Nonlinear Systems Analysis

Lecture Note

Section 4.8
(Ultimate) Boundedness (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

- Lyapunov analysis can be used to show the boundedness of the solution of the state equation, even when there is no E.P. at the origin.

- For example: consider the scalar eqn.:

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no E.P.s and whose solution is given by

$$x(t) = e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \, d\tau$$

- If $f(t,0) = 0, \forall t \geq 0$, the origin is an E.P. for $\dot{x} = f(t,x)$ at $t = 0$

- The solution satisfies the bound

$$\begin{aligned} |x(t)| &\leq e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau \\ &= e^{-(t-t_0)} a + \delta [1 - e^{-(t-t_0)}] \\ &= \delta + (a - \delta) e^{-(t-t_0)} \\ &\leq a, \quad \forall t \geq t_0 \end{aligned}$$

- Which shows that the solution is bouded for all $t \geq t_0$, uniformly in t_0, that is, with a bound independent of t_0.

- While this bound is valid for all $t \geq t_0$, it becomes a conservative estimate of the solution as time progresses, because it does not take into consideration the exponentially decaying term.

- If we pick any number b such that $\delta < b < a$, it can be easily seen that

$$|x(t)| \leq b, \quad \forall t \geq t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

(Ultimate) Boundedness

- The bound b,
  which again is independent of t_0,
  gives a better estimate of the sol.
  after a transient period has passes.

- In this case, the sol. is said to be
  uniformly ultimately bounded and
  b is called the ultimate bound.

via Lyapunov Analysis

- Showing that the sol. of $\dot{x} = -x + \delta \sin t$
  has the uniform boundedness and
  ultimate boundedness properties
  can be done via Lyapunov analysis
  w/o using the explicit sol. of the state eqn.

- Starting with $V(x) = x^2/2$,
  we calculate the derivative of V
  along the traj. of the system, to obtain

$$\begin{aligned} \dot{V} &= x\,\dot{x} \\ &= -x^2 + x\delta \sin t \\ &\leq -x^2 + \delta|x| \end{aligned}$$

- The RHS is not N.D.
  because, near the origin,
  $\delta|x|$ dominates $-x^2$.

- However, outside the set $\{|x| \leq \delta\}$, i.e., $\{|x| > \delta\}$, $\dot{V}$ is negative. $\dot{V} \;\leq\; -x^2 + \delta|x|$

- Choose $c > \delta^2/2$.

- Since $\dot{V}$ is negative on the boundary $V = c$, sols. starting in the set $\{V(x) \leq c\}$ will remain therein for all future time.

- Hence, the sol. are uniformly bounded.

- Moreover, if we pick any number ϵ such that $(\delta^2/2) < \epsilon < c$, then $\dot{V}$ will be negative in $\{\epsilon \leq V \leq c\}$, which shows that, in this set, V will decrease monotonically until the sol. enters the set $\{V \leq \epsilon\}$.

- Form the time on, the sol. cannot leave the set $\{V \leq \epsilon\}$ because $\dot{V}$ is negative on the boundary $V = \epsilon$.

- Thus, we can conclude that the sol. is uniformly ultimately bounded with the ultimate bound $|x| \leq \sqrt{2\epsilon}$. $V(x) \;=\; x^2/2 \;\leq\; \epsilon$

- Consider the following system:

$$\dot{x} = f(t, x) \quad (4.32)$$

where $f : [0, \infty) \times D \to R^n$
is piecewise continuous in t and
locally Lipschitz in x on $[0, \infty) \times D$,
and $D \subset R^n$ is a domain
containing the origin.

- Definition 4.6:

  The solutions of $\dot{x} = f(t, x)$ are

- uniformly bounded

  if there exists a positive constant c,
  independent of $t_0 \geq 0$, and
  for every $a \in (0, c)$,
  there is $\beta = \beta(a) > 0$,
  independent of t_0, such that

$$||x(t_0)|| \leq a \Rightarrow ||x(t)|| \leq \beta, \quad \forall t \geq t_0 \quad (4.33)$$

- globally uniformly bounded

  if (4.33) holds for arbitrarily large a.

Definitions of UB, GUB, UUB, GUUB

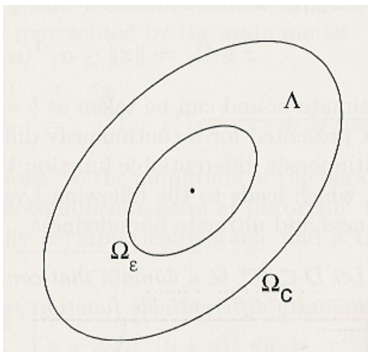
- uniformly ultimately bounded
  with ultimate bound b
  if there exist positive constants b and c,
  independent of $t_0 \geq 0$,
  and for every $a \in (0, c)$,
  there is $T = T(a, b) \geq 0$,
  independent of t_0, such that

  $$||x(t_0)|| \leq a \Rightarrow ||x(t)|| \leq b, \quad \forall t \geq t_0 + T \quad (4.34)$$

- globally uniformly ultimately bounded
  if (4.34) holds for arbitrarily large a.

- For autonomous systems,
  we may drop the word "uniformly"
  since the solution depends only on $t - t_0$.

Lyapunov Analysis - 1



- Consider a cont. diff., P.D. fun. $V(x)$ and
  suppose that $\{V(x) \leq c\}$ is compact,
  for some $c > 0$.

- Let $\Lambda = \{\epsilon \leq V(x) \leq c\}$
  for some positive constant $\epsilon < c$.

- Suppose the derivative of V
  along the traj. of $\dot{x} = f(t, x)$ satisfies

  $$\dot{V}(t, x) \leq -W_3(x), \; \forall x \in \Lambda, \; \forall t \geq t_0 \quad (4.35)$$

  where $W_3(x)$ is a cont. P.D. function.

- Inequality (4.35) implies that
  $\Omega_c = \{V(x) \leq c\}$ and $\Omega_\epsilon = \{V(x) \leq \epsilon\}$
  are positively invariant
  since on the boundaries $\partial\Omega_c$ and $\partial\Omega_\epsilon$,
  the derivative $\dot{V}$ is negative.

- Since $\dot{V}$ is negative in Λ,
  a trajectory starting in Λ must
  move in a direction of decreasing $V(x(t))$.

- In fact, while in Λ,
  V satisfies (4.22), (4.24) of Thm 4.9.

$$W_1(x) \le V(t,x) \le W_2(x), \quad (4.22)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad (4.24)$$

- Therefore, the trajectory behaves
  as if the origin was U.A.S.
  and satisfies an inequality of the form

  $$||x(t)|| \le \beta(||x(t_0)||, t - t_0)$$

  for some class $\mathcal{KL}$ function β.

- $V(x(t))$ will continue decreasing
  until the traj. enters Ω_ϵ in finite time
  and stays therein for all future time.

- The fact that
  the trajectory enters Ω_ϵ in finite time
  can be shown as follows:

- Because $W_3(x)$ is continuous
  and Λ is compact,
  let $k = \min_{x \in \Lambda} W_3(x) > 0$.

- It is positive since $W_3(x)$ is P.D.

- Hence, $W_3(x) \ge k$, $\forall x \in \Lambda$ (4.36)

Lyapunov Analysis - 4

- Inequalities (4.35) and (4.36) imply that

$$\dot{V}(t,x) \le -k,\ \forall x \in \Lambda,\ \forall t \ge t_0$$

$$\dot{V}(t,x) \le -W_3(x), \quad (4.35)$$

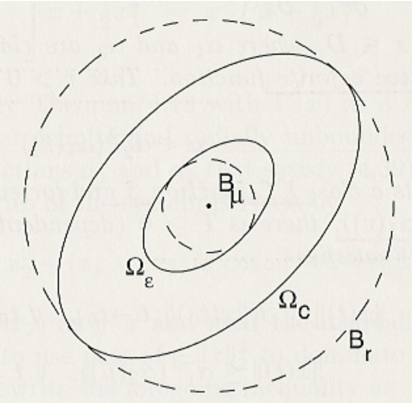
$$W_3(x) \ge k, \quad (4.36)$$

- Therefore,

$$\begin{aligned} V(x(t)) &\le V(x(t_0)) - k(t - t_0) \\ &\le c - k(t - t_0) \end{aligned}$$

which shows that $V(x(t))$ reduces to ϵ within the time interval $[t_0, t_0 + (c-\epsilon)/k]$.

$$\Lambda = \{\epsilon \le V(x) \le c\}$$

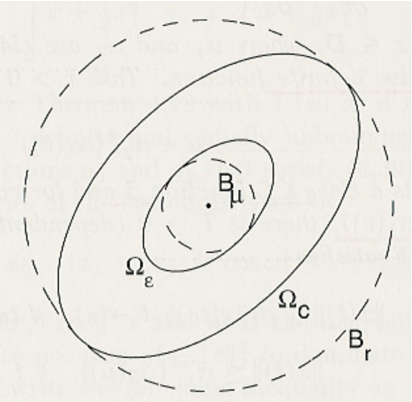


- In many problems, $\dot{V} \le -W_3$ is obtained by using norm inequalities.
- In such cases, it is more likely that we arrive at (4.37)

$$\dot{V}(t,x) \le -W_3(x),\ \forall \mu \le ||x|| \le r,\ \forall t \ge t_0$$

Lyapunov Analysis - 5

- If r is sufficiently larger than μ, we can choose c and ϵ such that Λ is nonempty and contained in $\{\mu \le ||x|| \le r\}$.



- In particular, let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

$$W_1(x) \le V(t,x) \le W_2(x), \quad (4.22)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad (4.24)$$

$$\Lambda = \{\epsilon \le V(x) \le c\}$$

- From the left inequality of (4.38), we have

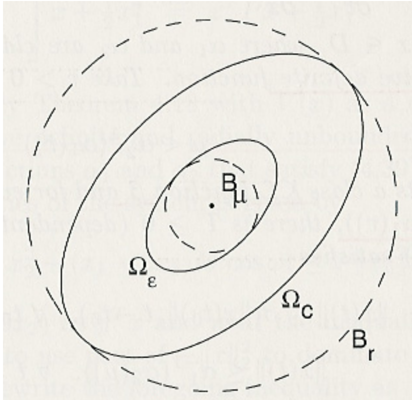
$$V(x) \le c \Rightarrow \alpha_1(||x||) \le c \Leftrightarrow ||x|| \le \alpha_1^{-1}(c)$$

- Therefore, taking $c = \alpha_1(r)$ ensures that $\Omega_c \subset B_r$.

- From the right inequality of (4.38), we have

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

$$||x|| \le \mu \Rightarrow V(x) \le \alpha_2(\mu)$$



- Consequently, taking $\epsilon = \alpha_2(\mu)$ ensures that $B_\mu \subset \Omega_\epsilon$.
- To obtain $\epsilon < c$, we must have $\mu < \alpha_2^{-1}(\alpha_1(r))$.
- All trajectories starting in Ω_c enter Ω_ϵ within a finite time T.
- To calculate the ultimate bound on $x(t)$, we use the left inequality of (4.38) to write

$$V(x) \le \epsilon \Rightarrow \alpha_1(||x||) \le \epsilon \Leftrightarrow ||x|| \le \alpha_1^{-1}(\epsilon)$$

- Recalling that $\epsilon = \alpha_2(\mu)$, we see that

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||) \quad (4.38)$$

$$x \in \Omega_\epsilon \Rightarrow ||x|| \le \alpha_1^{-1}(\alpha_2(\mu))$$

- Hence, the ultimate bound can be taken as $b = \alpha_1^{-1}(\alpha_2(\mu))$.
- The ideas just presented for a cont. diff. functon $V(x)$ can be extended to $V(t,x)$, as long as $V(t,x)$ satisfies inequality (4.38), which leads to the Lyapunov-like theorem for showing uniform boundedness and ultimate boundedness.

- Theorem 4.18:
- Let $D \subset R^n$ be a domain
  that contains the origin and
  $V : [0, \infty) \times D \rightarrow$
  be a cont. diff. func. such that

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||) \quad (4.39)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \ \forall ||x|| \geq \mu > 0 \quad (4.40)$$

$\forall t \geq 0$ and $\forall x \in D$,
where α_1 and α_2 are class $\mathcal{K}$ functions
and $W_3(x)$ is a cont. P.D. functon.

- Take $r > 0$ such that $B_r \subset D$ and
  suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$ (4.41)

- Then, there exists a class $\mathcal{KL}$ function β
  and for every initial state $x(t_0)$,
  satisfying $||x(t_0)|| \leq \alpha_2^{-1}(\alpha_1(r))$,
  there is $T \geq 0$ (dependent on $x(t_0)$ and μ)
  such that the soluton of $\dot{x} = f(t, x)$ satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0), \ \forall t_0 \leq t \leq t_0 + T \quad (4.42)$$

$$||x(t)|| \leq \alpha_1^{-1}(\alpha_2(\mu)), \ \forall t \geq t_0 + T \quad (4.43)$$

- Hence, $x(t)$ is U.B. (4.42) & U.U.B. (4.43).

- Moreover, if $D = R^n$ and
  α_1 belongs to class $\mathcal{K}_\infty$,
  then (4.42) and (4.43) hold
  for any initial state $x(t_0)$,
  with no restriction on how large μ is.

- Proof: See Appendix C. 9.

Theorem 4.18: G.U.U.B.

- Inequalities (4.42) and (4.43) show that $x(t)$ is uniformly bounded for all $t \geq t_0$ and uniformly ultimately bounded with the ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$.
- The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound.
- As $\mu \to 0$, the ultimate bound approaches zero.
- The main application of Thm 4.18 arises in studying the stability of perturbed syst.

Example 4.24

- Example 4.24:
- Consider a mass-spring system with a hardening spring, linear viscous damping, and a periodic external force can be represented by the Duffings equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos wt$$

- Taking $x_1 = y, x_2 = \dot{y}$ and assuming certain numerical values for the various constants, the system is represented by the state model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M\cos wt \end{aligned}$$

where $M \geq 0$ is proportional to the amplitude of the periodic external force.

Example 4.24

- When $M = 0$,
  the system has an E.P. at the origin.
- It is shown in Example 4.6 that
  the origin is G.A.S and
  a Lyapunov function can be taken as

$$
\begin{aligned}
V(x) &= x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3) dy \\
&= x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + x_1^2 + \frac{1}{2} x_1^4 \\
&= x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2} x_1^4 \\
&\triangleq x^T P x + \frac{1}{2} x_1^4 \\
&= \frac{3}{2} x_1^2 + x_2^2 + x_1 \, x_2 + \frac{1}{2} x_1^4
\end{aligned}
$$

- When $M > 0$, we apply Thm 4.18
  with $V(x)$ as a candidate func.

Example 4.24

- $V(x)$ is P.D. and R.U.;
  hence, by Lemma 4.3,
  there exist class $\mathcal{K}_\infty$ functions α_1 and α_2
  that satisfy (4.39) globally.

$$\alpha_1(||x||) \le V(t,x) \le \alpha_2(||x||) \quad (4.39)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall ||x|| \ge \mu > 0 \quad (4.40)$$

- The derivative of V along the traj.
  of the system is given by

$$V(x) = \frac{3}{2} x_1^2 + x_2^2 + x_1 \, x_2 + \frac{1}{2} x_1^4$$

$$
\begin{aligned}
\dot{V} &= 3\, x_1 \, \dot{x}_1 + 2\, x_2 \, \dot{x}_2 + \dot{x}_1 \, x_2 + x_1 \, \dot{x}_2 + 2\, x_1^3 \, \dot{x}_1 \\
&= -x_1^2 - x_2^2 - x_1^4 + (x_1 + 2x_2) \, M \, \cos wt \\
&\le -||x||_2^2 - x_1^4 + M\sqrt{5}\, ||x||_2
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(1 + x_1^2) x_1 - x_2 + M \cos wt
\end{aligned}
$$

where we wrote $(x_1 + 2x_2)$ as $y^T x$
and used the inequality $y^T x \le ||x||_2 ||y||_2$.

- To satisfy (4.40), we want to use part of $-||x||_2^2$ to dominate $M\sqrt{5}||x||_2$ for large $||x||$.

- Toward that end, we rewrite the foregoing inequality as

$$\dot{V} \le -(1-\theta)||x||_2^2 - x_1^4 - \theta||x||_2^2 + M\sqrt{5}||x||_2$$

where $0 < \theta < 1$.

$$\begin{aligned} -\theta||x||_2^2 + M\sqrt{5}||x||_2 &\le 0 \\ M\sqrt{5}||x||_2 &\le \theta||x||_2^2 \\ \frac{M\sqrt{5}}{\theta} &\le ||x||_2 \end{aligned}$$

- Then,

$$\dot{V} \le -(1-\theta)||x||_2^2 - x_1^4, \ \forall ||x||_2 \ge \frac{M\sqrt{5}}{\theta}$$

which shows that inequality (4.40) is satisfied globally with $\mu = M\sqrt{5}/\theta$.

- We conclude that the solutions are globally uniformly ultimately bounded.

- Suppose we want to go the extra step of calculating the ultimate bound.

- First to find the functions α_1 and α_2.

- From the inequalities $V(x) = x^TPx + \frac{1}{2}x_1^4$

$$V(x) \ge x^TPx \ge \lambda_{min}(P)||x||_2^2$$

$$V(x) \le x^TPx + \tfrac{1}{2}||x||_2^4 \le \lambda_{max}(P)||x||_2^2 + \tfrac{1}{2}||x||_2^4$$

- We see that α_1 and α_2 can be taken as

$$\begin{aligned} \alpha_1(r) &= \lambda_{min}(P)r^2, \\ \alpha_2(r) &= \lambda_{max}(P)r^2 + \frac{1}{2}r^4 \end{aligned}$$

Example 4.24

- Thus, the ultimate bound is given by

$$
\begin{aligned}
b &= \alpha_1^{-1}(\alpha_2(\mu)) \\
&= \sqrt{\frac{\alpha_2(\mu)}{\lambda_{min}(P)}} \\
&= \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}
\end{aligned}
$$