Nonlinear Systems Analysis

Lecture Note

Section 8.3

Invariance-like Theorems (Advanced Stability Analysis)

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Outline

- The Center Manifold Theorem (8.1)
- Region of Attraction (8.2)
- Invariance-Like Theorems (8.3)
- Stability of Periodic Systems (8.4)

Invariance-like Theorems (8.3)

- For **autonomous systems**

  LaSalle's invariance theorem shows that

  trajectory $\longrightarrow$ $E = \{\dot{V}(x) = 0\}$

- For **non-autonomous systems**

  $$\{\dot{V}(t, x)\}$$

- If it can be shown that

  $$\dot{V}(t, x) \leq -W(x) \leq 0$$

  $$E = \{W(x) = 0\}$$

  trajectory $\longrightarrow$ E as $t \to \infty$

Barbalat's Lemma

- **Lemma 8.2**

- Let $\phi : R \to R$ be a **unif. cont.** func. on $[0, \infty)$.

- Suppose that

  $$\lim_{t \to \infty} \int_0^t \phi(\tau) d\tau$$

  exists and is **finite**.

- Then,

  $\phi(t) \to 0$ as $t \to \infty$

- Proof:
- If is not true,
  then there is a positive constant k_1
  such that for every $T > 0$,
  we can find $T_1 \geq T$ with $|\phi(T_1)| \geq k_1$.

- Since $\phi(t)$ is unif. cont.,
  there is a positive constant k_2
  such that $|\phi(t+\tau) - \phi(t)| < k_1/2$
  for all $t \geq 0$ and all $0 \leq \tau \leq k_2$.

- Hence,

$$\begin{aligned} |\phi(t)| &= |\phi(t) - \phi(T_1) + \phi(T_1)| \\ &\geq |\phi(T_1)| - |\phi(t) - \phi(T_1)| \\ &> k_1 - \frac{1}{2}k_1 \\ &= \frac{1}{2}k_1, \ \forall t \in [T_1, T_1 + k_2] \end{aligned}$$

- Since $\phi(t)$ retains the same sign
  for $T_1 \leq t \leq T_1 + k_2$, then

$$\left| \int_{T_1}^{T_1+k_2} \phi(t)dt \right| = \int_{T_1}^{T_1+k_2} |\phi(t)|dt > \frac{1}{2}k_1 k_2$$

- Thus $\int_0^t \phi(\tau)d\tau$ cannot converge to
  a finite limit as $t \to \infty$,
  a contradiction.

- Theorem 8.4
- Let $D \subset R^n$ be a domain containing $x = 0$ and suppose $f(t, x)$ is piecewise cont. in t and locally Lipschitz in x, uniformly in t, on $[0, \infty) \times D$.
- Furthermore, $\forall t \geq 0$, suppose $f(t, 0)$ is unif. bdd.

- Let $V : [0, \infty) \times D \rightarrow R$ be a cont. diff. func such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W(x)$$

$\forall t \geq 0, \forall x \in D$,
where $W_1(x), W_2(x)$ are cont. P.D. func.
and $W(x)$ is cont. P.S.D. func. on D.

- Choose $r > 0$ such that $B_r \subset D$ and let $\rho < \min_{||x||=r} W_1(x)$.

Theorem 8.4

- Then, with $x(t_0) \in \{x \in B_r \mid W_2(x) \le \rho\}$ all sol. of $\dot{x} = f(t,x)$ are bdd and satisfy

$$W(x(t)) \to 0 \text{ as } t \to \infty$$

- Moreover, if all the assumptions hold globallly and $W_1(x)$ is radially unbounded, the statement is true for all $x(t_0) \in R^n$.

Theorem 8.4: Part of Proof

- Part of Proof:
- Since $V(t, x(t))$ is monotonically nonincreasing and bounded from below by zero, it converges as $t \to \infty$.
- So, for $W(x)$,

$$\int_{t_0}^{t} W(x(\tau))d\tau \le -\int_{t_0}^{t} \dot{V}(\tau, x(\tau))d\tau$$

$$= V(t_0, x(t_0)) - V(t, x(t))$$

- Therefore, $\lim_{t\to\infty} \int_{t_0}^{t} W(x(\tau))d\tau$ exists and is finite.

Theorem 8.4: Part of Proof

- Since $x(t)$ is bdd., $\forall t \geq t_0$
  $\dot{x} = f(t, x(t))$ is bdd., uniformly in t.
- Hence, $x(t)$ is unif. cont. in t on $[t_0, \infty)$.
- So, because $W(x)$ is unif. cont. in x
  on the compact set B_r,
  consequently,
  $W(x(t))$ is unif. cont. in t on $[t_0, \infty)$.
- Therefore, by Lemma 8.2,
  $W(x(t)) \to 0$ as $t \to \infty$.
- The limit $W(x(t)) \to 0$ implies that
  $x(t)$ approaches E as $t \to \infty$,
  where

$$E = \{x \in D \mid W(x) = 0\}$$

- Therefore, the positive limit set of $x(t)$ is a subset of E.

Theorem 8.5: U.A.S.

- Theorem 8.5
- Let $D \subset R^n$ be a domain containing
  $x = 0$
  and suppose $f(t, x)$ is piecewise cont.
  in t
  and locally Lipschitz in x
  for all $t \geq 0$ and $x \in D$.
- Let $x = 0$ be an E.P. for $\dot{x} = f(t, x)$
  at $t = 0$.

- Let $V : [0,\infty) \times D \to R$ be a cont. diff. func such that

$$W_1(x) \le V(t,x) \le W_2(x)$$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0$$

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) \le -\lambda V(t,x), \qquad 0 < \lambda < 1$$

$\forall t \ge 0, \forall x \in D$, for some $\delta > 0$,
where
$W_1(x), W_2(x)$ are cont. P.D. func. on D
and $\phi(\tau; t, x)$ is the sol. of the system starts at (t,x).

- Then, the origin is U.A.S.

- If all assumptions hod globally and $W_1(x)$ is radially unbounded, then the origin is G.U.A.S.

- If

$$W_1(x) \ge k_1 ||x||^c, \quad W_2(x) \le k_2 ||x||^c,$$

where $k_1, k_2, c > 0$,
then the origin is E.S.

- **Example 8.11**
- Consider the LTV system

$$\dot{x} = A(t)x$$

where $A(t)$ is cont. for all $t \geq 0$.

- Suppose that there is a cont. diff. symm. $P(t)$ that satisfies

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0$$

as well as matrix diff. eq.

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t)$$

where $C(t)$ is cont. in t.

- The derivative of the quadratic func.

$$V(t,x) = x^T P(t) x$$

along the traj. of the system is

$$\dot{V} = -x^T C^T(t) C(t) x \leq 0$$

- Let the sol. be $\phi(\tau; t, x) = \Phi(\tau, t)x$, where $\Phi(\tau, t)$ is state transition matrix.

- Therefore,

$$\begin{aligned} V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) &= \int_t^{t+\delta} \dot{V}(\tau, \phi(\tau; t, x)) d\tau \\ &= -x^T \int_t^{t+\delta} \Phi^T(\tau, t) C^T C \Phi(\tau, t) d\tau \\ &= -x^T W(t, t+\delta) x \end{aligned}$$

where $W(t, t+\delta) = \int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau$

Example 8.11: L.T.V.

- Suppose there is positive a constant $k \leq c_2$ such that

$$W(t, t+\delta) \geq kI, \ \forall t \geq 0$$

then

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t, x) \quad \leq \quad -k||x||_2^2 \ \leq -\frac{k}{c_2} V(t, x)$$

- Thus, all assumptions of Thm 8.5 are satisfied globally with

$$W_i(x) = c_i ||x||_2^2, \ i = 1, 2, \ \lambda = \frac{k}{c_2} < 1$$

- Then, $x = 0$ is G.E.S.

- Note that $W(t, t+\delta)$ is the obvervability Gramian of $(A(t), C(t))$ and $W(t, t+\delta) \geq kI$ is implied by uniform observability of $(A(t), C(t))$.

Example 8.12: Adaptive Control

- Example 8.12 (from Sec 1.2.6)
- Model reference adaptive control:

$$\begin{aligned} \text{plant model: } \dot{y}_p &= a_p y_p + k_p u \\ \text{reference model: } \dot{y}_m &= a_m y_m + k_m r \end{aligned}$$

- If $\gamma > 0$ is the adaptation gain, $e_o = y_p - y_m$ is the output error, and ϕ_1, ϕ_2 are the parameter errors.

- The closed-loop eq.

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)] \\ \dot{\phi}_1 &= -\gamma e_o r(t) \\ \dot{\phi}_2 &= -\gamma e_o [e_o + y_m(t)] \end{aligned}$$

- Assume that $k_p > 0$, $a_m < 0$
- $r(t)$ is piecewise cont. and bdd.
- Using

$$V = \frac{1}{2}\left[\frac{e_o^2}{k_p} + \frac{1}{\gamma}(\phi_1^2 + \phi_2^2)\right]$$

as a Lyapunov function candidate,

- We obtain

$$\begin{aligned}\dot{V} &= \frac{a_m}{k_p}e_o^2 + e_o(\phi_1 r + \phi_2 e_o + \phi_2 y_m) - \phi_1 e_o r - \phi_2 e_o(e_o + y_m) \\ &= \frac{a_m}{k_p}e_o^2 \leq 0\end{aligned}$$

- By applying Thm 8.4, we conclude that for any $c > 0$ and for all initial states in $\{V \leq c\}$, all state variables are bounded for all $t \geq t_0$ and $\lim_{t\to\infty} e_o(t) = 0$.
- This shows that $y_p \to y_m$, but it says nothing about $\phi_1, \phi_2 \to 0$!
- In fact, they may not converge to zero.
- If r, y_m are nonzero constant signals, the closed-loop system will have an equilibrium subspace $\{e_0 = 0, \phi_2 = (a_m/k_p)\phi_1\}$.

- Hence, we need to apply Thm 8.5
  to derive the conditions of $\phi_1, \phi_2 \to 0$.
  That is, the conditions which the origin
  $(e_o = 0, \phi_1 = 0, \phi_2 = 0)$ is **U.A.S.**

- Reformulate the system as:

$$\dot{x} = \begin{bmatrix} a_m & k_p r(t) & k_p y_p(t) \\ -\gamma r(t) & 0 & 0 \\ -\gamma y_p(t) & 0 & 0 \end{bmatrix} x,$$

where $$x = \begin{bmatrix} e_o \\ \phi_1 \\ \phi_2 \end{bmatrix}$$

- Suppose $\lim_{t\to\infty}[r(t) - r_{ss}(t)] = 0$,
  Then, $\lim_{t\to\infty}[y_m(t) - y_{ss}(t)] = 0$,
  Together with $\lim_{t\to\infty} e_o(t) = 0$,
  the above linear system can be
  represented by

$$\dot{x} = [A(t) + B(t)]x$$

where

$$A(t) = \begin{bmatrix} a_m & k_p r_{ss}(t) & k_p y_{ss}(t) \\ -\gamma r_{ss}(t) & 0 & 0 \\ -\gamma y_{ss}(t) & 0 & 0 \end{bmatrix}$$

and $\lim_{t\to\infty} B(t) = 0$

- Because $\lim_{t\to\infty} B(t) = 0$,
  if $\dot{x} = A(t)x$ is **U.A.S.**,
  then $\dot{x} = [A(t) + B(t)]x$ is **U.A.S.**.

- Using V as a Lypunov function candidate, we obtain

$$\dot{V} = \frac{a_m}{k_p} e_o^2 = -x^T C^T C x,$$

where $C = \sqrt{\frac{-a_m}{k_p}}[1\ 0\ 0]$

- From Example 8.11, if $(A(t), C)$ is uniformly observable, then the origin will be U.A.S.

- And, uniform observability of $(A(t), C)$ implies uniform observability of $(A(t) - K(t)C, C)$ for any piecewise cont., bdd $K(t)$.

- Take

$$K(t) = \sqrt{\frac{k_p}{-a_m}}\,[a_m \quad -\gamma r_{ss}(t) \quad -\gamma t_{ss}(t)]^T$$

and obtain

$$A(t) - K(t)C = \begin{bmatrix} 0 & k_p r_{ss}(t) & k_p y_{ss}(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \sqrt{\tfrac{-a_m}{k_p}}\,[1 \quad 0 \quad 0]$$

- Hence, by investigating observability of this pair for a given reference signal, we can determine whether the conditions of Thm 8.5 are satisfied.

Example 8.12: Adaptive Control

- For example,
  if r is a nonzero constant signal,
  it can be easily seen that
  the pair is not observable.

- On the other hand,
  if $r(t) = a \sin wt$ with positive a, w,
  we have
  $r_{ss}(t) = r(t)$ and $y_{ss} = aM \sin(wt + \delta)$,
  where M, δ are determined
  by the transfer func. of the ref. model.

Example 8.12: Adaptive Control

- It can be verified that
  the pair is uniformly observable;
  hence, the origin $(e_o = 0, \phi_1 = 0, \phi_2 = 0)$
  is U.A.S. and
  the parameter errors $\phi_1(t), \phi_2(t) \to 0$
  as $t \to \infty$.

- Note that $r(t) = a \sin wt$ is said to be
  persistently exciting,
  while a constant reference is
  not persistently exciting.

- Consider zero-input LTV system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) \end{aligned}$$

and let $\phi_i(t, t_0, x_0^i)$ the associated sol.
or,

$$x(t) = \begin{bmatrix} \phi_1(t, t_0, x_0^1) \\ \vdots \\ \phi_n(t, t_0, x_0^n) \end{bmatrix} \in R^{n \times 1}$$

- Note that

$$x(t) \text{ or } [\phi_i(t, t_0, x_0^i)] = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is
the state transition matrix
from t_0 to t.

- Let

$$y^i(t) = C(t)\phi_i(t, t_0, x_o^i)$$

So, over $[t_0, t_1]$
the pair $(C(t), A(t))$ is observable
iff $y^i(\cdot)$ are linear indep. vector func.

- Note that $y(t) = C(t)\Phi(t, t_0)x(t_0)$

- That is,
the columns of $C(t)\Phi(t, t_0)$ are
linear indep.

- That is, there are distint points $t_1, \ldots, t_p$ such that

$$\text{rank} \begin{bmatrix} C(t_1)\Phi(t_1, t_0) \\ C(t_2)\Phi(t_2, t_0) \\ \vdots \\ C(t_p)\Phi(t_p, t_0) \end{bmatrix} = n$$

- For LTI systems, because

$$\Phi(t_i, t_0) = \exp(A(t_i - t_0))$$

$$= I + A(t_i - t_0) + \frac{A^2}{2!}(t_i - t_0)^2 + \ldots$$

the Obervability Matrix becomes

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$