Nonlinear Systems Analysis

Lecture Note

Section 4.3
Linear Systems & Linearization (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

Linear Time-Invariant Systems (§4.3)

- Consider the linear time-invariant system

$$\dot{x} = Ax \quad (4.9)$$

has an E.P. at the origin.

- The E.P. is isolated iff $\det(A) \neq 0$.

- If $\det(A) = 0$,
  the matrix A has a nontrivial null space.

- Every point in the null space of A is
  an E.P. for the system (4.9).

- If $\det(A) = 0$,
  the system has an equilibrium subspace.

Linear Time-Invariant Systems (4.3)

- Notice that a linear system cannot have
  multiple isolated equilibrium points.

- Stability properties of the origin
  can be characterized by
  the locations of the eigenvalues of A.

Property of Matrix A

- From linear system theory that the solution of (4.9) for a given $x(0)$ is given by

$$x(t) = \exp(At)x(0) \quad (4.10)$$

and that for any matrix A there is nonsingular matrix P that transforms A into its Jordan form;

$$P^{-1}AP = J = \text{ block diag } [J_1, J_2, \cdots, J_r]$$

where J_i is a Jordan block associated with the eigenvalue λ_i of A.

Property of Matrix A

- A Jordan block of order one takes the form $J_i = \lambda_i$, while a Jordan block of order $m > 1$ takes the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

- Therefore,

$$\begin{aligned} \exp(At) &= P\exp(Jt)P^{-1} \\ &= \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} \exp(\lambda_i t) R_{ik} \end{aligned}$$

where m_i is the order of the Jordan block J_i.

- If an $n \times n$ matrix A has
  a repeated eigenvlaue λ_i
  of algebraic multiplicity q_i,
  then the Jordan blocks associated with λ_i
  have order one
  iff rank $(A - \lambda_i I) = n - q_i$.

- Theorem 4.5:
- The E.P. $x = 0$ of $\dot{x} = Ax$ is stable
  iff all eigenvalues of A satisfy
  Re $\lambda_i \leq 0$ and
  for every eigenvalue with Re $\lambda_i = 0$ and
  algebraic multiplicity $q_i \geq 2$,
  rank $(A - \lambda_i I) = n - q_i$,
  where n is the dimension of x.

- The E.P. $x = 0$ is
  (globally) asymptotically stable
  iff all eigenvalues of A satisfy Re $\lambda_i < 0$.

- Proof:
- From (4.10), we can see that the origin is stable iff $\exp(At)$ is a bounded fun of t, $\forall t \geq 0$.
- If one of the eigenvalues of A is in the open right-half complex plane, the corresponding $\exp(\lambda_i t)$ in (4.11) will grow unbounded as $t \to \infty$.
- Therefore, we must restrict the eigenvalues to be in the closed left-half complex plane.

- But, those e-values on the I-axis (if any) could give rise to unbounded terms if the order of an associated Jordan block is higher than one, due to the term t^{k-1} in (4.11).
- So, we must restrict e-values on the I-axis to have Jordan blocks of order one, which is equivalent to $\text{rank}\ (A - \lambda_i I) = n - q_i$.
- Thus, we conclude that the condition for stability is a necessity.
- The condition is also sufficient to ensure that $\exp(At)$ is bounded.

Theorem 4.5

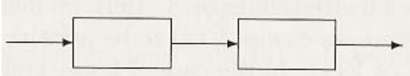
- For asymptotic stability of the origin,
  $\exp(At)$ must approach 0 as $t \to \infty$.

- This is, iff Re $\lambda_i < 0, \forall i$.

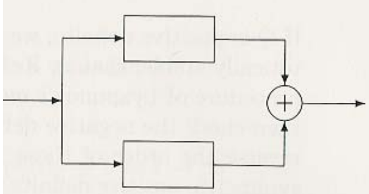
- Since $x(t)$ depends linearly on
  the initial state $x(0)$,
  asymptotic stability of the origin is global.

Example 4.12

- Example 4.12:
- Consider a series connection and a parallel connection
  of two identical systems.



(a)



(b)

- Each system is represented by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [1 \;\; 0] x$$

  where u and y are
  the input and output, respectively.

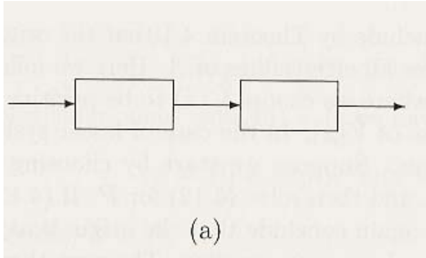
- Let A_s and A_p be the matrices of
  the series and parallel connections,
  when modeled without driving inputs.

Example 4.12

- Then

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$A_s = \; = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$



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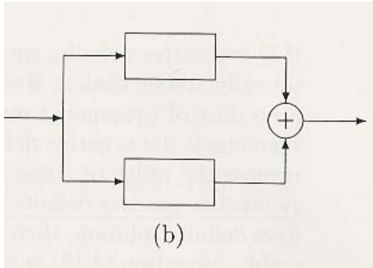


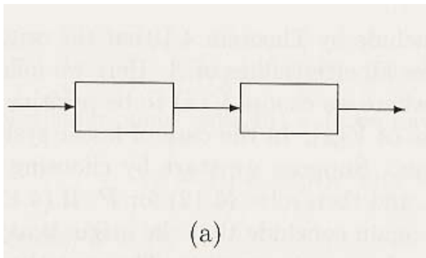
(b)

- The matrices A_p and A_s have the same e-values on the I-axis, $\pm j$, with algebraic multiplicity $q_i = 2$.

- Also, rank $(A_p - jI) = 2 = n - q_i$, while rank $(A_s - jI) = 3 \neq n - q_i$.

Example 4.12

- Thus, by Theorem 4.5, the origin of the parallel connection is stable, while the origin of the series connection is unstable.



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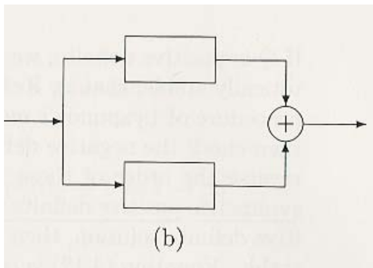


(b)

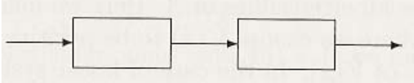
- To physically see the difference between the two cases, notice that in the parallel connection, nonzero initial conditions produce sinusoidal oscillations of freq 1 rad/sec, which are bounded functions of time.

Example 4.12

- The sum of these sinusoidal signals remains bounded.

- On the other hand, nonzero initial conditions in the first component of the series connection produce a sinusoidal oscillation of freq 1 rad/sec, which acts as a driving input for the second component.

- Since the second component has an undamped natural freq of 1 rad/sec, the driving input causes "resonance" and the response grows unbounded.



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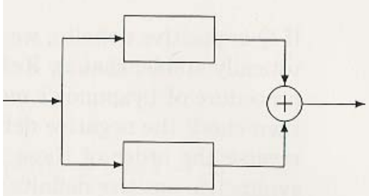

(b)

Hurwitz Matrix & Lyapunov Equation

- A is called a Hurwitz (stability) matrix if all eigenvalues of A satisfy Re $\lambda_i < 0$,

- The origin of (4.9) is asymptotically stable iff A is Hurwitz.

- Use Lyapunovs method to investigate the asymptotic stability of the origin.

- Consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x$$

where P is a real symmetric P.D. matrix.

- The derivative of V along the trajectories of (4.9) $\dot{x} = Ax$ is given by

$$\begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P_x \\ &= x^T (PA + A^T P) x = -x^T Q_x \end{aligned}$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \quad (4.12)$$

- If Q is P.D., we can conclude by Thm 4.1 that the origin is asymptotically stable; that is, $\text{Re}\lambda_i < 0$ for all eigenvalues of A.

- The usual procedure of Lyapunov's method is to choose $V(x)$ to be P.D. first and then check the N.D. of $\dot{V}(x)$.

- In the case of linear systems, we can reverse the order of these steps.

- Suppose we start by choosing Q as a real symmetric P.D. matrix, and then solve (4.12) for P.

- If (4.12) has a P.D. solution, then we can again conclude that the origin is asymptotically stable.

- Equation (4.12) is called the Lyapunov equation.

Theorem 4.6

- Theorem 4.6:
- A matrix A is Hurwitz; that is,
  Re $\lambda_i < 0$ for all e-values of A,
  iff for any given P.D. symmetric matrix Q
  there exists a P.D. symmetric matrix P and
  that satisfies the Lyapunov equation (4.12).

- Moreover, if A is Hurwitz,
  then P is the unique solution of (4.12).

Theorem 4.6 & Proof

- Proof:
- Sufficiency follows from Theorem 4.1
  with the Lyapunov function $V(x) = x^T P_x$,
  as we have already shown.

- To prove necessity, assume that
  all eigenvalues of A satisfy Re $\lambda_i < 0$ and
  consider the matrix P, defined by

$$P = \int_0^\infty \exp(A^T t) Q \exp(At) dt \quad (4.13)$$

- The integrand is a sum of terms
  of the form $t^{k-1} \exp(\lambda_i t)$,
  where Re $\lambda_i < 0$.

- Therefore, the integral exists.

Theorem 4.6 & Proof

- Next, we need to show that the matrix P is symmetric and P.D.
  Symmetric is from the form of P;
  P.D. will be proved in the following.

- Supposing it is not so,
  there is a vector $x \neq 0$
  such that $x^T P_x = 0$.

- However,

$$\begin{aligned} & x^T P_x = 0 \\ & \Rightarrow \int_0^\infty x^T \exp(A^T t) Q \exp(At) x dt = 0 \\ & \Rightarrow \exp(At)x \equiv 0, \forall t \geq 0 \\ & \Rightarrow x = 0 \end{aligned}$$

since $\exp(At)$ is nonsingular for all t.

Theorem 4.6 & Proof

- This contradiction shows that P is P.D.

- Next, we show that
  P is the unique solution of (4.12)

- Now, substituting (4.13) in the LHS of (4.12) yields

$$\begin{aligned} & PA + A^T P \\ = & \int_0^\infty \exp(A^T t) Q \exp(At) A dt \\ & + \int_0^\infty A^T \exp(A^T t) Q \exp(At) dt \\ = & \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) dt \\ = & \exp(A^T t) Q \exp(At) \Big|_0^\infty \\ = & -Q \end{aligned}$$

which shows that
P is indeed a solution of (4.12).

Theorem 4.6 & Proof

- To show that it is the unique solution, suppose there is another solution $\tilde{P} \neq P$.

- Then,

$$(P - \tilde{P})A + A^T(P - \tilde{P}) = 0$$

- Premultiplying by $\exp(A^T t)$ and Postmultiplying by $\exp(At)$, we obtain

$$\begin{aligned} 0 &= \exp(A^T t)\left[(P - \tilde{P})A + A^T(P - \tilde{P})\right] \\ &\quad \exp(At) \\ &= \frac{d}{dt}\left\{\exp(A^T t)(P - \tilde{P})\exp(At)\right\} \end{aligned}$$

- Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{ a constant } \forall t$$

Theorem 4.6 & Proof

- In particular, since $\exp(A0) = I$, we have

$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P})\exp(At) \to 0$$

$$\text{as } t \to \infty$$

- Therefore, $\tilde{P} = P$.

- Q.E.D.

- The P.D. requirement on Q can be relaxed.

- That is, Q can be taken as a P.S.D. matrix of the form $Q = C^T C$, where the pair (A, C) is observable.

- Example 4.13:
- Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

where, due to symmetry, $p_{12} = p_{12}$.

- The Lyapunov equation (4.12) can be rewritten as

$$\begin{bmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

- The unique solution of this equation is given by

$$\begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \\ 1.0 \end{bmatrix}$$
$$\Rightarrow \quad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}$$

- The matrix P is P.D. since its leading principal minors (1.5 and 1.25) are positive.

- Hence, all eigenvalues of A are in the open left-half complex plane.

Property after Linearization

- Let us go back to the nonlinear system

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is
a continuously differentiable map
from a domain $D \subset R^n$ into R^n.

- Suppose the origin $x = 0$ is in D and
an E.P. for the system;
that is, $f(0) = 0$.

- By the mean value theorem,

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)x$$

where z_i is a point on the line segment
connecting x to the origin.

Property after Linearization

- The foregoing equality is valid
for any point $x \in D$ such that
the line segment connecting x to the origin
lies entirely in D.

- Since $f(0) = 0$, we can write

$$\begin{aligned} f_i(x) &= \frac{\partial f_i}{\partial x}(z_i)x \\ &= \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]x \end{aligned}$$

- Hence,

$$f(x) = Ax + g(x)$$

where

$$A = \frac{\partial f}{\partial x}(0) \text{ and } g_i(x) = \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]x$$

Property after Linearization

- The function $g_i(x)$ satisfies

$$|g_i(x)| \le \left|\left|\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right|\right| \, ||x||$$

- By continuity of $[\partial f/\partial x]$,
  we see that

$$\frac{||g(x)||}{||x||} \to 0 \text{ as } ||x|| \to 0$$

- This suggests that
  in a small neighborhood of the origin
  we can approximate the nonlinear system
  by its linearization about the origin

$$\dot{x} = Ax, \text{ where } A = \frac{\partial f}{\partial x}(0)$$

Theorem 4.7: Lyapunov Indirect Method

- Theorem 4.7:
- Let $x = 0$ be an E.P. for the NL syst

$$\dot{x} = f(x) \quad (\text{or} = Ax + g(x))$$

  where $f : D \to R^n$ is
  continuously differentiable
  and D is a neighborhood of the origin.

- Let

$$A = \frac{\partial f}{\partial x}(x)\Big|_{x=0}$$

- Then,
  1. If Re $\lambda_i < 0$ for all eigenvalues of A,
     the origin is asymptotically stable.
  2. If Re $\lambda_i > 0$ for one or more of the eigenvalues of A,
     the origin is unstable.

- Proof:
- 1st part: (asymptotic stable)
  - use $V(x) = x^T P x$
  - Thm 4.6
  - Thm 4.1
- 2nd part: (unstable)
  - $=$ no eig(A) on the I-axis
  - assme in open RHP and open LHP
  - use Thm 4.6, Thm 4.3
  - $=$ some eig(A) on the I-axis
  - shift the I-axis, then work like the above
  - use Thm 4.6, Thm 4.3

- 1st part: (asymptotic stable)
- Let A be a Hurwitz matrix.
- Then, by Theorem 4.6,
  for any P.D. sym. matrix Q,
  the solution P of the Lyapunov equation
  is P.D.
- We use $V(x) = x^T P_x$
  as a Lyapunov function candidate
  for the nonlinear system.

- The derivative of V(x) along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T(PA + A^T P)x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x)\end{aligned}$$

- The 1st term on the RHS is N.D., while the 2nd term is indefinite (in general).

- The function $g(x)$ satisfies

$$\frac{||g(x)||_2}{||x||_2} \to 0 \text{ as } ||x||_2 \to 0$$

- Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$||g(x)||_2 < \gamma ||x||_2, \ \forall ||x||_2 < r$$

- Hence,

$$\dot{V}(x) < -x^T Q x + 2\gamma ||P||2||x||_2^2, \ \forall ||x||_2 < r$$

- But

$$x^T Q x \geq \lambda_{min}(Q)||x||_2^2$$

where $\lambda_{\text{min}}(\cdot)$ denotes the minimum eigenvalue of a matrix.

- Note that since Q is sym and P.D., $\lambda_{\text{min}}(Q)$ is real and positive.

Theorem 4.7 & Proof:

- Thus,

$$\dot{V}(x) < -\left[\lambda_{\min}(Q) - 2\gamma ||P||_2\right] \; ||x||_2^2,$$

$$\forall ||x||_2 < r$$

- Choosing $\gamma < (1/2)\lambda_{min}(Q)/||P||_2$ ensures that $\dot{V}(x)$ is N.D.

- By Theorem 4.1, we conclude that the origin is asymptotically stable.

Theorem 4.7 & Proof:

- 2nd part: (unstable)
- case 1: (no $\pm j$ e-values)

- Cluster eig(A) into
a group of e-values in the open RHP and
a group of e-values in the open LHP,
then there is a nonsingular matrix T
such that

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_1 and A_2 are Hurwitz matrices.

- Let

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where the partition of z is compatible
with the dimensions of A_1 and A_2.

Theorem 4.7 & Proof:

- The change of variables $z = Tx$
  transforms the system

$$\dot{x} = Ax + g(x)$$

into the form

$$\dot{z}_1 = -A_1 z_1 + g_1(z)$$

$$\dot{z}_2 = A_2 z_2 + g_2(z)$$

where the functions $g_i(z)$ have the property
that for any $\gamma > 0$,
there exists $r > 0$ such that

$$||g_i(z)||_2 < \gamma ||z||_2, \ \forall ||z||_2 \leq r, i = 1, 2$$

Theorem 4.7 & Proof:

- The origin $z = 0$ is an E.P.
  for the system in the z-coordinates.

- Since T is nonsingular,
  the stability properties of $z = 0$ implies
  those of the E.P. $x = 0$ in the x-coordinates.

- To show the origin is unstable, we apply Thm 4.3.

- Let Q_1 and Q_2 be P.D. sym matrices of the dimensions of A_1 and A_2, respectively.

- Since A_1 and A_2 are Hurwitz, we know from Theorem 4.6 that the Lyapunov equations

$$P_i A_i + A_i^T P_i = -Q_i, \; i = 1, 2$$

have unique P.D. solutions P_1 and P_2.

- Let

$$\begin{aligned} V(z) &= z_1^T P_1 z_1 - z_2^T P_2 z_2 \\ &= z^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} z \end{aligned}$$

- In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin.

- Let

$$U = \{ z \in R^n \mid ||z||_2 \leq r \text{ and } V(z) > 0 \}$$

- In U,

$$\begin{aligned}\dot{V}(z) &= -z_1^T(P_1A_1 + A_1^TP_1)z_1 + 2z_1^TP_1g_1(z) \\ &\quad -z_2^T(P_2A_2 + A_2^TP_2)z_2 - 2z_2^TP_2g_2(z) \\ &= z_1^TQ_1z_1 + z_2^TQ_2z_2 + 2z^T\begin{bmatrix} P_1g_1(z) \\ -P_2g_2(z)\end{bmatrix} \\ &\geq \lambda_{\min}(Q_1)\|z_1\|_2^2 + \lambda_{\min}(Q_2)\|z_2\|_2^2 \\ &\quad -2\|z\|_2\sqrt{\|P_1\|_2^2\|g_1(z)\|_2^2 + \|P_2\|_2^2\|g_2(z)\|_2^2} \\ &> (\alpha - 2\sqrt{2}\beta\gamma)\|z\|_2^2\end{aligned}$$

where $\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}$
and $\beta = \max\{\|P_1\|_2, \|P_2\|_2\}$.

- Thus, choosing $\gamma < \alpha/(2\sqrt{2}\beta)$
ensures that $\dot{V}(z) > 0$ in U.

- Therefore, by Theorem 4.3,
the origin is unstable.

- For the original coordinates
define the matrices

$$P = T^T\begin{bmatrix} P_1 & 0 \\ 0 & -P_2\end{bmatrix}T; \quad Q = T^T\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2\end{bmatrix}T$$

which satisfy the equation

$$PA + A^TP = Q$$

- The matrix Q is P.D.,
and $V(x) = x^TP_x$ is positive
for points arbitrarily close to the origin.

- case 2: (with e-values on I-axis)
- By a simple trick of shifting the I-axis,
  we can reduce this case to the special case.

- Suppose A has m e-values
  with Re $\lambda_i > \delta > 0$.

- Then, the matrix $[A - (\delta/2)I]$
  has m eigenvalues in the open RHP,
  but no eigenvalues on the I-axis.

- By previous arguments,
  there exist matrices
  $P = P^T$ and $Q = Q^T > 0$
  such that

$$P\left[A - \frac{\delta}{2}I\right] + \left[A - \frac{\delta}{2}I\right]^T P = Q$$

  where $V(x) = x^T P x$ is positive
  for points arbitrarily close to the origin.

- The derivative of $V(x)$
  along the trajectories of the system
  is given by

$$\begin{aligned}\dot{V}(x) &= x^T(PA + A^T P)x + 2x^T P g(x) \\ &= x^T \left[P(A - \frac{\delta}{2}I) + (A - \frac{\delta}{2}I)^T P\right] x \\ &\quad + \delta x^T P x + 2x^T P g(x) \\ &= x^T Q x + \delta V(x) + 2x^T P g(x)\end{aligned}$$

- In the set

$$\{x \in R^n \mid ||x||_2 \le r \text{ and } V(x) > 0\}$$

where r is chosen such that
$||g(x)||_2 \le \gamma ||x||_2$ for $||x||_2 < r$,
$\dot{V}(x)$ satisfies

$$\begin{aligned}\dot{V}(x) &\ge \lambda_{\min}(Q)||x||_2^2 - 2||P||_2\,||x||_2\,||g(x)||_2 \\ &\ge (\lambda_{\min}(Q) - 2\gamma||P||_2)||x||_2^2\end{aligned}$$

which is positive
for $\gamma < (1/2)\lambda_{\min}(Q)/||P||_2$.

- Applying Theorem 4.3 concludes the proof.

- Example 4.14:
- Consider the scalar system

$$\dot{x} = ax^3$$

- One eig(A) on the I-axis.

- If $a < 0$,
choose $V(x) = x^4$,
then $\dot{V}(x) = 4ax^6 < 0$, for $x \ne 0$,
then the origin is asymptotically stable

- If $a = 0$,
the system is linear, and
is stable by Thm 4.5.

- If $a > 0$,
choose $V(x) = x^4$,
then $\dot{V}(x) = 4ax^6 > 0$, for $x \ne 0$,
then the origin is unstable by Thm 4.3.

Examples 4.15

- Example 4.15 (Pendulum Eqn):

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

E.P. $= (0,0)$ and $(\pi, 0)$.

- For $(0,0)$
- For all $a, b > 0$,
  Re $\lambda_i < 0$,
  $(0,0)$ is asympototically stable.

- For all $a > 0, b = 0$,
  Re $\lambda_i = 0$,
  $(0,0)$ is stable from Ex 4.3.

- For $(\pi, 0)$
- For all $a > 0, b \geq 0$,
  one e-value on the RHP
  $(\pi, 0)$ is unstable.