Nonlinear Systems Analysis

Lecture Note

Section 4.4

Comparison Functions (Lyapunov Stability)

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Sep04 – Jan05

Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

Comparison Functions (4.4)

- From autononous to nonautonomous
- The sol of the nonautonomous syst
  $\dot{x} = f(t, x)$,
  starting at $x(t_0) = x_0$,
  depends on both t and t_0.
- Should refine the definitions
  to let stability hold
  uniformly in the initial time t_0.
- Hence, we need some special comparison functions.

Class K & KL Functions

- Definition 4.2 (Class $\mathcal{K}$):
- A continuous function
  $\alpha : [0, a) \to [0, \infty)$
  is said to belong to class $\mathcal{K}$
  if it is strictly increasing and $\alpha(0) = 0$.
- It is said to belong to class $\mathcal{K}_\infty$
  if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

- Definition 4.3 (Class $\mathcal{KL}$):

- A continuous function
  $\beta : [0, a) \times [0, \infty) \to [0, \infty)$
  is said to belong to class $\mathcal{KL}$
  if, for each fixed s,
  the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$
  w.r.t. r and,
  for each fixed r,
  the mapping $\beta(r, s)$ is decreasing w.r.t. s
  and $\beta(r, s) \to 0$ as $s \to \infty$.

Example 4.16

- Example 4.16:
- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing
  since $\alpha'(r) = 1/(1 + r^2) > 0$.
  It belongs to class $\mathcal{K}$,
  but not class $\mathcal{KL}$
  since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$.

- $\alpha(r) = r^c$,
  for any positive real number c,
  is strictly increasing
  since $\alpha'(r) = cr^{c-1} > 0$.
  Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$;
  thus, it belongs to class $\mathcal{K}_\infty$.

Example 4.16

- $\alpha(r) = \min\{r, r^2\}$
  is continuous, strictly increasing,
  and $\lim_{r\to\infty} \alpha(r) = \infty$.
  Hence, it belongs to class $\mathcal{K}_\infty$.

- Notice that $\alpha(r)$ is
  not continuously differentiable at $r = 1$.
  Continuous differentiability is not required
  for a class $\mathcal{K}$ function.

Example 4.16

- $\beta(r, s) = r/(ksr + 1)$,
  for any positive real number k,
  is strictly increasing in r
  since
  $$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$
  and strictly decreasing in s
  since
  $$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$
  Moreover, $\beta(r, s) \to 0$ as $s \to \infty$.
  Therefore, it belongs to class $\mathcal{KL}$.

- $\beta(r, s) = r^c e^{-s}$,
  for any positive real number c,
  belongs to class $\mathcal{KL}$.

Lemma 4.2

- Lemma 4.2:
- Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$,
  α_3 and α_4 be class $\mathcal{K}_\infty$ functions
  and β be a class $\mathcal{KL}$ function.
- Denote the inverse of α_i by α_i^{-1}.
- Then,
  - α_1^{-1} is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.
  - α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$.
  - $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$.
  - $\alpha_3 \circ \alpha_4$ belongs to class $\mathcal{K}_\infty$.
  - $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ belongs to class $\mathcal{KL}$.

Lemma 4.3

- Lemma 4.3:
- Let $V : D \to R$ be a continuous P.D. function defined on a domain $D \subset R^n$ that contains the origin.
- Let $B_r \subset D$ for some $r > 0$.
- Then, there exist class $\mathcal{K}$ functions α_1, α_2, defined on $[0, r]$, such that

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$$

for all $x \in B_r$.

Lemma 4.3

- If $D = R^n$,
  α_1, α_2 will be defined on $[0, \infty)$ and
  the foregoing inequality will hold $\forall x \in R^n$.

- Moreover, if $V(x)$ is radially unbounded,
  then α_1, α_2 can be chosen
  to belong to class $\mathcal{K}_\infty$.

- If $V(x) = x^T P x$,

$$\lambda_{\min}(P)||x||_2^2 \leq x^T P x \leq \lambda_{\max}(P)||x||_2^2$$

Lemma 4.4

- Lemma 4.4:
- Consider the scalar autonomous D.E.

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

  where $\alpha(\cdot)$ is a local Lipschitz class $\mathcal{K}$ func
  defined on $[0, a)$.

- For all $0 \leq y_0 < a$,
  this equation has a unique solution $y(t)$
  defined for all $t \geq t_0$.

- Moreover,

$$y(t) = \sigma(y_0, t - t_0)$$

  where σ is a class $\mathcal{KL}$ function
  defined on $[0, a) \times [0, \infty)$.

Lemma 4.4

- Examples:
- If $\dot{y} = -ky$, $k > 0$, then

$$y(t) = y_0 \exp[-k(t - t_0)]$$

$$\sigma(r, s) = r \exp(-ks)$$

- If $\dot{y} = -ky^2$, $k > 0$, then

$$y(t) = \frac{y_0}{ky_0(t - t_0) + 1}$$

$$\sigma(r, s) = \frac{r}{krs + 1}$$

Comparison Funs & Lyapunov Analysis

- For the proof of Thm 4.1:
- Want to choose β, δ
  such that $B_\delta \subset \Omega_\beta \subset B_r$.
- For a P.D. function $V(x)$, it satisfies

$$\alpha_1(||x||) \le V(x) \le \alpha_2(||x||)$$

  we can choose $\beta \le \alpha_1(r)$ and $\delta \le \alpha_2^{-1}(\beta)$.

- This is so because

$$V(x) \le \beta \Rightarrow \alpha_1(||x||) \le \alpha_1(r) \text{ iff } ||x|| \le r$$

  and $||x|| \le \delta \Rightarrow V(x) \le \alpha_2(\delta) \le \beta$.

- For $\dot{V}(x)$ is N.D.,
  $x(t) \to 0$ as $t \to \infty$.

- Using Lemma 4.3 we see that there is a class $\mathcal{K}$ function α_3 such that $\dot{V}(x) \leq -\alpha_3(||x||)$.

- Hence, V satisifies the diff. inequality

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$$

- Comparison lemma (Lma 3.4) shows that $V(x(t))$ is bounded by the solution of

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))$$

- Lemma 4.2 shows that $\alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$ function.

- Lemma 4.4 shows that the solution is $y(t) = \beta(y(0), t)$, where β is a class $\mathcal{KL}$ function.

- Consequently, $V(x(t))$ satisfies $V(x(t)) \leq \beta(V(x(0)), t)$, which shows that $V(x(t)) \to 0$ as $t \to \infty$.

- Estimates of $x(t)$
- $V(x(t)) \leq V(x(0))$ implies

$$\alpha_1(||x(t)||) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(||x(0)||)$$

- Hence, $||x(t)|| \leq \alpha_1^{-1}(\alpha_2(||x(0)||))$, where $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function.

- Similarly, $V(x(t)) \leq \beta(V(x(0)), t)$ implies

$$\alpha_1(||x(t)||) \leq V(x(t)) \leq \beta(V(x(0)), t)$$
$$\leq \beta(\alpha_2(||x(0)||), t)$$

- Therefore, $||x(t)|| \leq \alpha_1^{-1}(\beta(\alpha_2(||x(0)||), t))$, where $\alpha_1^{-1}(\beta(\alpha_2(r), t))$ is a class $\mathcal{KL}$ func.