,1), x(:,2), '-' );
end
```

Filename: funVanderPol.m

```
%%% Nonlinear Systems Analysis, NTU-EE, 91 Fall
%%% Dynamic Equations of Van der Pol oscillator
%%% Date: Sep. 26, 2002
%%% By Feng-Li Lian, fengli@ntu.edu.tw

function dx = funVanderPol(t,x)

dx = zeros(2,1);
epsilon = 1.0;
dx(1) = x(2);
dx(2) = - x(1) + epsilon * ( 1 - x(1)^2 ) * x(2);
```