Nonlinear Systems Analysis

Lecture Note

Section 4.2

Invariance Principle (Lyapunov Stability)

Feng-Li Lian

NTU-EE

Sep04 – Jan05

Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

LaSalle's Invariance Principle (§4.2)

- The pendulum equation with friction (Example 4.4):

- The energy Lyapunov function fails to satisfy the asymptotic cond. of Thm 4.1 because $\dot{V}(x) = -bx_2^2$ is only negative semidefinite.

- But, $\dot{V}(x)$ is negative everywhere, except on the line $x_2 = 0$, where $\dot{V}(x) = 0$.

- For the system to maintain $\dot{V}(x) = 0$, the trajectory of the system must be confined to the line $x_2 = 0$.

LaSalle's Invariance Principle (4.2)

- Unless $x_1 = 0$, this is impossible because from the pendulum equation

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow \sin x_1(t) \equiv 0$$

- Hence, on $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain $\dot{V}(x) = 0$ only at the origin $x = 0$.

- So, $V(x(t))$ must decrease toward 0 and, consequently, $x(t) \to 0$ as $t \to \infty$, which is consistent with the fact that, due to friction, energy cannot remain constant while the system is in motion.

LaSalle's Invariance Principle (4.2)

- LaSalles invariance principle:

- If in a domain about the origin
  we can find a Lyapunov function
  whose derivative
  along the trajectories of the systems
  is negative semidefinite,
  and if we can establish that
  no trajectory can stay identically at points
  where $\dot{V}(x) = 0$,
  except at the origin,
  then the origin is asymptotically stable.

Positive Limit Set & Invariant Set

- Let $x(t)$ be a solution of (4.1).

- A point p is said to be
  a positive limit point of $x(t)$
  if there is a sequence $\{t_n\}$,
  with $t_n \to \infty$ as $n \to \infty$,
  such that $x(t_n) \to p$ as $n \to \infty$.

- The set of all positive limit points of $x(t)$
  is called the positive limit set of $x(t)$.

- A set M is said to be
  an invariant set with respect to (4.1)

  if $\quad x(0) \in M \Rightarrow x(t) \in M, \ \forall t \in R$

Positive Limit Set & Invariant Set

- That is, if a solution belongs to M at some time instant, then it belongs to M for all future and past time.
- A set M is said to be a positively invariant set if

$$x(0) \in M \Rightarrow x(t) \in M, \ \forall t \geq 0$$

- We also say that $x(t)$ approaches a set M as $t \to \infty$, if for each $\epsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \epsilon, \ \forall t > T$$

Positive Limit Set & Invariant Set

- Where $\text{dist}(p, M)$ denotes the distance from a point p to a set M, that is, the smallest distance from p to any point in M.
- More precisely, $\text{dist}(p, M) = \inf_{x \in M} ||p - x||$.
- The equilibrium point and the limit cycle are invariant sets, since any solution starting in either set remains in the set for all $t \in R$.
- $x(t)$ approaches M as $t \to \infty$ does not imply that $\lim_{t \to \infty} x(t)$ exists.

Lemma 4.1:

- Lemma 4.1:
- If a solution $x(t)$ of (4.1) is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nenempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$.

- Proof: See Appendix C.3.

Theorem 4.4: LaSalle's Theorem

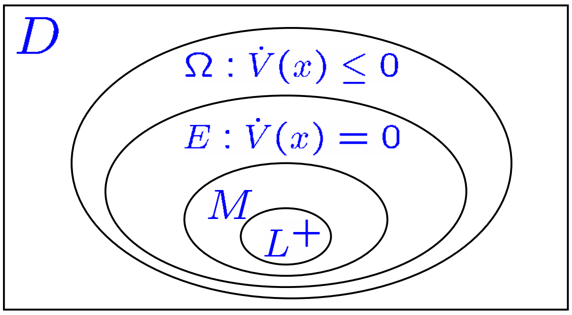
- Theorem 4.4 (LaSalle's theorem):
- Let $\Omega \subset D$ be a compact set that is positively invariant w.r.t. (4.1).

- Let $V : D \to R$ be a cont. diff. func. such that $\dot{V}(x) \leq 0$ in Ω.

- Let E be the set of all points in Ω where $\dot{V}(x) = 0$.

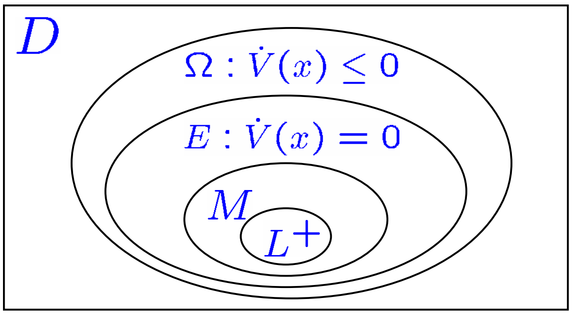
- Let M be the largest invariant set in E.

- Then every solution starting in Ω approaches M as $t \to \infty$.



Theorem 4.4: LaSalle's Theorem

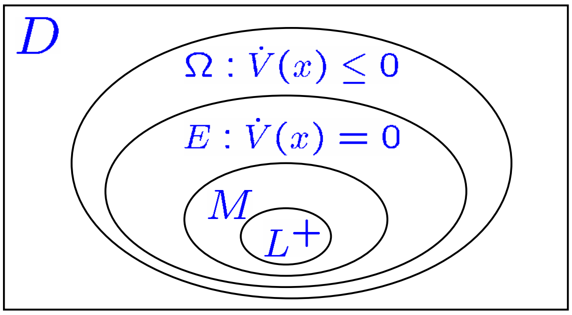
- Proof:
- Let $x(t)$ be a solution of (4.1) starting in Ω.



- Since $\dot{V}(x) \leq 0$ in Ω, $V(x(t))$ is a decreasing function of t.
- Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω.
- Therefore, $V(x(t))$ has a limit a as $t \to \infty$.
- Also note that, because Ω is a closed set, the positive limit set L^+ is in Ω

Theorem 4.4: LaSalle's Theorem

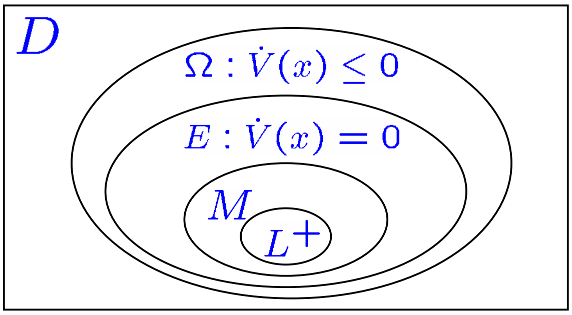
- For any $p \in L^+$, there is a sequence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$.



- By continuity of $V(x)$, $V(p) = \lim_{n\to\infty} V(x(t_n)) = a$.
- Hence, $V(x) = a$ on L^+.
- By Lemma 4.1, since L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+.
- Thus, $L^+ \subset M \subset E \subset \Omega$

Theorem 4.4: LaSalle's Theorem

- Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 4.1).
- Hence, $x(t)$ approaches M as $t \to \infty$.
- Q.E.D.



Theorem 4.4: LaSalle's Theorem

- Thm 4.4 does not require the function $V(x)$ to be positive definite.
- Also, note that the construction of the set Ω does not have to be tied in with the construction of the function $V(x)$
- The consctruction of $V(x)$ will itself guarantee the existence of a set Ω.
- When $V(x)$ is positive definite, Ω_c is bounded for sufficiently small $c > 0$.

Theorem 4.4: LaSalle's Theorem

- This is not necessarily true when $V(x)$ is not positive definite.

- For example, if $V(x) = (x_1 - x_2)^2$, the set Ω_c is not bounded no matter how small c is.

- If $V(x)$ is radially unbounded - that is, $V(x) \to \infty$ as $||x|| \to \infty$ - the set Ω_c is bounded for all values of c.

- This is true whether or not $V(x)$ is positive definite.

Corollaries 4.1 & 4.2:

- Showing that $x(t) \to 0$ as $t \to \infty$.

- By showing that no solution can stay identically in E, other than the trivial solution $x(t) = 0$.

Corollary 4.1:

- Corollary 4.1 (Barbashin's theorem) :
- Let $x = 0$ be an E.P. for (4.1).
- Let $V : D \to R$ be
  a continuously differentiable
  positive definite function
  on a domain D containing the origin $x = 0$,
  such that $\dot{V}(x) \leq 0$ in D.
- Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and
  suppose that
  no solution can stay identically in S,
  other than the trivial solution $x(t) \equiv 0$.
- Then, the origin is asymptotically stable.

Corollary 4.2:

- Corollary 4.2 (Krasovskii's theorem) :
- Let $x = 0$ be an E.P. for (4.1).
- Let $V : R^n \to R$ be
  a continuously differentiable,
  radially unbounded,
  positive definite function
  such that $\dot{V}(x) \leq 0$ for all $x \in R^n$.
- Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and
  suppose that
  no solution can stay identically in S,
  other than the trivial solution $x(t) \equiv 0$.
- Then, the origin is
  globally asymptotically stable.

Corollaries 4.1 & 4.2:

- When $\dot{V}(x)$ is negative definite,
  $S = \{0\}$.

- Then, Corollaries 4.1 and 4.2 coincide with
  Theorems 4.1 and 4.2 respectively.

Example 4.8:

- Example 4.8:
- Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2)\end{aligned}$$

where $h_1(\cdot)$ and $h_2(\cdot)$
are locally Lipschitz and satisfy

$$h_i(0) = 0,$$

$$yh_i(y) > 0, \ \forall y \neq 0 \text{ and } y \in (-a, a)$$

- The system has an isolated E.P.
  at the origin.

- Depending upon $h_1(\cdot)$ and $h_2(\cdot)$,
  it might have other equilibrium points.

Example 4.8:

- The system can be viewed as a generalized pendulum with $h_2(x_2)$ as the friction term.

- Therefore, a Lyapunov function candidate may be taken as the energy-like function

$$V(x) = \int_0^{x_1} h_1(y)dy + \frac{1}{2}x_2^2$$

- Let $D = \{x \in R^2 \mid -a < x_i < a\}$; $V(x)$ is positive definite in D and

$$\begin{aligned} \dot{V}(x) &= h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] \\ &= -x_2 h_2(x_2) \le 0 \end{aligned}$$

is negative semidefinite.

Example 4.8:

- To find $S = \{x \in D \mid \dot{V}(x) = 0\}$, note that

$$\dot{V}(x) = 0 \Rightarrow x_2 h_2(x_2) = 0 \Rightarrow x_2 = 0,$$

since $-a < x_2 < a$

- Hence, $S = \{x \in D \mid x_2 = 0\}$.

- Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow$$

$$h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

- Therefore, the only solution that can stay identically in S is the trivial solution $x(t) \equiv 0$.

- Thus, the origin is asymptotically stable.

- Example 4.9:
- Consider again the system of Example 4.8,
  but this time let $a = \infty$ and assume that
  $h_1(\cdot)$ satisfies the additional condition:

$$\int_0^y h_1(z) \to \infty \text{ as } |y| \to \infty$$

- The Lyapunov function

$$V(x) = \int_0^{x_1} h_1(y)dy + (1/2)x_2^2$$

  is radially unbounded.

- Similar to the previous example,
  it can be shown that
  $\dot{V}(x) \leq 0$ in R^2,
  and the set

$$\begin{aligned} S &= \{x \in R^2 \mid \dot{V}(x) = 0\} \\ &= \{x \in R^2 \mid x_2 = 0\} \end{aligned}$$

  contains no solutions
  other than the trivial solution.

- Hence, the origin is
  globally asymptotically stable.

- Not only does LaSalle's theorem
  relax the negative definiteness requirement
  of Lyapunovs theorem,
  but it also extends Lyapunov's theorem
  in three different directions.

Example 4.9:

- First, it gives an estimate of the region of attraction, which is not necessarily of the form $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$.

- The set Ω of Theorem 4.4 can be any compact positively invariant set.

- Second, LaSalle's theorem can be used in cases where the system has an equilibrium set, rather than an isolated equilibrium point.

- This will be illustrated by an application to a simple adaptive control example.

Example 4.9:

- Third, the function $V(x)$ does not have to be positive definite.

- The utility of this feature will be illustrated by an application to the neural network example.