Nonlinear Systems Analysis

Lecture Note

Section 4.7

Converse Theorems (Lyapunov Stability)

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Outline

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Converse Theorems (4.7)

- Two Questions:
  - Is there a function
    that satisfies the conditions of the Thms?
    (Thm 4.9, 4.10, e.x.)
  - How can we search for such a function?
- In many cases,
  Lyapunov theory provides an affirmative
  answer to the first question.
- The answer takes the form of
  a converse Lyapunov theorem, which is
  the inverse of one of Lyapunov's theorems.
- Most of these converse theorems are proven
  by actually constructing auxiliary functions
  that satisfy the conditions of the respective
  theorems.

Converse Theorems

- But, the construction almost always
  assumes the knowledge of
  the sol. of the diff. eqn.
- In this section,
  we give three converse Lyapunov theorems.
- The first one is a converse Lyapunov thm
  when the origin is exponentially stable and,
- The second,
  when it is uniformly asymptotically stable.
- The third thm applies to autonomous syst.
  and defines the converse Lyapunov func.
  for the whole region of attraction of
  an asymptotically stable equilibrium point.

Theorem 4.14: E.S.

- Theorem 4.14:
- Let $x = 0$ be an EP for the NL system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is cont. diff., $D = \{x \in R^n \mid ||x|| < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bdd on D, uniformly in t.

- Let k, λ, and r_0 be positive const. with $r_0 < r/k$.

- Let $D_0 = \{x \in R^n \mid ||x|| < r_0\}$.

- Assume that the traj. of the syst. satisfy

$$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)},$$

$$\forall x(t_0) \in D_0, \; \forall t \geq t_0 \geq 0$$

Theorem 4.14: E.S.

- Then, there is a function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$c_1||x||^2 \leq V(t, x) \leq c_2||x||^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -c_3||x||^2$$

$$\left|\left|\frac{\partial V}{\partial x}\right|\right| \leq c_4||x||$$

for some positive const. $c_1, c_2, c_3,$ and c_4.

- Moreover, if $r = \infty$ and the origin is G.E.S., then $V(t, x)$ is defined and satisfies the aforementioned inequalities on R^n.

- Furthermore, if the system is autonomous, V can be chosen independent of t.

- Proof:
- Due to the equivalence of norms, it is sufficient to prove the thm for the 2-norm.
- Let $\phi(\tau; t, x)$ denote the sol. of the syst. that starts at (t, x); that is, $\phi(t; t, x) = x$.
- For all $x \in D_0$, $\phi(\tau; t, x) \in D$ for all $\tau \geq t$.
- Let

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau; t, x)\phi(\tau; t, x) d\tau$$

where δ is a positive constant to be chosen.

- Due to the exponentially decaying bound on the trajectories, we have

$$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)},$$

$$\forall x(t_0) \in D_0, \ \forall t \geq t_0 \geq 0$$

$$\begin{aligned} V(t,x) &= \int_t^{t+\delta} \phi^T(\tau; t, x)\phi(\tau; t, x) d\tau \\ &= \int_t^{t+\delta} ||\phi(\tau; t, x)||_2^2 d\tau \\ &\leq \int_t^{t+\delta} k^2 e^{-2\lambda(\tau - t)} d\tau ||x||_2^2 \\ &= \frac{k^2}{2\lambda}(1 - e^{-2\lambda\delta})||x||_2^2 \end{aligned}$$

- On the other hand, the Jacobian matrix $[\partial f / \partial x]$ is bdd on D.

- Let

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|_2 \leq L,\ \forall x \in D$$

- Then, $||f(t,x)||_2 \leq L||x||_2$ and
  $\phi(\tau;t,x)$ satisfies the lower bound

$$||\phi(\tau;t,x)||_2^2 \geq ||x||_2^2 e^{-2L(\tau-t)}$$

- Hence,

$$\begin{aligned} V(t,x) &\geq \int_t^{t+\delta} e^{-2L(\tau-t)} d\tau ||x||_2^2 \\ &= \frac{1}{2L}(1-e^{-2L\delta})||x||_2^2 \end{aligned}$$

- Thus, $V(t,x)$ satisfies
  the first inequality of the theorem with

$$c_1 = \frac{1-e^{-2L\delta}}{2L} \text{ and } c_2 = \frac{k^2(1-e^{-2\lambda\delta})}{2\lambda}$$

- To calculate the derivative of V
  along the trajectories of the system,
  define the sensitivity functions

$$\phi_t(\tau;t,x) = \frac{\partial}{\partial t}\phi(\tau;t,x)$$

$$\phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x)$$

- Then,

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x)$$

$$\begin{aligned} &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - \phi^T(t;t,x)\phi(t;t,x) \\ &+ \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_t(\tau;t,x)d\tau + \int_t^{t+\delta} 2\phi^T(\tau;t,x)\phi_x(\tau;t,x)d\tau f(t,x) \end{aligned}$$

$$\begin{aligned} &= \phi^T(t+\delta;t,x)\phi(t+\delta;t,x) - ||x||_2^2 \\ &+ \int_t^{t+\delta} 2\phi^T(\tau;t,x)\left[\phi_t(\tau;t,x) + \phi_x(\tau;t,x)f(t,x)\right] d\tau \end{aligned}$$

- It is not difficult to show that (Ex 3.30)

$$\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t, x) \equiv 0, \ \forall \tau \geq t$$

- Therefore,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) = \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - ||x||_2^2$$
$$\leq -(1 - k^2 e^{-2\lambda\delta})||x||_2^2$$

- By choosing $\delta = \ln(2k^2)/(2\lambda)$,
  the second inequality of the thm.
  is satisfied with $c_3 = 1/2$.

$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}$,

$\forall x(t_0) \in D_0, \ \forall t \geq t_0 \geq 0$

- To show the last inequality, let us note that $\phi_x(\tau; t, x)$ satisfies the sensitivity eqn.

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau; t, x))\phi_x, \ \phi_x(t; t, x) = I$$

- Since $||\frac{\partial f}{\partial x}(t,x)||_2 \leq L$ on D,
  ϕ_x satisfies the bound

$$||\phi_x(\tau; t, x)||_2 \leq e^{L(\tau - t)}$$

- Therefore,

$$\left\|\frac{\partial V}{\partial x}\right\|_2 = \left\|\int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x) d\tau\right\|_2$$
$$\leq \int_t^{t+\delta} 2\left\|\phi(\tau; t, x)\right\|_2 \left\|\phi_x(\tau; t, x)\right\|_2 d\tau$$
$$\leq \int_t^{t+\delta} 2ke^{-\lambda(\tau-t)}e^{L(\tau-t)} d\tau ||x||_2$$
$$= \frac{2k}{\lambda - L}[1 - e^{-(\lambda - L)\delta}]||x||_2$$

- The last inequality of the thm. is satisfies

$$\text{with} \quad c_4 = \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]$$

- If all the assumptions hold globally, then r_0 can be chosen arbitrarily large.

- If the system is autonomous, then $\phi(\tau; t, x)$ depends only on $(\tau - t)$; i.e.,

$$\phi(\tau; t, x) = \psi(\tau - t; x)$$

$$V(t, x) = \int_t^{t+\delta} \phi^T(\tau; t, x)\phi(\tau; t, x) d\tau$$

- Then,

$$\begin{aligned} V(t, x) &= \int_t^{t+\delta} \psi^T(\tau - t; x)\psi(\tau - t; x) d\tau \\ &= \int_0^{\delta} \psi^T(s; x)\psi(s; x) ds \end{aligned}$$

which is independent of t.

- QED

Theorem 4.15: E.S. of NL & L Systems

- Theorem 4.15:
- Let $x = 0$ be an E.P. for the NL syst.

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is cont. diff., $D = \{x \in R^n \mid ||x||_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bdd and Lipschitz on D, uniformly in t.

- Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)\Big|_{x=0}$$

- Then, $x = 0$ is an E.S. E.P. for the NL syst. iff it is an E.S. E.P for the L syst.

$$\dot{x} = A(t)x$$

- Proof:
- The "if" part follows from Thm 4.13.
- To prove the "only if" part,
  write the linear system as

  $\dot{x} = f(t, x) - [f(t, x) - A(t)x] = f(t, x) - g(t, x)$

- Recalling the argument preceding
  Thm 4.13, we know that

  $$||g(t, x)||_2 \leq L||x||_2^2, \ \forall x \in D, \ \forall t \geq 0$$

- Since $x = 0$ is an E.S. E.P. of
  the NL syst.,
  there are positive const $k, \lambda,$ and c
  such that

  $$||x(t)||_2 \leq k||x(t_0)||_2 e^{-\lambda(t-t_0)},$$

  $$\forall t \geq t_0 \geq 0, \forall ||x(t_0)||_2 < c$$

- Choosing $r_0 < \min\{c, r/k\}$,
  all the conditions of Thm 4.14 are satisfied.
- Let $V(t, x)$ be the function
  provided by Thm 4.14 and
  use it as a Lyapunov function candidate
  for the L syst.

Theorem 4.15: E.S. of NL & L Systems

- Then,

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) - \frac{\partial V}{\partial x}g(t,x) \\ &\le -c_3||x||_2^2 + c_4L||x||_2^3 \\ &< -(c_3 - c_4L\rho)||x||_2^2, \qquad \forall ||x||_2 < \rho\end{aligned}$$

- The choice $\rho < \min\{r_0, c_3/(c_4L)\}$ ensures that $\dot{V}(t,x)$ is N.D. in $||x||_2 < \rho$.

- Consequently, all the conditions of Thm 4.10 are satisfied in $||x||_2 < \rho$, and we conclude that the origin is an E.S. E.P. for the L. syst. QED

Corollary 4.3

- Corollary 4.3:
- Let $x = 0$ be an E.P. of the NL syst.

$$\dot{x} = f(x)$$

where $f(x)$ is cont. diff. in some nbhd of $x = 0$.

- Let

$$A = \left[\frac{\partial f}{\partial x}\right](0)$$

- Then, $x = 0$ is an E.S. E.P. for the NL system iff A is Hurwitz.

Theorem 4.16: U.A.S.

- Theorem 4.16:
- Let $x = 0$ be an E.P. for the NL syst.

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is cont. diff.,
$D = \{x \in R^n \mid ||x||_2 < r\}$, and
the Jacobian matrix $[\partial f / \partial x]$ is
bdd on D, uniformly in t.

- Let β be a class $\mathcal{KL}$ function and
r_0 be a positive constant
such that $\beta(r_0, 0) < r$.

- Let $D_0 = \{x \in R^n \mid ||x|| < r_0\}$.

Theorem 4.16: U.A.S.

- Assume that the traj. of the syst. satisfies

$$||x(t)|| \leq \beta(||x(t_0)||, t - t_0),$$

$$\forall x(t_0) \in D_0, \quad \forall t \geq t_0 \geq 0$$

- Then, there is a cont. diff. function
$V : [0, \infty) \times D_0 \to R$
that satisfies the inequalities

$$\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(||x||)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(||x||)$$

where $\alpha_1, \alpha_2, \alpha_3,$ and α_4 are
class $\mathcal{K}$ functions defined on $[0, r_0]$.

- If the system is autonomous,
V can be chosen independent of t.
- Proof: See Appendix C.7.

Theorem 4.17: Region of Attraction

- Theorem 4.17:
- Let $x = 0$ be an AS EP for the NL syst

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is locally Lipschitz and $D \subset R^n$ is a domain that contains $x = 0$.

- Let $R_A \subset D$ be the region of attraction of $x = 0$.
- Then, there is a smooth, PD function $V(x)$ and a cont., PD function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \le -W(x), \quad \forall x \in R_A$$

and for any $c > 0$, $\{V(x) \le c\}$ is a compact subset of R_A.

- When $R_A = R^n$, $V(x)$ is radially unbounded.
- Proof: See Appendix C.8.

Theorem 4.17: Region of Attraction

- An interesting feeture of Thm 4.17 is that any bounded subset S of the region of attraction can be included in a compact set of the form $\{V(x) \le c\}$ for some constant $c > 0$.
- This feature is useful because quite often we have to limit our analysis to a positively invariant, compact set of the form $\{V(x) \le c\}$.
- With the property $S \subset \{V(x) \le c\}$, our analysis will be valid for the whole set S.

Theorem 4.17: Region of Attraction

- If, on the other hand,
  all we know is the existence
  of a Lyapunov function $V_1(x)$ on S,
  we will have to choose a constant c_1
  such that $\{V_1(x) \leq c_1\}$ is compact and
  included in S;
  then our analysis will be limited to
  $\{V_1(x) \leq c_1\}$, which is only a subset of S.

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