Nonlinear Systems Analysis

Lecture 4

2.3: Qualitative Behavior Near EP
2.2: Multiple Equilibria

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Outline

- 2.3: Qualitative Behavior Near Equilibrium Points
  - Linearization, Jacobian Matrix
- 2.2: Multiple Equilibria
  - Tunnel-diode circuit, Pendulum
- 2.1: Perturbed Linear Systems

- Consider the state model:

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

- f_1, f_2 are continuously differentiable.

- E.P.: $p = (p_1, p_2)$.
  That is,

- Expand the RHS
  into its Taylor series about p:

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

- Let $y_1 = x_1 - p_1, y_2 = x_2 - p_2$
  analyze the trajectory near (p_1, p_2).

- New state equation:

$$\dot{y}_1 =$$

$$\dot{y}_2 =$$

- New state equation:

$$\dot{y} = A\, y$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \right|_{x=p}$$

$$= \left. \frac{\partial f}{\partial x} \right|_{x=p}$$

- $\frac{\partial f}{\partial x}$ is called the Jacobian matrix of $f(x)$
  A is the Jacobian matrix
  evaluated at the E.P. p.

- If the origion of the linearized state eqn is

(1) a stable/unstable node with distinct eigen-values,

(2) a stable/unstable focus, or

(3) a saddle point,

- Then in a small neighborhood of the E.P., the trajectories of the nonlinear state eqn will behave like

(1) a stable/unstable node,

(2) a stable/unstable focus, or

(3) a saddle point.

- How conclusive the linearization approach is depends to a great extent on how the various qualitative phase portraits of a linear system persis under perturbations.

- For example,
  suppose A has distinct eigenvalues and
  consider $A + \Delta A$
  ΔA: 2×2 real matrix
  its elememts have arbitrarily small magnitudes.

- From the purterbation theory of matrices,
  the eigenvalues of a matrix
  depend continuously on its parameters.

- That is,
  given an $\epsilon > 0$,
  exist a corresponding $\delta > 0$
  the magnitude of the perturbation
  in each element of A is less than δ,
  the eigenvalues of ($A + \Delta A$) will lie in B_ϵ,
  B_ϵ = open discs of radius ϵ centered at the
  the eigenvalues of A.

- Hence, after arbitrarily small perturbations,
  eigenvalues of A
  in open RHP remain in open RHP
  in open LHP remain in open LHP

- However, when perturbated,
  eigenvalues on the imaginary axis might
  go into either the RHP or LHP.

- If the EP $x = 0$ of $\dot{x} = Ax$ is
  a node, focus, or saddle point,
  then the EP $x = 0$ of
  $\dot{x} = (A + \Delta A)x$ will be of the same type
  for sufficiently small perturbations.

- It is quite different
  if the EP is a center.

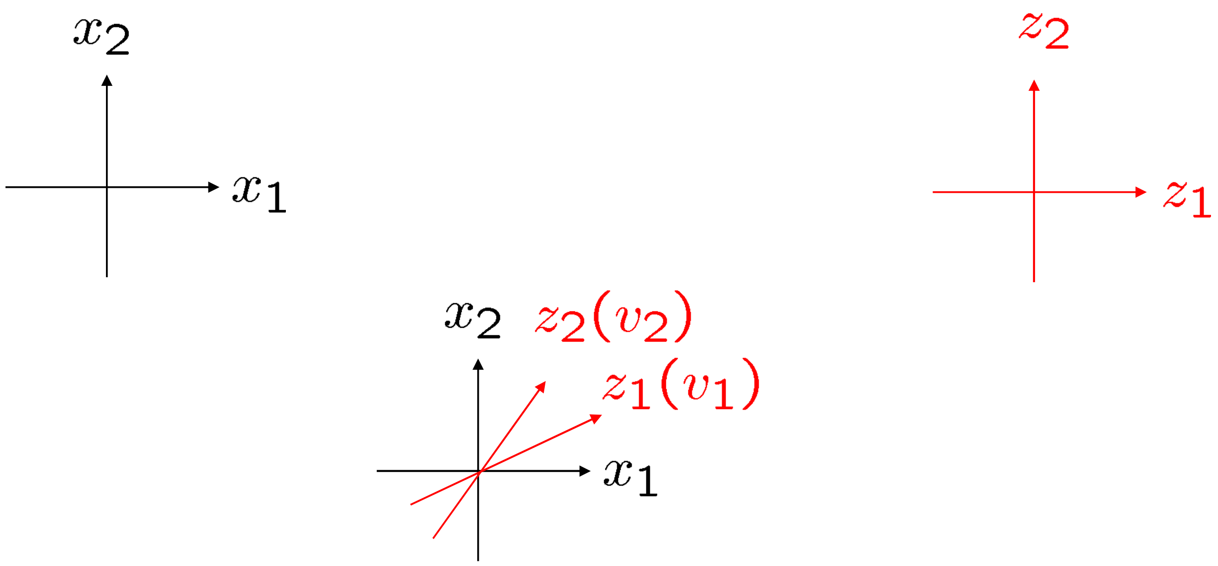
- The node, focus, and saddle EPs are said
  to be structurally stable,
  while the center EP is not.

2.1: Qualitative Behavior of Nonlinear Systems

Linearization
at the E.P.

Change of Coordinate
$z = M^{-1}x$
$J_r = M^{-1}AM$

- Nonlinear Systems: $\dot{x} = f(x)$ $\Longrightarrow$
- Linear Systems: $\dot{x} = Ax$ $\Longrightarrow$
- In z-coordinate: $\dot{z} = J_r z$



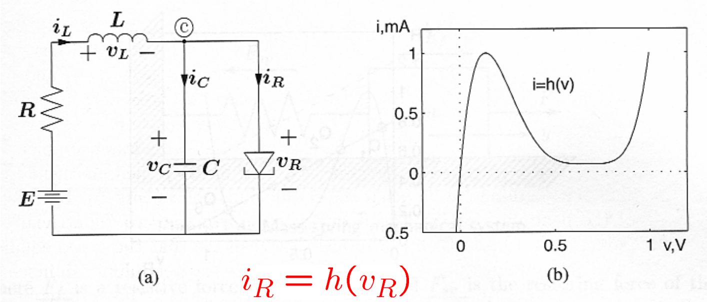
2.2: Multiple Equilibria – 1

- For linear systems,
  - det $A \neq 0$
    (A has no zero eigenvalues),
    $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$.
  - det $A = 0$, the system has a continuum of equilibrium points.
  - There are the only possible patterns.

- For nonlinear systems,
  - it can have multiple isolated equilibrium points.

- the tunnel-diode circuit

- the pendulum euqation



$i_R = h(v_R)$

Figure 1.2: (a) Tunnel-diode circuit; (b) Tunnel-diode v_R–i_R characteristic.

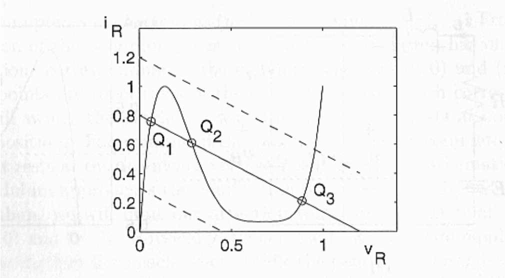


Figure 1.3: Equilibrium points of the tunnel-diode circuit.

Kirchhoff's current/voltage law:

$$
\begin{aligned}
i_C + i_R - i_L &= 0 \quad \text{(KCL)} \\
v_C - E + Ri_L + v_L &= 0 \quad \text{(KVL)}
\end{aligned}
$$

State model:

- state: $x_1 = v_C, x_2 = i_L$, and
- input: $u = E$,
- $i_C = C\frac{dv_C}{dt}$, $v_L = L\frac{di_L}{dt}$

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\
\dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u]
\end{aligned}
$$

Equilibrium points:

$$
\begin{aligned}
0 &= -h(x_1) + x_2 \\
0 &= -x_1 - Rx_2 + u
\end{aligned}
$$

That is, the roots of:

$$
h(x_1) = \frac{E}{R} - \frac{1}{R}x_1
$$

- Example 2.1:

  State Model:

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]$$

$$\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

- Assume that the circuit parameters are:

$$u = 1.2V, R = 1.5k\Omega, C = 2pF, L = 5\mu H$$

- time t in nanoseconds
  $x_2, h(x_1)$ in mA

- State Model:

$$\dot{x}_1 = 0.5[-h(x_1) + x_2]$$

$$\dot{x}_2 = 0.2[-x_1 - 1.5x_2 + 1.2]$$

and

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$

- Equilibrium Points: (let $\dot{x}_1 = \dot{x}_2 = 0$)

$$Q_1 = \begin{bmatrix} 0.063 \\ 0.758 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 0.285 \\ 0.61 \end{bmatrix} \quad Q_3 = \begin{bmatrix} 0.884 \\ 0.21 \end{bmatrix}$$

2.2: Tunnel-Diode Circuit – 4

- **Example 2.3**:

  The Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

- Evaluated at E.P. Q_1, Q_2, Q_3:

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (-3.57, -0.33) \quad V_1 = \begin{bmatrix} -.99 & -0.15 \\ -0.06 & -0.99 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (1.77, -0.25) \quad V_2 = \begin{bmatrix} 0.99 & -0.23 \\ -0.09 & 0.97 \end{bmatrix}$$

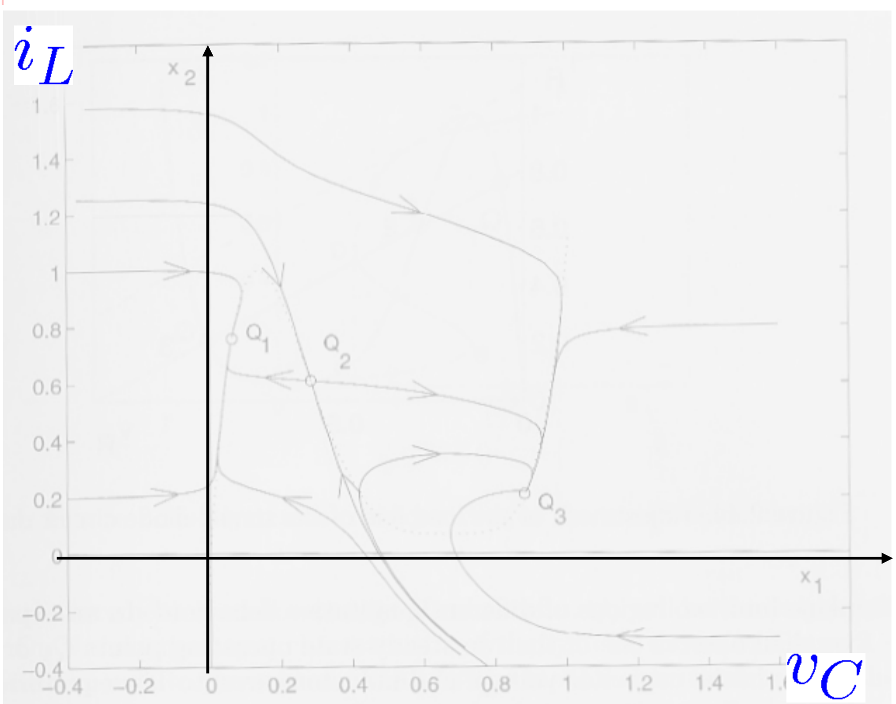
$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (-1.33, -0.4) \quad V_3 = \begin{bmatrix} -0.98 & -0.43 \\ -0.19 & -0.89 \end{bmatrix}$$

2.2: Tunnel-Diode Circuit – 5

- Q_1 is a stable node

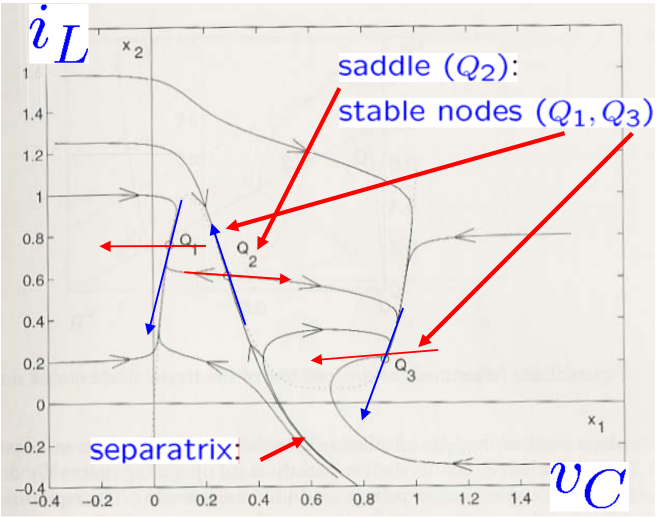
  Q_2 is a saddle

  Q_3 is a stable node



- The two special trajectories, which approach Q_2, are the stable trajectories of the saddle. They form a curve that divides the plane into two halves. Which is called a separatrix.

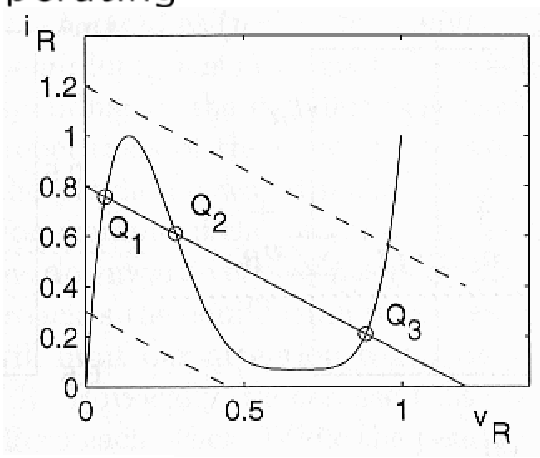
- The separatrix partitions the plane into two regions of different qualitative behavior.



- In an experimental setup, we shall observe one of the two steady-state operating points Q_1 or Q_3, depending on the initial capacitor voltage and inductor current.

- The equilibrium point at Q_2 is never observed in practice because the ever-present physical noise would cause the trajectories to diverge from Q_2 even if it were possible to set up the exact initial conditions corresponding to Q_2.

- The tunnel-diode circuit is refered as a bistable circuit, because it has two steady-state operating points.



- Used in computer memory,
  $Q_1 \rightarrow'' 0''$
  $Q_3 \rightarrow'' 1''$

- Triggering from Q_1 to Q_3 or vice versa is achieved by a triggering signal of sufficiently amplitude and duration that allows the trajectory to move to the other side of the separatrix.

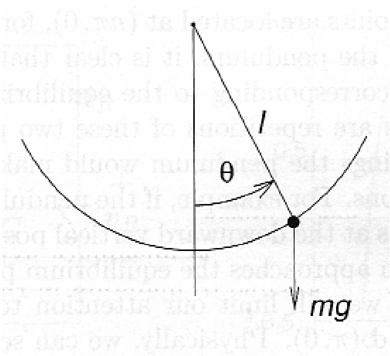
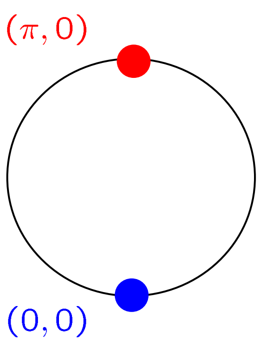


Figure 1.1: Pendulum.



Using Newton's Second Law,
Write the equation of motion
in the tangential direction:

$$ml\ddot{\theta} = -mg \sin\theta - kl\dot{\theta}$$

State model (let $x_1 = \theta, x_2 = \dot{\theta}$):

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

Equilibrium points (let $\dot{x}_1 = \dot{x}_2 = 0$):

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

Equilibrium points are $(n\pi, 0), n = 0, \pm 1, \pm 2, ...$,
or, physically, $(0, 0)$ and $(\pi, 0)$.

Question? Which one is stable or unstable?

- **Example 2.2**:
  State model:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10\sin x_1 - x_2\end{aligned}$$

- (0,0): or $(0,0), (2\pi,0), (-2\pi,0)$, etc.
  a stable focus.

- (π,0): or $(\pi,0), (-\pi,0)$, etc.
  a saddle.

- This picture is repeated periodically.
  Trajectories approach different E.P.,
  corresponding to # of full swings.

- **Example 2.4**:
  The Jacobian matrix:

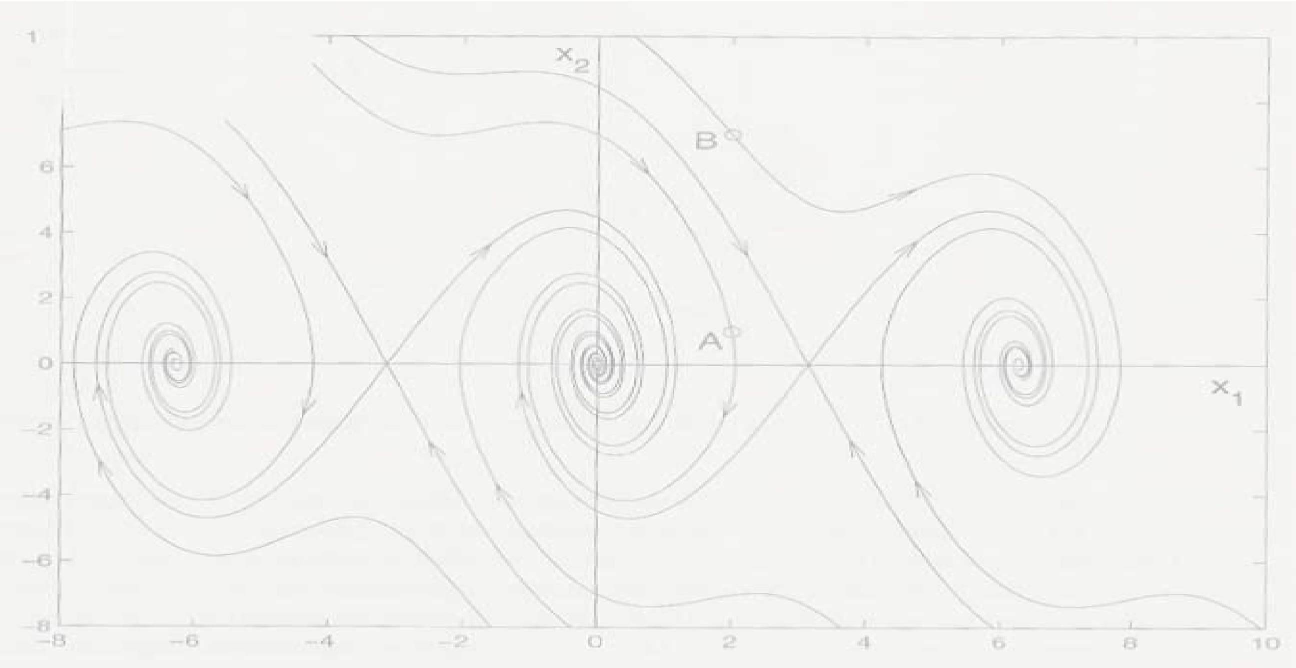
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}$$

- Evaluated at E.P. $Q_1 = (0,0)$, $Q_2 = (\pi,0)$:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad (-0.5 \pm j3.12)$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \quad (2.7, -3.7)$$

$$\begin{aligned} V_1 &= \begin{bmatrix} 0.30 - j0.05 & 0.30 + j0.05 \\ 0.01 + j0.98 & 0.01 - j0.98 \end{bmatrix}, \\ V_2 &= \begin{bmatrix} 0.37 & -0.27 \\ 1 & 1 \end{bmatrix}, \end{aligned}$$



- Phase portraits of Tunnel-Diode Circuit and Pendulum Equation show that the qualitative behavior in the vicinity of each E.P. looks just like those for linear systems.

- Tunnel-Diode circuit:
  The trajectories near Q_1, Q_2, Q_3 are similar to those associated with a stable node, saddle, and stable node, respectively.

- Pendulum:

  The trajectories near $(0, 0), (\pi, 0)$
  are similar to those associated with
  a stable focus and saddle, respectively.

- **Example 2.5**:

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = \quad x_1 - \mu x_2(x_1^2 + x_2^2)$$

$$A = \frac{\partial f}{\partial x}\Big|_{0,0} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

- It has an E.P. at the origin.

  The linearized state equation at the origin
  has eigenvalues $\pm j$.
  $\Rightarrow$ A center E.P.

- The qualitative behavior of
  the nonlinear system
  can be examinated by the new variables

  - a stable focus when $\mu > 0$
  - an unstable focus when $\mu < 0$