Nonlinear Systems Analysis

Lecture Note

Section 4.5

Non-autonomous Systems (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

E.P. of Non-Autonomous Systems (4.5)

- Consider the nonautonomous system:

$$\dot{x} = f(t, x) \qquad (4.15)$$

where $f : [0, \infty] \times D \to R^n$
is piecewise continuous in t and
locally Lipschitz in x on $[0, \infty] \times D$,
and $D \subset R^n$ is a domain
that contains the origin $x = 0$.

- If $f(t, 0) = 0, \forall t \geq 0$,
the origin is an E.P. for (4.15) at $t = 0$

- An equilibrium point at the origin
could be a translation of a nonzero E.P. or,
a translation of a nonzero sol. of the syst.

E.P. of Non-Autonomous Systems

- To see the latter point,
suppose $\bar{y}(\tau)$ is a solution of the system

$$\frac{dy}{d\tau} = g(\tau, y)$$

defined for all $\tau \geq a$.

- The change of variables

$$x = y - \bar{y}(\tau); \quad t = \tau - a$$

- Transforms the system into the form

$$\begin{aligned} \dot{x} &= g(\tau, y) - \dot{\bar{y}}(\tau) \\ &= g(t + a, x + \bar{y}(t + a)) - \dot{\bar{y}}(t + a) \\ &\triangleq f(t, x) \end{aligned}$$

E.P. of Non-Autonomous Systems

- Since

$$\dot{\bar{y}}(t+a) = g(t+a, \bar{y}(t+a)), \forall t \geq 0$$

the origin $x = 0$ is an E.P. of
the transformed system at $t = 0$.

- So, by examining the stability behavior of
the origin
as an E.P. for the transformed system,
we determine the stability behavior of
the solution $\bar{y}(\tau)$ of the original system.

- Notice that if $\bar{y}(\tau)$ is not constant,
the transformed system will be
nonautonomous
even when the original system is
antonomous,
that is, even when $g(\tau, y) = g(y)$.

Stability and Asymptotic Stability

- This is why
studying the stability behavior of solutions
in the sense of Lyapunov can be done
only in the context of
studying the stability behavior of
the equilibria of nonautonomous systems.

- The notions of stability and
asymp. stability of
E.P. of nonautonomous systems
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- The notions of stability and
  asymp. stability of
  E.P. of nonautonomous systems
  are basically the same as those
  introduced in Definition 4.1
  for autonomous systems.

- The new element here is that,
  while the sol. of an autonomous system
  depends only on $(t - t_0)$,
  the sol. of a nonautonomous system
  may depend on both t and t_0.

- So, the stability behavior of E.P. will,
  in general, depend on t_0.

- The origin $x = 0$ is a stable E.P. for (4.15)
  if, for each $\varepsilon > 0$, and any $t_0 \geq 0$
  there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon, \quad \forall t \geq t_0$$

Stability and Asymptotic Stability

- The constant δ, in general, depends on the initial time t_0.

- The existence of δ for every t_0 does not necessarily guarantee that there is one constant δ, dependent only on ε, that would work for all t_0, as illustrated by the next example.

Example 4.17: Stability Case

- Example 4.17:
- The linear first-order system

$$\dot{x} = (6t \sin t - 2t)x$$

has the solution

$$\begin{aligned} x(t) &= x(t_0) \exp\left[\int_{t_0}^{t} (6\tau \sin \tau - 2\tau) d\tau\right] \\ &= x(t_0) \exp[6 \sin t - 6t \cos t - t^2 \\ &\quad -6 \sin t_0 + 6t_0 \cos t_0 + t_0^2] \end{aligned}$$

- For any t_0, the term $-t^2$ will evertually dominate, which shows that the exp term is bounded for all $t \geq t_0$ by a constant $c(t_0)$ dependent on t_0.

Example 4.17: Stability Case

- Hence,

$$|x(t)| < |x(t_0)|c(t_0), \quad \forall t \geq t_0$$

- For any $\varepsilon > 0$,
  the choice $\delta = \varepsilon / c(t_0)$ shows that
  the origin is stable.

- Suppose t_0 takes on the successive values
  $t_0 = 2n\pi$, for $n = 0, 1, 2, ...$, and
  $x(t)$ is evaluated π seconds later
  in each case.

Example 4.17: Stability Case

- Then,

$$x(t_0 + \pi) = x(t_0) \exp[(4n+1)(6-\pi)\pi]$$

  which implies that, for $x(t_0) \neq 0$,

$$\frac{x(t_0 + \pi)}{x(t_0)} \rightarrow \infty \text{ as } n \rightarrow \infty$$

- Thus, given $\varepsilon > 0$,
  there is no δ independent of t_0
  that would satisfy the stability requirement
  uniformly in t_0.

Example 4.18: Asymp. Stability Case

- Example 4.18:
- The linear first-order system

$$\dot{x} = -\frac{x}{1+t}$$

has the solution

$$\begin{aligned} x(t) &= x(t_0)\exp\left(\int_{t_0}^{t} \frac{-1}{1+\tau} d\tau\right) \\ &= x(t_0)\frac{1+t_0}{1+t} \end{aligned}$$

- Since $|x(t)| \le |x(t_0)|$, $\forall t \ge t_0$, the origin is clearly stable.
- Actually, given any $\varepsilon > 0$, we can choose δ independent of t_0.
- It is also clear that

$$x(t) \to 0 \text{ as } t \to \infty$$

Example 4.18: Asymp. Stability Case

- Consequently, according to Definition 4.1, the origin is asymptotically stable.
- Notice, however, that the convergence of x(t) to the origin is not uniform wrt the initial time t_0.
- Recall that convergence of x(t) to the origin is equivalent to saying that, given any $\varepsilon > 0$, there's $T = T(\varepsilon, t_0) > 0$ such that $|x(t)| < \varepsilon$, for all $t \ge t_0 + T$.
- Alghough this is ture for every t_0, the constant T cannot be chosen independent of t_0.

- Definition 4.4:
- The equilibrium point $x = 0$ of (4.15) is
  - stable if, for each $\varepsilon > 0$,
    there is $\delta = \delta(\varepsilon, t_0) > 0$
    such that $||x(t_0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$,
    $\forall t \geq t_0 \geq 0$ (4.16)
  - uniformly stable if, for each $\varepsilon > 0$,
    there is $\delta = \delta(\varepsilon) > 0$, independent of t_0,
    such that (4.16) is satisfied.
  - unstable if it is not stable.
  - asymptotically stable
    if it is stable and
    there is a positive constant $c = c(t_0)$
    such that $x(t) \to 0$ as $t \to \infty$,
    for all $||x(t_0)|| < c$.

- – uniformly asymptotically stable
    if it is uniformly stable and
    there is a positive constant c, in. of t_0,
    such that for all $||x(t_0)|| < c$,
    $x(t) \to 0$ as $t \to \infty$, uniformly in t_0;
    that is, for each $\eta > 0$,
    there is $T = T(\eta) > 0$ such that

    $$||x(t)|| < \eta, \ \forall t \geq t_0 + T(\eta), \ \forall ||x(t_0)|| < c$$

  - globally uniformly asymptotically stable
    if it is uniformly stable,
    $\delta(\varepsilon)$ can be chosen to satisfy
    $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, and,
    for each pair of positive numbers η & c,
    there is $T = T(\eta, c) > 0$ such that

    $$||x(t)|| < \eta, \ \forall t \geq t_0 + T(\eta, c), \ \forall ||x(t_0)|| < c$$

Lemma 4.5

- Lemma 4.5:
- The E.P. $x = 0$ of (4.15) is
  - uniformly stable

    iff there exist a class $\mathcal{K}$ function α and

    a positive constant c,

    independent of t_0, such that

    $$\forall t \geq t_0 \geq 0, \quad \forall ||x(t_0)|| < c$$

    $$||x(t)|| \leq \alpha(||x(t_0)||) \quad (4.19)$$

  - uniformly asymptotically stable

    iff there exist a class $\mathcal{KL}$ function β and

    a positive constant c,

    independent of t_0, such that

    $$\forall t \geq t_0 \geq 0, \quad \forall ||x(t_0)|| < c$$

    $$||x(t)|| \leq \beta(||x(t_0)||, t - t_0) \quad (4.20)$$

Lemma 4.5

- – globally uniformly asymptotically stable

  iff inequality (4.20) is satisfied

  for any initial state $x(t_0)$.

Definition 4.5

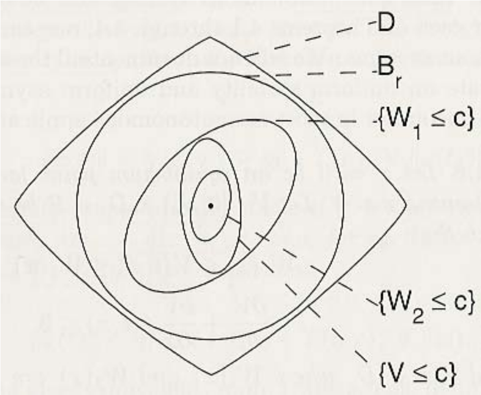
- Definition 4.5
- The E.P. $x = 0$ of (4.15) is
  - exponentially stable
    if there exist positive constants $c, k,$ &
    λ
    such that

    $$||x(t)|| \leq k||x(t_0)||e^{-\lambda(t-t_0)}, \ \forall ||x(t_0)|| < c$$

  - and globally exponentially stable
    if the above inequality is satisfied
    for any initial state $x(t_0)$.

Theorem 4.8: U.S. & Proof

- Theorem 4.8:
- Let $x = 0$ be an E.P. for (4.15) and
  $D \subset R^n$ be a domain containing $x = 0$.



- Let $V : [0, \infty] \times D \rightarrow R$ be
  a continuously differentiable func such that

  $$W_1(x) \leq V(t,x) \leq W_2(x)$$

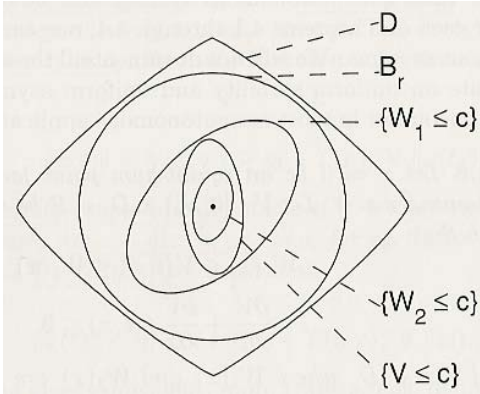
  $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$

  $\forall t \geq 0$ and $\forall x \in D$,
  where $W_1(x)$ and $W_2(x)$ are
  continuous P.D. functions on D.

- Then, $x = 0$ is uniformly stable.

- Proof:
- The derivative of V along the trajectories of (4.15) is given by

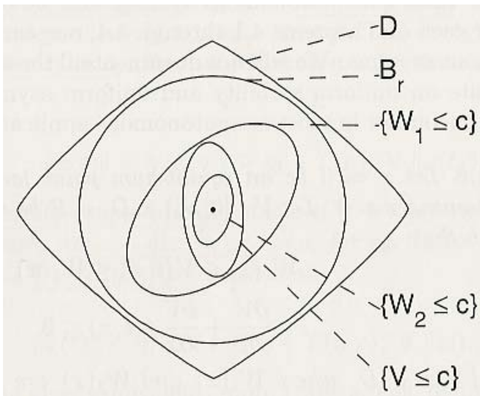
$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$$



- Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and $c < min_{||x||=r} W_1(x)$.
- Then, $\{x \in B_r \mid W_1(x) \leq c\}$ is in the interior of B_r.
- Define a time-dependent set $\Omega_{t,c}$ by

$$\Omega_{t,c} = \{x \in B_r \mid V(t,x) \leq c\}$$

- Since $W_2(x) \leq c \Rightarrow V(t,x) \leq c$, the set $\Omega_{t,c}$ contains $\{x \in B_r \mid W_2(x) \leq c\}$

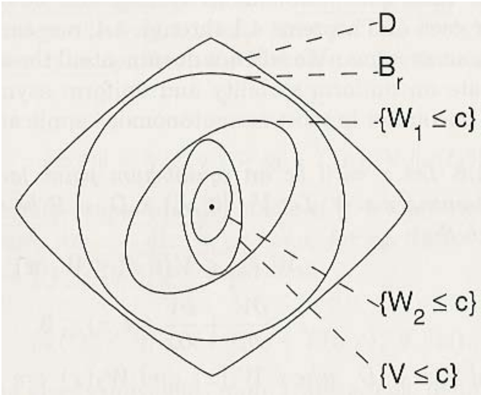


- On the other hand, since $V(t,x) \leq c \Rightarrow W_1(x) \leq c$, $\Omega_{t,c}$ is a subset of $\{x \in B_r \mid W_1(x) \leq c\}$.
- Thus,

$$\{x \in B_r \mid W_2(x) \leq c\} \subset \Omega_{t,c} \subset$$
$$\{x \in B_r \mid W_1(x) \leq c\} \subset B_r \subset D$$

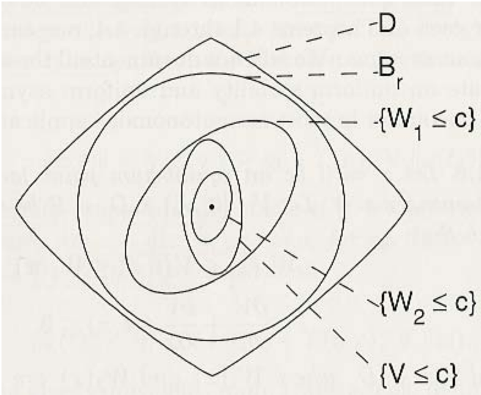
for all $t \geq 0$.

Theorem 4.8: U.S. & Proof



- In Figure, the surface $V(t,x) = c$ is now dependent on t, and that is why it is surrounded by the time-independent surfaces $W_1(x) = c$ and $W_2(x) = c$.
- Since $\dot{V}(t,x) \le 0$ on D, for any $t_0 \ge 0$ and any $x_0 \in \Omega_{t_0,c}$, the solution starting at (t_0, x_0) stays in $\Omega_{t,c}$ for all $t \ge t_0$.
- Therefore, any solution starting in $\{x \in B_r \mid W_2(x) \le c\}$ stays in $\Omega_{t,c}$, and consequently in $\{x \in B_r \mid W_1(x) \le c\}$, for all futute time.

Theorem 4.8: U.S. & Proof



- Hence, the solution is bounded and difined for all $t \ge t_0$.
- Moreover, since $\dot{V} \le 0$,

$$V(t, x(t)) \le V(t_0, x(t_0)), \ \forall t \ge t_0$$

- By Lemma 4.3, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(||x||) \le W_1(x) \le$$

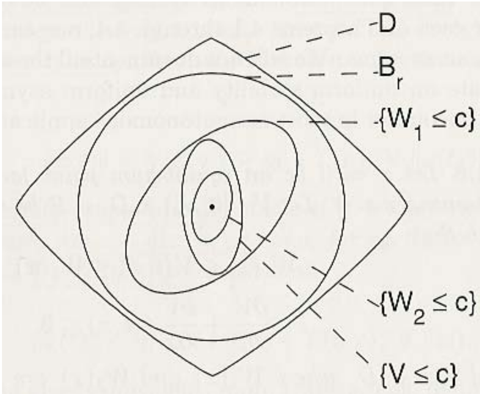
$$V(t,x) \le W_2(x) \le \alpha_2(||x||)$$

Theorem 4.8: U.S. & Proof

- Combining the preceding two inequalities, we see that

$$\|x(t)\| \leq \alpha_1^{-1}(V(t,x(t))) \leq$$

$$\alpha_1^{-1}(V(t_0,x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$



- Since $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function (by Lemma 4.2), the inequality $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$ shows that the origin is uniformly stable.

- QED

Theorem 4.9: U.A.S. & G.U.A.S.

- Theorem 4.9:
- Suppose the assumptions of Theorem 4.8 are satisfied with strengthened inequality:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x)$$

$\forall t \geq 0$ and $\forall x \in D$, where $W_3(x)$ is a continuous P.D. func on D.

- Then, $x = 0$ is uniformly asymptotically stable.

- Moreover, if r and c are chosen such that
  $B_r = \{||x|| \leq r\} \subset D$ & $c < \min_{||x||=r} W_1(x)$,
  then every trajectory starting
  in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

  $$||x(t)|| \leq \beta(||x(t_0)||, t - t_0, \ \forall t \geq t_0 \geq 0$$

  for some class $\mathcal{KL}$ function β.

- Finally, if $D = R^n$ and $W_1(x)$ is
  radially unbounded,
  then $x = 0$ is
  globally uniformly asymptotically stable.

- Proof:
- Continuing with the proof of Thm 4.8,
  we know that trajectories starting
  in $\{x \in B_r \mid W_2(x) \leq c\}$ stay
  in $\{x \in B_r \mid W_1(x) \leq c\}$ for all $t \geq t_0$.

- By Lemma 4.3,
  there exists a class $\mathcal{K}$ function α_3,
  defined on $[0, r]$, such that

  $$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \\ &\leq -W_3(x) \\ &\leq -\alpha_3(||x||) \end{aligned}$$

- Using the inequality

  $$V \leq \alpha_2(||x||) \iff \alpha_2^{-1}(V) \leq ||x|| \iff$$
  $$\alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(||x||)$$

- We see that
  V satisfies the differential inequality

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)) \triangleq -\alpha(V)$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function defined on $[0, r]$. (See Lemma 4.2.)

- Assume, without loss of generality, that
  α is locally Lipschitz.

- Let $y(t)$ satisfy the autonomous equation

$$\dot{y} = -\alpha(y), \; y(t_0) = V(t_0, x(t_0)) \geq 0$$

- By (the comparison) Lemma 3.4,

$$V(t, x(t)) \leq y(t), \; \forall t \geq t_0$$

- By Lemma 4.4,
  there exists a class $\mathcal{KL}$ function $\sigma(r, s)$
  defined on $[0, r] \times [0, \infty]$ such that

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0),$$

$\forall V(t_0, x(t_0)) \in [0, c]$

- Therefore, any solution starting in
  $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies the inequality

$$\begin{aligned}
||x(t)|| &\leq \alpha_1^{-1}(V(t, x(t))) \\
&\leq \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t - t_0)) \\
&\leq \alpha_1^{-1}(\sigma(\alpha_2(||x(t_0)||), t - t_0)) \\
&\triangleq \beta(||x(t_0)||, t - t_0)
\end{aligned}$$

Theorem 4.9: U.A.S. & G.U.A.S.

- Lemma 4.2 shows that
  β is a class $\mathcal{KL}$ function.

- Thus, inequality (4.20) is satisfied,
  which implies that
  $x = 0$ is uniformly asymptotically stable.

Theorem 4.9: U.A.S. & G.U.A.S.

- If $D = R^n$,
  the functions α_1, α_2, and α_3
  are defined on $[0, \infty)$.

- Hence, α, and consequently β,
  are independent of c.

- As $W_1(x)$ is radially unbounded,
  c can be chosen arbitrarily large
  to include any initial state in $\{W_2(x) \le c\}$.

- Thus, (4.20) holds for any initial state,
  showing that the origin is
  globally uniformly asymptotically stable.

- QED

- A function $V(t, x)$ is said to be
- positive semidefinite

  if $V(t, x) \geq 0$,
- positive definite

  if $V(t, x) \geq W_1(x)$

  for some positive definite function $W_1(x)$,
- radially unbounded

  if $W_1(x)$ is so,
- decrescent

  if $V(t, x) \leq W_2(x)$.
- negative definite (semidefinite)

  if $-V(t, x)$ is positive definite (semidefinite).

- Therefore, Theorems 4.8 and 4.9 say that the origin is
  - uniformly stable

    if there is a continuously differentiable, PD, decresscent function $V(t, x)$, whose derivative along the trajectories of the system is negative semidefinite.
  - uniformly asymptototically stable

    if the derivative is negative definite, and
  - globally uniformly asymptotically stable

    if the conditions for uniform asymptotic stability hold globally with a radially unbounded $V(t, x)$.

Theorem 4.10: E.S. & G.E.S.

- Theorem 4.10:
- Let $x = 0$ be an E.P. for (4.15) and
  $D \subset R^n$ be a domain containing $x = 0$.
- Let $V : [0, \infty] \times D \to R$ be
  a continuously differentiable function s.t.

$$k_1 ||x||^a \le V(t, x) \le k_2 ||x||^a \quad (4.25)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -k_3 ||x||^a \quad (4.26)$$

  $\forall t \ge 0$ and $\forall x \in D$,
  where k_1, k_2, k_3, and a are + constants.
- Then, $x = 0$ is exponentially stable.
- If the assumptions hold globally,
  then $x = 0$ is globally exponentially stable.

Theorem 4.10: E.S. & G.E.S.

- Proof:
- With the help of Figure 4.7,
  it can be seen that for sufficiently small c,
  trajectories starting in $\{k_2 ||x||^a \le c\}$,
  remain bounded for all $t \ge t_0$.
- Inequalities (4.25) and (4.26) show that
  V satisfies the differential inequality

$$\dot{V} \le -\frac{k_3}{k_2} V$$

- By (the comparison) Lemma 3.4,

$$V(t, x(t)) \le V(t_0, x(t_0)) e^{-(\frac{k_3}{k_2})(t - t_0)}$$

Theorem 4.10: E.S. & G.E.S.

- Hence,

$$\begin{aligned}||x(t)|| &\le [\frac{V(t,x(t))}{k_1}]^{1/a} \\ &\le [\frac{V(t_0,x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}]^{1/a} \\ &\le [\frac{k_2||x(t_0)||^a e^{-(k_3/k_2)(t-t_0)}}{k_1}]^{1/a} \\ &= (\frac{k_2}{k_1})^{1/a}||x(t_0)||e^{-(k_3/k_2 a)(t-t_0)}\end{aligned}$$

- Thus, the origin is exponentially stable.

- If all the assumptions hold globally, c can be chosen arbitrarily large and the foregoing inequality holds for all $x(t_0) \in R^n$.

- QED

Example 4.19: G.U.A.S.

- Example 4.19 :
- Consider the scalar system

$$\dot{x} = -[1+g(t)]x^3$$

where $g(t)$ is continuous and $g(t) \ge 0$ for all $t \ge 0$.

- Using the Lyapunov function candidate $V(x) = x^2/2$, we obtain

$$\dot{V}(t,x) = -[1+g(t)]x^4 \le -x^4,$$

$\forall x \in R,\ \forall t \ge 0$

- The assumptions of Theorem 4.9 are satisfied globally with $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$.

- Hence, the origin is globally uniformly asymptotically stable.

- Example 4.20 :
- Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

where $g(t)$ is cont. diff. and satisfies

$$0 \le g(t) \le k \text{ and } \dot{g}(t) \le g(t),\ \forall t \ge 0$$

- Taking $V(t,x) = x_1^2 + [1+g(t)]x_2^2$ as a Lyapunov function candidate, it can be easily seen that

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2,\ \forall x \in R^2$$

- Hence, $V(t,x)$ is PD, decrescent, and radially unbounded.

- The derivative of V along the trajectories of the system is given by

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2+2g(t) - \dot{g}(t)]x_2^2$$

- Using the inequality

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

we obtain

$$\begin{aligned}\dot{V}(t,x) &\le -2x_1^2 + 2x_1x_2 - 2x_2^2 \\ &= -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\triangleq -x^T Q x\end{aligned}$$

where Q is PD; therefore, $\dot{V}(t,x)$ is ND.

Example 4.20: G.E.S.

- Thus, all the assumptions of Thm 4.9
  are satisfied globally
  with PD quadratic func W_1, W_2, and W_3.

- Recalling that
  a PD quadratic function $x^T P x$ satisfies

  $$\lambda_{\min}(P) x^T x \leq x^T P x \leq \lambda_{\max}(P) x^T x$$

  we see that the conditions of Thm 4.10
  are satisfied globally with $a = 2$.

- Hence, the origin is
  globally exponentially stable.

Example 4.21: G.E.S. of LTV System

- Example 4.21:
- The linear time-varying system

  $$\dot{x} = A(t)x$$

  has an E.P. at $x = 0$.

- Let $A(t)$ be continuous for all $t \geq 0$.

- Suppose there is a
  cont. diff., sym., bdd, PD matrix $P(t)$;
  that is,

  $$0 < c_1 I \leq P(t) \leq c_2 I, \ \forall t \geq 0$$

  which satisfies the matrix diff. eqn (4.28)

  $$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

  where $Q(t)$ is cont., sym., and PD;
  that is,

  $$Q(t) \geq c_3 I > 0, \ \forall t \geq 0$$

Example 4.21: G.E.S. of LTV System

- The Lyapunov function candidate

$$V(t,x) = x^T P(t) x$$

satisfies

$$c_1 ||x||_2^2 \le V(t,x) \le c_2 ||x||_2^2$$

and its derivative along the trajectories of the system (4.27) is given by

$$\begin{aligned} \dot{V}(t,x) &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\ &= x^T [\dot{P}(t) + P(t)A(t) + A^T(t)P(t)] x \\ &= -x^T Q(t) x \\ &\le -c_3 ||x||_2^2 \end{aligned}$$

- Thus, all the assumptions of Thm 4.10 are satisfied globally with $a = 2$, and we conclude that the origin is globally exponentially stable.