Nonlinear Systems Analysis

Lecture Note

Section 4.9

Input-to-State Stability (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

- Consider the system

$$\dot{x} = f(t, x, u) \quad (4.44)$$

where $f : [0, \infty) \times R^n \times R^m \to R^n$ is piecewise continuous in t and locally Lipschitz in x and u.

- The input $u(t)$ is a piecewise continuous, bdd function of t for all $t \geq 0$.

- Suppose the unforced system

$$\dot{x} = f(t, x, 0) \quad (4.45)$$

has a G.U.A.S. E.P. at $x = 0$.

- What can we say about the behavior of the system (4.44) in the presence of a bounded input $u(t)$?

- For the L.T.I. system

$$\dot{x} = Ax + Bu$$

with a Hurwitz matrix A, we can write the solution as

$$x(t) = e^{(t-t_0)A} x(t_0) + \int_{t_0}^{t} e^{(t-\tau)A} Bu(\tau) d\tau$$

- And use the bound $||e^{(t-t_0)A}|| \leq ke^{-\lambda(t-t_0)}$ to estimate the solution by

$$\begin{aligned} ||x(t)|| &\leq ke^{-\lambda(t-t_0)} ||x(t_0)|| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)} ||B|| \; ||u(\tau)|| d\tau \\ &\leq ke^{-\lambda(t-t_0)} ||x(t_0)|| + \frac{k||B||}{\lambda} \sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \end{aligned}$$

- This estimate shows that
  the zero-input response
  decays to zero exponentially fast,
  while the zero-state response is bounded
  for every bounded input.

- In fact, the estimate shows more than
  a bounded-input-bounded-state (BIBO)
  property.

- It shows that
  the bound on the zero-state response
  is proportional to
  the bound on the input.

For General Nonlinear Systems

- For a general nonlinear system,
  it should not be surprising that
  these properties may not hold even when
  the origin of the unforced syst. is G.U.A.S.

- e.g., consider the scalar system

$$\dot{x} = -3x + (1 + 2x^2)u$$

  which has a G.E.S. origin when $u = 0$.

- Yet, when $x(0) = 2$ and $u(t) \equiv 1$,
  the solution $x(t) = \frac{(3-e^t)}{(3-2e^t)}$
  is unbounded;
  it even has a finite escape time.

- Let us view the system
  $\dot{x} = f(t, x, u)$ as
  a perturbation of the unforced syst
  $\dot{x} = f(t, x, 0)$.

- Supose we have a Lyapunov func $V(t, x)$
  for the unforced system and
  let us calculate the derivative of V
  in the presence of u.

- Due to the boundedness of u,
  it is plausible that in some cases
  it should be possible to show that
  $\dot{V}$ is negative outside a ball of radius μ,
  where μ depends on $\sup ||u||$.

- This would be expected, for example,
  when the functon $f(t, x, u)$ satisfies
  the Lipschitz condition

$$||f(t, x, u) - f(t, x, 0)|| \leq L||u||, \qquad (4.46)$$

- Showing that $\dot{V}$ is negative
  outside a ball of radius μ
  would enable us to apply Thm 4.18
  to show that $x(t)$ satisfies (4.42), (4.43).

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$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \quad \forall t_0 \leq t \leq t_0 + T \qquad (4.42)$$

$$||x(t)|| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \geq t_0 + T \qquad (4.43)$$

For General Nonlinear Systems

- These inequalities show that

  $||x(t)||$ is bdd by a class $\mathcal{KL}$ function

  $\beta(||x(t_0)||, t - t_0)$ over $[t_0, t_0 + T]$ and

  by a class $\mathcal{K}$ function $\alpha_1^{-1}(\alpha_2(\mu))$

  for $t \geq t_0 + T$.

- Consequently,

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu))$$

  is valid for all $t \geq t_0$.

Definition of ISS

- **Definition 4.7:**
- The system $\dot{x} = f(t, x, u)$ is said to be

  input-to-state stable

  if there exist a class $\mathcal{KL}$ function β

  and a class $\mathcal{K}$ function γ

  such that for any initial state $x(t_0)$

  and any bdd input $u(t)$,

  the sol. $x(t)$ exists for all $t \geq t_0$ and

  satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} ||u(\tau)|| \right), \quad (4.47)$$

- Inequality (4.47) guarantees that

  for any bdd input $u(t)$,

  the state $x(t)$ will be bounded.

- Furthermore, as t increases,
  the state $x(t)$ will be ultimately bounded
  by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} ||u(t)||$.

- Ex 4.58 uses inequality (4.47) to show
  that
  if $u(t)$ converges to zero as $t \to \infty$,
  so does $x(t)$.

- Since, with $u(t) \equiv 0$,
  (4.47) reduces to

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0)$$

input-to-state stability implies that
the origin of the unforced system (4.45)
is G.U.A.S.

$$\dot{x} = f(t, x, u) \quad (4.44)$$

$$\dot{x} = f(t, x, 0) \quad (4.45)$$

- The notion of input-to-state stability
  is defined for the global case
  where the initial state and the input
  can be arbitrarily large.

- A local version of this notion
  is presented in Ex 4.60.

- **Theorem 4.19:**
- Let $V : [0,\infty) \times R^n \to R$
  be a cont. diff. func. such that
  $\forall (t,x,u) \in [0,\infty) \times R^n \times R^m$

$$\alpha_1(||x||) \le V(t,x) \le \alpha_2(||x||) \quad (4.48)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \le -W_3(x), \quad \forall ||x|| \ge \rho(||u||) > 0, \quad (4.49)$$

  where α_1, α_2 are class $\mathcal{K}_\infty$ functions,
  ρ is a class $\mathcal{K}$ function, and
  $W_3(x)$ is a cont. P.D. func. on R^n.
- Then, the system (4.44) is ISS
  with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

- **Proof:**
- By applying the global version of Thm 4.18
  we find that
  the sol. $x(t)$ exists and satisfies, $\forall t \ge t_0$,

$$||x(t)|| \le \beta(||x(t_0)||, t-t_0) + \gamma(\sup_{\tau \ge t_0} ||u(\tau)||), \quad (4.50)$$

- Since $x(t)$ depends only on $u(\tau)$
  for $t_0 \le \tau \le t$,
  the supremum on the RHS of (4.50)
  can be taken over $[t_0, t]$,
  which yields (4.47).

Lemma 4.6: Converse Theorem for G.E.S.

- **Lemma 4.6:**
- Suppose $f(t,x,u)$ is cont. diff. and globally Lipshitz in (x,u), uniformly in t.
- If the unforced syst (4.45), i.e., $u \equiv 0$ has a GES EP at the origin $x = 0$, then the system (4.44) is ISS.

$$\dot{x} = f(t,x,u) \quad (4.44)$$

$$\dot{x} = f(t,x,0) \quad (4.45)$$

Lemma 4.6: Converse Thm for G.E.S.

- **Proof:**
- View (4.44) as a perturbation of the unforced system (4.45).

$$\dot{x} = f(t,x,u) \quad (4.44)$$

$$\dot{x} = f(t,x,0) \quad (4.45)$$

- (The converse Lyapunov) Thm 4.14 shows that the unforced system (4.45) has a Lyapunov function $V(t,x)$ that satisfies (4.10)-(4.12) globally.

$$c_1||x||^2 \leq V(t,x) \leq c_2||x||^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \leq -c_3||x||^2$$

$$\left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x||$$

- Due to the uniform global Lipschitz property of f, the perturbation term satisfies (4.46) for all $t \geq t_0$ and all (x,u).

$$||f(t,x,u) - f(t,x,0)|| \leq L||u||, \quad (4.46)$$

- The derivative of V

  with respect to (4.44) satisfies

$$\dot{x} = f(t,x,u) \quad (4.44)$$

$$\dot{x} = f(t,x,0) \quad (4.45)$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)]$$

$$\leq -c_3\,||x||^2 + c_4\,||x||\,L\,||u||$$

$$c_1||x||^2 \leq V(t,x) \leq c_2||x||^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) \leq -c_3||x||^2$$

$$\left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x||$$

- To use the term $-c_3||x||^2$

  to dominate $c_4\,L\,||x||\,||u||$, for large $||x||$,

  we rewrite the foregoing inequality as

$$\dot{V} \leq -c_3\,(1-\theta)\,||x||^2 - c_3\,\theta\,||x||^2 + c_4\,L\,||x||\,||u||$$

  where $0 < \theta < 1$.

- Then,

$$\dot{V} \leq -c_3\,(1-\theta)\,||x||^2, \qquad \forall ||x|| \geq \frac{c_4\,L\,||u||}{c_3\,\theta}, \qquad \forall (t,x,u)$$

- Hence, the conditions of Thm 4.19

  are satisfied with

  $\alpha_1(r) = c_1 r^2$,

  $\alpha_2(r) = c_2 r^2$, and

  $\rho(r) = (c_4 L / c_3 \theta) r$,

  and

  we conclude that the system is ISS

  with $\gamma(r) = \sqrt{c_2/c_1}(c_4 L / c_3 \theta) r$.

- QED

- Lemma 4.6 requires
  a globally Lipschitz function f and
  G.E.S. of $x = 0$ of the unforced system
  to conclude input-to-state stability.

- It is easy to construct examples
  where the lemma does not hold
  in absence of one of these 2 conditions.

- The system $\dot{x} = -3x + (1 + x^2)u$,
  which we discussed earlier in the Sec,
  doesn't satisfy the global Lipschitz cond.

- The system $\dot{x} = -\frac{x}{1+x^2} + u =^{def} f(x, u)$
  has a globally Lipschitz f
  since the partial derivatives of f
  w.r.t. x & u are globally bounded.

- The origin of $\dot{x} = -\frac{x}{1+x^2}$ is G.A.S.,
  as it can be seen by the Lyapunov function $V(x) = x^2/2$,
  whose derivative $\dot{V}(x) = -\frac{x^2}{1+x^2}$
  is N.D. for all x.

- It is locally E.S.
  because the linearization at the origin
  is $\dot{x} = -x$.

Discussion of Lemma 4.6

- However, it is not G.E.S.
- It is easiest seen through the fact that the system is not I.S.S..
- Notice that with $u(t) \equiv 1$, $f(x, u) \geq 1/2$.
- Hence, $x(t) \geq x(t_0) + t/2$ for all $t \geq 0$, which shows that the sol. is unbounded.
- In the absence of G.E.S. or globally Lipschitz functions, we may still be able to show ISS by applying Thm 4.19.
- This process is illustrated by the three examples that follow.

Example 4.25

- Example 4.25:
- The system $\dot{x} = -x^3 + u$ has a GAS origin when $u = 0$.
- Taking $V = \frac{1}{2}x^2$, the $\dot{V}$ along the traj. of the syst is given by

$$\begin{aligned}\dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\qquad \forall |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3} \quad \text{where } 0 < \theta < 1 \\ &\leq -(1-\theta)x^4\end{aligned}$$

- Thus, the syst is input-to-state stable with $\gamma(r) = (r/\theta)^{1/3}$.

Example 4.26

- **Example 4.26:**
- The system

$$\dot{x} = f(x,u) = -x - 2x^3 + (1+x^2)u^2$$

has a GES origin when $u = 0$,

but Lemma 4.6 does not apply

since f is not globally Lipschitz.

- Taking $V = \frac{1}{2}x^2$, we obtain

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1+x^2)u^2 \\ &\qquad \forall |x| \geq u^2 \\ &\leq -x^4, \end{aligned}$$

- Thus, the syst is input-to-state stable

with $\gamma(r) = r^2$.

Examples 4.25 & 4.26

- Note that, in examples 4.25 & 4.26,

$V(x) = x^2/2$ satisfies

(4.48) of Thm 4.19

with $\alpha_1(r) = \alpha_2(r) = r^2/2$.

- Hence, $\alpha_1^{-1}(\alpha_2(r)) = r$

and $\gamma(r)$ reduces to $\rho(r)$.

$$\alpha_1(||x||) \leq V(t,x) \leq \alpha_2(||x||) \quad (4.48)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W_3(x), \quad \forall ||x|| \geq \rho(||u||) > 0, \quad (4.49)$$

- Applications of I.S.S. to

  stability analysis of cascade systems

- Consider

$$\dot{x}_1 = f_1(t, x_1, x_2) \quad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \quad (4.52)$$

where

$f_1 : [0, \infty) \times R^{n_1} \times R^{n_2} \to R^{n_1}$ and

$f_2 : [0, \infty) \times R^{n_2} \to R^{n_2}$

are piecewise cont. in t

and locally Lipschitz in $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Suppose both

$$\dot{x}_1 = f_1(t, x_1, 0)$$
$$\dot{x}_2 = f_2(t, x_2)$$

have G.U.A.S. E.P.

at their respective origins.

- Under what condition

  will the origin $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  of the cascade system

  posses the same property?

- **Lemma 4.7:**
- **Under the stated assumptions,**

  and with x_2 as input

  if $\dot{x}_1 = f_1(t, x_1, x_2)$ is ISS

  and $x_2 = 0$ of $\dot{x}_2 = f_2(t, x_2)$ is GUAS,

  then $x = 0$ of the cascade system:

$$\dot{x}_1 = f_1(t, x_1, x_2) \qquad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \qquad (4.52)$$

  is GUAS.

$$\dot{x}_1 = f_1(t, x_1, x_2) \qquad (4.51)$$

$$\dot{x}_2 = f_2(t, x_2) \qquad (4.52)$$

- **Proof:**
- **Let $t_0 \geq 0$ be the initial time.**
- **The sol. of (4.51) & (4.52) satisfy**

$$||x_1(t)|| \;\leq\; \beta_1\left(||x_1(s)||, t-s\right) + \gamma_1\left(\sup_{s \leq \tau \leq t} ||x_2(\tau)||\right) \qquad (4.53)$$

$$||x_2(t)|| \;\leq\; \beta_2\left(||x_2(s)||, t-s\right) \qquad (4.54)$$

  globally, where $t \geq s \geq t_0$,

  $\beta_1, \; \beta_2$ are class $\mathcal{KL}$ functions

  and γ_1 is a class $\mathcal{K}$ function.

- **Apply (4.53) with $s = (t+t_0)/2$**

$$||x_1(t)|| \le \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{\frac{t+t_0}{2}\le\tau\le t} ||x_2(\tau)||\right) \qquad \textbf{(4.55)}$$

- **To estimate $x_1(\frac{t+t_0}{2})$,**

  apply (4.53) with $s = t_0$ and

  t replaced by $\frac{t+t_0}{2}$ to obtain

$$\left\|x_1\left(\frac{t+t_0}{2}\right)\right\| \le \beta_1\left(||x_1(t_0)||, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{t_0\le\tau\le\frac{t+t_0}{2}} ||x_2(\tau)||\right) \qquad \textbf{(4.56)}$$

- **Using (4.54), we obtain**

$$\sup_{t_0\le\tau\le\frac{t+t_0}{2}} ||x_2(\tau)|| \le \beta_2(||x_2(t_0)||, 0) \qquad \textbf{(4.57)}$$

$$\sup_{\frac{t+t_0}{2}\le\tau\le t} ||x_2(\tau)|| \le \beta_2(||x_2(t_0)||, \frac{t-t_0}{2}) \qquad \textbf{(4.58)}$$

- **Substituting (4.56) through (4.58)**

  into (4.55) and using the inequalities

$$||x_1(t_0)|| \leq ||x(t_0)||,$$

$$||x_2(t_0)|| \leq ||x(t_0)||, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$||x(t)|| \leq ||x_1(t)|| + ||x_2(t)||$$

yield

$$||x(t)|| \leq \beta\left(||x(t_0)||, t - t_0\right)$$

where

$$\beta\left(r, s\right) = \beta_1\left(\underline{\beta_1\left(r, s/2\right) + \gamma_1\left(\beta_2(r,0)\right)},\ s/2\right) + \gamma_1\left(\beta_2\left(r, s/2\right)\right) + \beta_2\left(r, s\right)$$

- **So, β is a class $\mathcal{KL}$ func for all $r \geq 0$.**

  Hence, $x = 0$ is GUAS