Nonlinear Systems Analysis

Lecture Note

Section 4.6

LTV Systems & Linearization (Lyapunov Stability)

Feng-Li Lian

NTU-EE

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

U.A.S. of LTV Systems (4.6)

- Consider the linear time-varying systems:

$$\dot{x} = A(t)x \qquad (4.29)$$

- $x = 0$ is an equilibrium point

- The stability behavior of the origin as an equilibrium point can be completely characterized in terms of the state transition matrix of the system.

$e^{A(t-t_0)}$

- Form linear system theory, we know that the solution is given by

$$x(t) = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix.

Theorem 4.11: G.U.A.S.

- Theorem 4.11

- The equilibrium point $x = 0$ of (4.29) is (globally) uniformly asymptotically stable

$||x(t)|| \le \beta\,(||x(t_0)||, t-t_0)$

- if and only if the state transition matrix satisfies the inequality

$$\boxed{||\Phi(t, t_0)|| \le k\, e^{-\lambda(t-t_0)},} \qquad \forall t \ge t_0 \ge 0$$

for some positive constants k and λ.

- Proof:
- Due to the linear dependence of $x(t)$ on $x(t_0)$,
  if the origin is U.A.S., it is globally so.
- Sufficiency:

$$\begin{aligned} ||x(t)|| &\le ||\Phi(t,t_0)||\,||x(t_0)|| \\ &\le k||x(t_0)||e^{-\lambda(t-t_0)} \end{aligned}$$

- Necessity:
  Suppose the origin is U.A.S.
- Then, there is a class $\mathcal{KL}$ function β such that

$$\underline{||x(t)|| \le \beta\Big(||x(t_0)||,\ t-t_0\Big)},$$

$$\forall t \ge t_0, \quad \forall x(t_0) \in R^n$$

- By the def. of an induced matrix norm (Appendix A), we have

$\mathbb{R}^n$ $x(t)$

$$\begin{aligned} ||\Phi(t,t_0)|| &= \max_{\underline{||x||=1}} ||\Phi(t,t_0)\boxed{x}|| \\ &\le \max_{||x||=1} \beta\Big(||x||,\ t-t_0\Big) \\ &= \beta\Big(1,\ t-t_0\Big) \end{aligned}$$

- Since $\beta(1,s) \to 0$ as $s \to \infty$,
  there exists $T > 0$
  such that $\beta(1,T) \le 1/e$.

- For any $t \ge t_0$,
  let N be the smallest positive integer
  such that $t \le t_0 + NT$.

Theorem 4.11: G.U.A.S.

- Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals of width T each.

$t_0 + (N-1)T \le t \le t_0 + NT$

- Using the transition property of $\Phi(t, t_0)$, we can write

$$
\begin{aligned}
\Phi(t, t_0) \;=\; & \Phi\left(t,\; t_0 + (N-1)T\right) \\
& \Phi\left(t_0 + (N-1)T,\; t_0 + (N-2)T\right) \\
& \cdots \\
& \Phi\left(t_0 + T,\; t_0\right)
\end{aligned}
$$

- Hence,

$$
\begin{aligned}
\|\Phi(t, t_0)\| \;&\le\; \|\Phi(t, t_0 + (N-1)T)\| \prod_{k=1}^{k=N-1} \|\Phi(t_0 + kT, t_0 + (k-1)T)\| \\
&\le\; \beta(1,0) \prod_{k=1}^{k=N-1} \frac{1}{e} = e\,\beta(1,0)\,e^{-N} \\
&\le\; e\,\beta(1,0)\,e^{-(t-t_0)/T} = k\,e^{-\lambda\,(t-t_0)}
\end{aligned}
$$

where $k = e\beta(1,0)$ and $\lambda = 1/T$. Q.E.D.

Theorem 4.11: G.U.A.S.

- Theorem 4.11 shows that, for linear systems, U.A.S. of the origin = E.S..

- Note that, for linear time-varying systems, U.A.S. cannot be characterized by the location of the eigenvalues of A.

- Thm 4.11 is not helpful as a stability test because it needs to solve the state eqn.

- However, it guarantees the existence of a Lyapunov function. See Example 4.21, for example.

Theorem 4.12: E.S.

- Theorem 4.12
- Let $x = 0$ be the E.S. E.P. of

$$\dot{x} = A(t)\,x(t), \quad (4.29)$$

- Suppose $A(t)$ is continuous and bounded.
- Let $Q(t)$ be a cont., bdd., P.D., symm. matrix.

$$0 \le c_3 I \le Q(t) \le c_4 I, \qquad \forall t \ge 0$$

- THEN, there is a cont. diff., bdd., P.D., symm. matrix $P(t)$ that satisfies:

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \quad (4.28)$$

- Hence, $V(t,x) = x^T P(t) x$ is a Lyapunov function of the system that satisfies the conditions of Thm 4.10.

Theorem 4.12: E.S.

- Proof:
- Let

$$\dot{x} = A(t)\,x(t), \quad (4.29)$$

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)d\tau$$

and $\phi(\tau;t,x)$ be the solution of (4.29) that starts at (t,x).

- Due to linearity, $\phi(\tau;t,x) = \Phi(\tau,t)\,x$.

- In view of the definition of $P(t)$, we have

$x(\tau)$

$$\boxed{x^T P(t) x} = \int_t^\infty x^T\,\Phi^T(\tau,t)\,Q(\tau)\,\Phi(\tau,t)\,x\,d\tau$$

$$= \int_t^\infty \phi^T(\tau;t,x)\,Q(\tau)\,\phi(\tau;t,x)d\tau$$

Theorem 4.12: E.S.

- Because $||\Phi(t,t_0)|| \le k\, e^{-\lambda(t-t_0)}$,
- And $0 < c_3 I \le Q(t) \le c_4 I$

$$
\begin{aligned}
x^T P(t) x &\le \int_t^\infty c_4 ||\Phi(\tau,t)||_2^2\, ||x||_2^2 d\tau \\
&\le \int_t^\infty k^2 e^{-2\lambda(\tau-t)} d\tau \quad c_4 ||x||_2^2 \\
&= \frac{k^2 c_4}{2\lambda} ||x||_2^2 \\
&\triangleq c_2\, ||x||_2^2
\end{aligned}
$$

- On the other hand, since

$$||A(t)||_2 \le L, \quad \forall t \ge 0$$

the solution $\phi(\tau; t, x)$ satisfies the lower bound

$$||\phi(\tau;t,x)||_2^2 \ge ||x||_2^2\, e^{-2\,L\,(\tau-t)}$$

Exercise 3.17:

$$
\begin{aligned}
\dot{x} &= a(t)x \\
-L \le\ & a(t) \le L \\
-Lx \le\ & \dot{x} \le Lx \\
e^{-L(t-t_0)} \le\ & x(t) \le e^{L(t-t_0)} \\
e^{-2L(t-t_0)} \le\ & x^2(t) \le e^{2L(t-t_0)}
\end{aligned}
$$

Theorem 4.12: E.S.

- Hence,

$$
\begin{aligned}
x^T P(t) x &\ge \int_t^\infty c_3 ||\phi(\tau;t,x)||_2^2 d\,\tau \\
&\ge \int_t^\infty e^{-2L(\tau-t)} d\,\tau\ c_3 ||x||_2^2 \\
&= \frac{c_3}{2L} ||x||_2^2 \\
&\triangleq c_1\, ||x||_2^2
\end{aligned}
$$

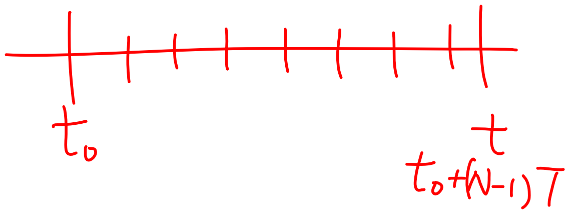
- Thus, $c_1\, ||x||_2^2 \le x^T P(t) x \le c_2\, ||x||_2^2$
  which shows that $P(t)$ is P.D. and bdd.

- The definition of P(t) shows that
  it is symm. and cont. diff..

- And

$$P(t) = \int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau$$

$$\begin{aligned}
x(\tau) &= \Phi(\tau,t)x(t) \\
\frac{d}{dt}x(\tau) &= \boxed{\frac{\partial}{\partial t}\Phi(\tau,t)}x(t) + \Phi(\tau,t)\boxed{\frac{d}{dt}x(t)} \\
0 &= \frac{\partial}{\partial t}\Phi(\tau,t)x(t) + \Phi(\tau,t)A(t)x(t) \\
\frac{\partial}{\partial t}\Phi(\tau,t) &= -\Phi(\tau,t)A(t)
\end{aligned}$$



- In particular,

$$\begin{aligned}
\dot{P}(t) &= \int_t^\infty \Phi^T(\tau,t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau,t)\,d\tau + \int_t^\infty \left[\frac{\partial}{\partial t}\Phi^T(\tau,t)\right]Q(\tau)\Phi(\tau,t)\,d\tau - Q(t) \\
&= -\boxed{\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau}A(t) - A^T(t)\boxed{\int_t^\infty \Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau} - Q(t) \\
&= -P(t)A(t) - A^T(t)P(t) - Q(t)
\end{aligned}$$

- The fact that $V(t,x) = x^T P(t) x$ is a Lyaunov function is shown in Ex 4.21.

Q.E.D.

Linearization to Non-autonomous System

- Consider the nonlinear nonautonomous sys

$$\dot{x} = f(t,x)$$

where $f : [0,\infty] \times D \to R^n$ is cont. diff.
and $D = \{x \in R^n \mid ||x||_2 < r\}$.

- Suppose the origin $x = 0$ is an E.P.
for the systems at $t = 0$;
that is, $f(t,0) = 0$ for all $t \geq 0$.

- Furthermore, suppose
the Jacobian matrix $[\partial f/\partial x]$ is bdd. and
Lipschitz on D, uniformly in t; thus,

$$\left|\left|\frac{\partial f_i}{\partial x}(t,x_1) - \frac{\partial f_i}{\partial x}(t,x_2)\right|\right|_2 \leq L_1||x_1 - x_2||_2,$$

$\forall x_1, x_2 \in D$, $\forall t \geq 0$ for all $1 \leq i \leq n$.

- By the mean value theorem,

$$f_i(t,x) = f_i(t,0) + \frac{\partial f_i}{\partial x}(t,z_i)x$$

where z_i is a point on the line segment connecting x to the origin.

- Since $f(t,0) = 0$, we can write $f_i(t,x)$ as

$$\begin{aligned} f_i(t,x) &= \frac{\partial f_i}{\partial x}(t,z_i)x \\ &= \frac{\partial f_i}{\partial x}(t,0)x + \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right]x \end{aligned}$$

- Hence, $f(t,x) = A(t)x + g(t,x)$

where

. $A(t) = \frac{\partial f}{\partial x}(t,0)$ and

. $g_i(t,x) = \left[\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right]x$

- The function $g(t,x)$ satisfies

$$\begin{aligned} ||g(t,x)||_2 &\le \left(\sum_{i=1}^{n} \left|\left|\frac{\partial f_i}{\partial x}(t,z_i) - \frac{\partial f_i}{\partial x}(t,0)\right|\right|_2^2\right)^{1/2} ||x||_2 \\ &\le L\,||x||_2^2 \end{aligned}$$

where $L = \sqrt{n}L_1$.

- Therefore, in a small nbhd of the origin, we may approximate the nonlinear system by its linearization about the origin.

- The next theorem states Lyapunov's indirect method for showing E.S. of the origin in the nonautonomous case.

- Theorem 4.13:
- Let $x = 0$ be an E.P. for the NL sys

$$\dot{x} = f(t, x)$$

where $f : [0, \infty] \times D \to R^n$ is cont. diff., $D = \{x \in R^n \mid ||x||_2 < r\}$, and the Jacobian matrix $[\partial f / \partial x]$ is bdd. and Lipschitz on D, uniformly in t.

- Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)\Big|_{x=0}$$

- Then, the origin is an E.S. E.P. for the nonlinear system if it is an E.S. E.P. for the linear system

$$\dot{x} = A(t)x$$

- Proof:
- Since the linear system has an E.S. E.P. at the origin and $A(t)$ is cont. and bdd., Them 4.12 ensures the existence of a cont.diff., bdd., P.D. symm. matrix $P(t)$ that satisfies (4.28), where $Q(t)$ is cont., P.D., and symm.
- We use $V(t, x) = x^T P(t) x$ as a Lyapunov func. candidate for the NL sys.

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t) \qquad (4.28)$$

Theorem 4.13: E.S. of Non-Auto Syst.

- The derivative of $V(t,x)$ along the trajectories of the sys is given by

$0 \le c_3 I \le Q(t) \le c_4 I$

$c_1\ I \le P(t) \le c_2\ I$

$||g(t,x)||_2 \ \le\ L\ ||x||_2^2$

$$
\begin{aligned}
\dot{V}(t,x) &= x^T P(t)\dot{x} + \dot{x}^T(t,x)P(t)x + x^T\dot{P}(t)x \\
&= x^T P(t) f(t,x) + f^T(t,x)P(t)x + x^T\dot{P}(t)x \\
&= x^T\left[P(t)A(t) + A^T(t)P(t) + \dot{P}(t)\right]x + 2x^T P(t) g(t,x) \\
&= -x^T Q(t) x + 2x^T P(t) g(t,x) \\
&\le -c_3\ ||x||_2^2 + 2\ c_2\ L\ ||x||_2^3 \\
&\le -(c_3 - 2c_2 L\rho)||x||_2^2, \quad \forall ||x||_2 < \rho
\end{aligned}
$$

- Choosing $\rho < \min\{r, c_3/(2c_2L)\}$ ensures that $\dot{V}(t,x)$ is N.D. in $||x||_2 < \rho$.

- Therefore, all the conditions of Thm 4.10 are satisfied in $||x||_2 < \rho$, and we conclude that the origin is E.S..