Nonlinear Systems Analysis

Lecture Note

Section 4.1

Autonomous Systems (Lyapunov Stability)

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Outline

- Introduction (L8)
- Autonomous Systems (4.1, L8, L9)
  - Basic Stability Definitions
  - Lyapunov's stability theorems
- The Invariance Principle (4.2, L9+L10)
  - LaSalle's theorem
- Linear Systems and Linearization (4.3, L10)
- Comparison Functions (4.4, L11)
- Non-autonomous Systems (4.5, L11)
- Linear Time-Varying Systems & Linearization (4.6, L11+0.5)
- Converse Theorems (4.7, L12)
- Boundedness & Ultimate Boundedness (4.8, L12)
- Input-to-State Stability (4.9, L13)

- Stability theory plays a central role in systems theory and engineering. In this book, we will discuss stability of equilibrium points (Chap 4), input-output stability, and stability of periodic orbits.

- Stability of equilibrium points is usually characterized in the sense of Lyapunov, a Russian mathematician and engineer.

- An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium points as time approaches infinity.

- **Section 4.1:** Basic theorems of Lyapunov's method for autonomous systems
- **Section 4.2:** An extension of the basic theory, LaSalle.
- **Section 4.3:** Stability of E.P. of $\dot{x}(t) = Ax(t)$: by the location of the eigenvalues of A.
- **Section 4.4:** Class $\mathcal{K}$ and class $\mathcal{KL}$ functions
- **Section 4.5:** Uniform stability, uniform asymptotic stability, and exponential stability for nonautonomous systems
- **Section 4.6:** Linear time-varying systems and linearization
- **Section 4.7:** Converse theorems
- **Section 4.8:** Boundedness and utlimate boundedness
- **Section 4.9:** Input-to-state stability

Autonomous Systems (§4.1)

- Consider the autonomous system

$$\dot{x} = f(x) \quad (4.1)$$

  where $f : D \to R^n$ is
  a locally Lipschitz map
  from a domain $D \subset R^n$ into R^n.

- Suppose $\bar{x} \in D$ is an equilibrium point
  of (4.1); that is, $f(\bar{x}) = 0$.
  Our goal is to characterize and study
  the stability of $\bar{x}$.

- For convenience,
  we state all definitions and theorems
  for the case
  when the equilibrium point is
  at the origin of R^n; that is, $\bar{x} = 0$.

Autonomous Systems

- Suppose $\bar{x} \neq 0$ and
  consider the change of variables $y = x - \bar{x}$.
  The derivative of y is given
  by $\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) := g(y)$,
  where $g(0) = 0$.

- In the new variable y,
  the system has equilibrium at the origin.
  Therefore, without loss of generality (wlog),
  we will always assume that
  $f(x)$ satisfies $f(0) = 0$
  and study the stability of the origin $x = 0$.

Basic Stability Definitions

- Definition 4.1

  The equilibrium point $x = 0$ of (4.1) is

  stable

  if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$

  such that

  $$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \geq 0$$

  unstable

  if it is not stable.

  asymptotically stable

  if it is stable and δ can be chosen

  such that

  $$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Pendulum Example

- the pendulum example.

  $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - b x_2 \end{aligned}$$

  has two equilibrium points

  at $(x_1 = 0, x_2 = 0)$ and $(x_1 = \pi, x_2 = 0)$.

- Let $b = 0$, (neglecting friction),

  trajectories in the neighborhood

  of the first equilibrium pt are closed orbits.

- Therefore, by starting sufficiently close to

  the equilibrium point,

  trajectories can be guaranteed to stay

  within any specified ball centered

  at the equilibrium point.

Pendulum Example

- Hence, the $\epsilon - \delta$ requirement for stability is satisfied.

- The equilibrium point, however, is not asymptotically stable since trajectories starting off the equilibrium point do not tend to it eventually. Instead, they remain in their closed orbits.

- Let $b > 0$, (friction is considered) the equilibrium point at the origin becomes a stable focus.

- Inspection of the phase portrait of a stable focus shows that the $\epsilon - \delta$ requirement for stability is satisfied.

Pendulum Example

- In addition, trajectories starting close to the equilibrium point tend to it as t tends to ∞.

- So, it is AS.

- The second equilibrium point at $x_1 = \pi$ is a saddle point.

- Clearly the $\epsilon - \delta$ requirement cannot be satisfied since, for any $\epsilon > 0$, there is always a trajectory that will leave the ball $\{x \in R^n \mid ||x - \bar{x}|| \leq \epsilon\}$ even when $x(0)$ is arbitrarily close to the equilibrium point $\bar{x}$.

Determining Stability

- Actually finding all solutions
  $\Rightarrow$ May be difficult or even impossible.
  $\Rightarrow$ Try energy concepts first.

- Define the energy of the pendulum $E(x)$ as potential energy + kinetic energy, with the reference of the potential energy chosen such that $E(0) = 0$; that is,

$$
\begin{aligned}
E(x) &= \int_0^{x_1} a \sin y dy + \frac{1}{2}x_2^2 \\
&= a(1 - \cos x_1) + \frac{1}{2}x_2^2
\end{aligned}
$$

Determining Stability

- When friction is neglected ($b = 0$), the system is conservative; that is, there is no dissipation of energy.

- Hence, $E =$ constant during the motion of the system or, in other words, $dE/dt = 0$ along the trajectories.

- Since $E(x) = c$ forms a closed contour around $x = 0$ for small c, we can again arrive at the conclusion that $x = 0$ is a stable equilibrium point.

$$
\begin{aligned}
\frac{d}{dt}E(x) &= \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 \\
&= (a \sin x_1)\dot{x}_1 + (x_2)\dot{x}_2 \\
&= (a \sin x_1)x_2 + (x_2)(-a \sin x_1 - bx_2) \\
&= -bx_2^2
\end{aligned}
$$

Determining Stability

- When friction is accounted for ($b > 0$), energy will dissipate during the motion of the system, that is, $dE/dt \leq 0$ along the trajectories of the system.

- Due to friction, E cannot remain constant indefinitely while the system is in motion.

- Hence, it keeps decreasing until it eventually reaches zero, showing that the trajectory tends to $x = 0$ as t tends to ∞.

Determining Stability

- Thus, by examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point.

- In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point.

Lyapunov Stability (§4.1)

- Let $V : D \to R$ be
  a continuously differentiable function
  defined in a domain $D \subset R^n$
  that contains the origin.

- The derivative of V
  along the trajectories of (4.1),
  denoted by $V(x)$, is given by

$$\begin{aligned}\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial xi}\dot{x}_i \\ &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\ &= \frac{\partial V}{\partial x} f(x)\end{aligned}$$

Lyapunov Stability

- The derivative of V
  along the trajectories of a system
  is dependent on the system's equation.

- Hence, $\dot{V}(x)$ will be different for different
  systems.

- If $\phi(t;x)$ is the solution of (4.1)
  that starts at initial state x at time $t = 0$,
  then

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t;x))\bigg|_{t=0}$$

  Therefore, if $\dot{V}(x)$ is negative,
  V will decrease along the solution of (4.1).

Lyapunov's Stability Theorem

- Theorem 4.1:

  Let $x = 0$ be an equilibrium point for (4.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function such that

  $$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \quad (4.2)$$

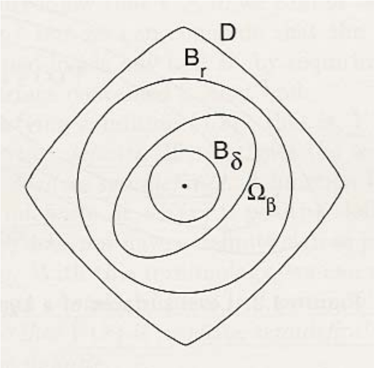
  $$\dot{V}(x) \leq 0 \text{ in } D \quad (4.3)$$

  Then, $x = 0$ is stable.

  Moreover, if

  $$\dot{V}(x) < 0 \text{ in } D - \{0\} \quad (4.4)$$

  then $x = 0$ is asymptotically stable.



Lyapunov's Stability Theorem: Proof - 1

- Proof:
- Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

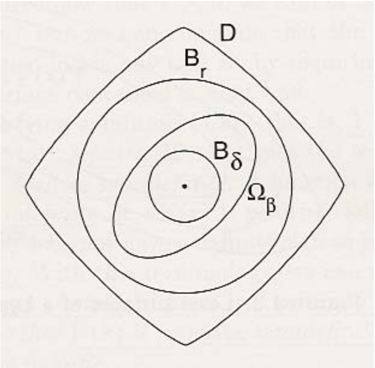
  $$B_r = \{x \in R^n \mid ||x|| \leq r\} \subset D$$

  Let $\alpha = \min_{||x||=r} V(x)$.
  Then, $\alpha > 0$ by (4.2).

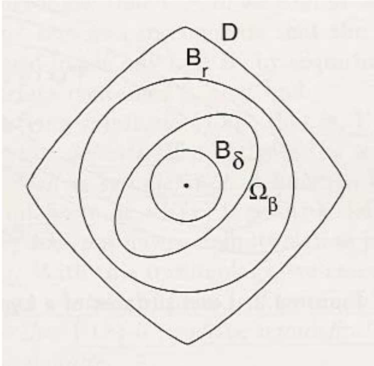
  Take $\beta \in (0, \alpha)$ and let

  $$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

  Then, Ω_β is in the interior of B_r.
  (See Figure 4.1.)



Lyapunov's Stability Theorem: Proof - 2



- Because in Ω_β,
  any trajectory starting in Ω_β at $t = 0$,
  and $\forall t \geq 0$

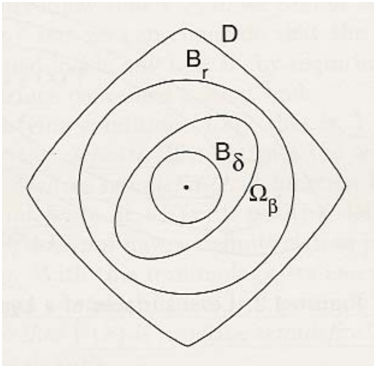
  $$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta$$

- In Ω_β, any trajectory starting in Ω_β at $t = 0$
  stays in Ω_β for all $t \geq 0$.

- Ω_β is closed by definition and
  bounded (contained in B_r).
  Hence, it is compact.

- We conclude from Theorem 3.3 that
  (4.1) has a unique solution defined $\forall t \geq 0$
  whenever $x(0) \in \Omega_\beta$.

Lyapunov's Stability Theorem: Proof - 3



- As $V(x)$ is continuous and $V(0) = 0$,
  there is $\delta > 0$ such that

  $$||x|| \leq \delta \Rightarrow V(x) < \beta$$

- Then,

  $$B_\delta \subset \Omega_\beta \subset B_r$$

  and

  $$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta$$

  $$\Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

- Therefore,

  $$||x(0)|| < \delta \Rightarrow ||x(t)|| < r \leq \epsilon, \forall t \geq 0$$

  which shows that
  the equilibrium point $x = 0$ is stable.

Lyapunov's Stability Theorem: Proof - 4

- Now, assume that (4.4) holds as well.

- To show asymptotic stability,
  we need to show that
  $x(t) \to 0$ as $t \to \infty$;
  that is, for every $a > 0$, there is $T > 0$
  such that $||x(t)|| < a$, for all $t > T$.

- By repetition of previous arguments,
  we know that for every $a > 0$,
  we can choose $b > 0$
  such that $\Omega_b \subset B_a$.

- Therefore, it is sufficient to show that
  $V(x(t)) \to 0$ as $t \to \infty$.

Lyapunov's Stability Theorem: Proof - 5

- Since $V(x(t))$ is monotonically decreasing
  and bounded from delow by zero.
  $V(x(t)) \to c \geq 0$ as $t \to \infty$

- To show that $c = 0$,
  we use a contradiction argument.

- Suppose $c > 0$.
  By continuity of V(x),
  there is $d > 0$
  such that $B_d \subset \Omega_c$.

- The limit $V(x(t)) \to c > 0$ implies that
  the trajectory $x(t)$ lies outside B_d, $\forall t \geq 0$.

Lyapunov's Stability Theorem: Proof - 6

- Because the continuous function $\dot{V}(x)$
  has a maximum
  over the compact set $\{d \le ||x|| \le r\}$.

  Let $-\gamma = \max_{d \le ||x|| \le r} \dot{V}(x)$.

- By (4.4), $-\gamma < 0$.
  It follows that

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ &\le V(x(0)) - \gamma t \end{aligned}$$

- Since the right-hand side will eventually
  become negative,
  the inequality contradicts the assumption
  that $c > 0$.

- QED

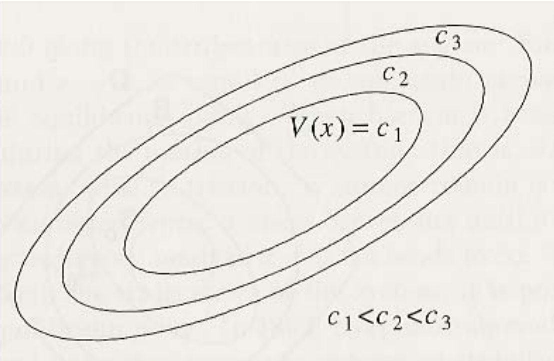
Lyapunov's Stability Theorem - 1

- A continuously differentiable function $V(x)$
  satisfying (4.2) and (4.3)
  is called a Lyapunov function.

- The surface $V(x) = c$, for some $c > 0$,
  is called a Lyapunov surface or
  a level surface.

- Using Lyapunov surfaces,
  we notice that
  Figure 4.2 makes the theorem
  intuitively clear.

- It shows Lyapunov surfaces
  for increasing values of c.



Lyapunov's Stability Theorem - 2

- The condition $\dot{V} \leq 0$ implies that
  when a trajectory crosses
  a Lyapunov surface $V(x) = c$,
  it moves inside the set
  $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$
  and can never come out again.

- When $\dot{V} < 0$,
  the trajectory moves
  from one Lyapunov surface
  to an inner Lyapunov surface
  with a smaller c.

Lyapunov's Stability Theorem - 3

- As c decreases,
  the Lyapunov surface $V(x) = c$
  shrinks to the origin,
  showing that
  the trajectory approaches the origin
  as time progresses.

- If we only know that $\dot{V} \leq 0$,
  we cannot be sure that
  the trajectory will approach the origin,
  but we can conclude that
  the origin is stable
  since the trajectory can be contained
  inside any ball B_ϵ
  by requiring the initial state $x(0)$
  to lie inside a Lyapunov surface
  contained in that ball.

Positive/Negative (Semi) Definiteness - 1 (§4.1)

- A function $V(x)$ satisfying condition (4.2) that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, is said to be positive definite.

- If it satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, it is said to be positive semidefinite.

- A function $V(x)$ is said to be negative definite or negative semidefinite if $-V(x)$ is positive definite or positive semidefinite, respectively.

- If $V(x)$ does not have a definite sign as per one of these four cases, it is said to be indefinite.

Positive/Negative (Semi) Definiteness - 2

- With this terminology, we can rephrase Lyapunov's theorem to say that the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

- A class of scalar functions for which sign definiteness can be easily checked is the class of functions of the quadratic form

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j$$

where P is a real symmetric matrix.

Positive/Negative (Semi) Definiteness - 3

- In this case,
  $V(x)$ is positive definite
  (positive semidefinite)

  iff all the eigenvalues of P are positive
  (nonnegative),
  which is true

  iff all the leading principal minors of P
  are positive
  (all principal minors of P are nonnegative).

- If $V(x) = x^T P x$ is positive definite
  (positive semidefinite),
  we say that the matrix P is positive definite
  (positive semidefinite)
  and write $P > 0$ ($P \geq 0$).

Example 4.1 - 1

- Example 4.1
  Consider

$$\begin{aligned} V(x) &= ax_1^2 + 2x_1x_3 + ax_2^2 + 4x_2x_3 + ax_3^2 \\ &= [x_1 x_2 x_3] \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x^T P x \end{aligned}$$

- The leading principal minors of P
  are a, a^2, and $a(a^2 - 5)$.

- Therefore, $V(x)$ is positive definite
  if $a > \sqrt{5}$.

Example 4.1 - 2

- For negative definiteness, the leading principal minors of $-P$ should be positive; that is , the leading principal minors of P should have alternating signs, with the odd-numbered minors being negative & the even-numbered moinors being positive.
- Consequently, $V(x)$ is negative definite if $a < -\sqrt{5}$.
- By calculating all principal minors, it can be seen that if $a \geq \sqrt{5}$, $V(x)$ is positive semidefinite and if $a \leq -\sqrt{5}$, negative semidefinite
- For $a \in (-\sqrt{5}, \sqrt{5})$, $V(x)$ is indefinite.

Example 4.1 – 3: Remarks

- Lyapunov's theorem can be applied without solving the differential equation.
- On the other hand, there is no systematic method for finding Lyapunov functions.
- In some cases, there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems.
- In other cases, it is basically a matter of trial and error.
- But, here, we will try to find some hints through various examples and applications.

Example 4.2 - 1

- Example 4.2

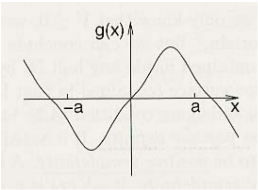
  Consider the differential equation

$$\dot{x} = -g(x)$$

  where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0;$$

$$xg(x) > 0, \quad \forall x \neq 0 \text{ and } x \in (-a, a)$$



- A a possible g(x) is shown is Fig. 4.3.

- The system has an isolated equilibrium point at the origin.

Example 4.2 - 2

- It is not difficult in this simple example to see that the origin is asymptotically stable, because solutions starting on either side of the origin will have to move toward the origin due to the sign of the derivative $\dot{x}$.

- To arrive at the same conclusion using Lyapunov's theorem, consider the function

$$V(x) = \int_0^x g(y)dy$$

- Over the domain $D = (-a, a)$, $V(x)$ is continuously differentiable, $V(0) = 0$, and $V(x) > 0$, $\forall x \neq 0$.

Example 4.2 - 3

- Thus, $V(x)$ is a valid Lyapunov function candidate.

- To see whether or not $V(x)$ is indeed a Lyapunov function, we calculate its derivative along the trajectories of the system.

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}[-g(x)] \\ &= -g^2(x) < 0, \forall x \in D - \{0\} \end{aligned}$$

- Hence, by Theorem 4.1 we conclude that the origin is asymptotically stable.

Example 4.3 - 1

- Example 4.3

  Consider the pendulum eqn w/o friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 \end{aligned}$$

  and let us study the stability of the equilibrium point at the origin.

- A natural Lyapunov function candidate is the energy function

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

- Clearly, $V(0) = 0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$.

Example 4.3 - 2

- The derivative of $V(x)$
  along the trajectories of the system
  is given by

$$\begin{aligned}\dot{V}(x) &= a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 \\ &= ax_2 \sin x_1 - ax_2 \sin x_1 = 0\end{aligned}$$

- Thus, conditions (4.2) and (4.3)
  of Theorem 4.1 are satisfied,
  and we conclude that the origin is stable.

- Since $\dot{V}(x) \equiv 0$,
  we can also conclude that
  the origin is not asymptotically stable;
  for trajectories starting
  on a Lyapunov surface $V(x) = c$
  remain on the same surface
  for all future time.

Example 4.4 - 1

- Example 4.4
  Consider the pendulum eqn with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2\end{aligned}$$

- Again, let us try
  $V(x) = a(1 - \cos x_1) + (1/2)x_2^2$
  as a Lyapunov function candidate.

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

- The derivative $\dot{V}(x)$ is
  negative semidefinite.

Example 4.4 - 2

- It is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1; that is, $\dot{V}(x) = 0$ along the x_1-axis.

- Therefore, we can only conclude that the origin is stable.

- However, using the phase portrait of the pendulum equation, we have seen that when $b > 0$, the origin is asymptotically stable.

- The energy Lyapunov function fails to show this fact.

Example 4.4 - 3

- We will see later in Section 4.2 that LaSalle's theorem will enable us to arrive at a different conclusion.

- For now, let us look for a Lyapunov function $V(x)$ that would have a negative definite $\dot{V}(x)$.

- Starting from the energy Lyapunov func let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^T Px$ for some 2×2 positive definite matrix P:

$$
\begin{aligned}
V(x) &= \frac{1}{2}x^T Px + a(1 - \cos x_1) \\
&= \frac{1}{2}[x_1 x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&+ a(1 - \cos x_1)
\end{aligned}
$$

Example 4.4 - 4

- For the quadratic form $\frac{1}{2}x^T P x$ to be positive definite, the elements of the matrix P must satisfy

$$p_{11} > 0, \; p_{11}p_{22} - p_{12}^2 > 0$$

- The derivative $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a\sin x_1)x_2 \\ &+ (p_{12}x_1 + p_{22}x_2)(-a\sin x_1 - bx_2) \\ &= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 \\ &+ (p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2^2\end{aligned}$$

- Now we want to choose p_{11}, p_{12}, and p_{22} such that $\dot{V}(x)$ is negative definite.

Example 4.4 - 5

- Since the cross product terms $x_2 \sin x_1$ and x_1x_2 are sign indefinite, we will cancel them by taking $p_{22} = 1$ and $p_{11} = bp_{12}$.

- With these choices, p_{12} must satisfy $0 < p_{12} < b$ for $V(x)$ to be positive definite.

- Let us take $p_{12} = b/2$.

- Then, $\dot{V}(x)$ is given by

$$\dot{V}(x) = -\frac{1}{2}abx_1 \sin x_1 - \frac{1}{2}bx_2^2$$

- The term $x_1 \sin x_1 > 0, \quad \forall\, 0 < |x_1| < \pi$.

Example 4.4 - 6

- Taking $D = \{x \in R^2 \mid |x_1| < \pi\}$, we see that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D.
- Thus, by Theorem 4.1, we conclude that the origin is asymptotically stable.
- The theorem's conditions are only sufficient.
- Failure of a Lyapunov function candidate does not mean that the equilibrium is not stable or asymptotically stable.
- It only means that such stability property cannot be established by using this Lyapunov function candidate.

Variable Gradient Method - 1 (§4.1): skip?

- The variable gradient method:
  A procedure that searches for a Lyapunov function in a backward manner.
  That is, investigate an expression for the derivative $\dot{V}(x)$ and go back to choose the parameters of $V(x)$ so as to make $\dot{V}(x)$ negative definite.
- To describe the procedure, let $V(x)$ be a scalar function of x and $g(x) = \nabla V = (\partial V / \partial x)^T$.
- The derivative $\dot{V}(x)$ along the trajectories of (4.1) is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

Variable Gradient Method - 2

- The idea now is to try to choose $g(x)$ such that it would be the gradient of a positive definite function $V(x)$ and, at the same time, $\dot{V}(x)$ would be negative definite.

- It is not difficult to verify that (Exercise 4.5) $g(x)$ is the gradient of a scalar function iff the Jacobian matrix $[\partial g/\partial x]$ is symmetric; that is,

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \ \forall i, j = 1, ..., n$$

- Under this constraint, we start by choosing $g(x)$ such that $g^T(x)f(x)$ is negative definite.

Variable Gradient Method - 3

- The function $V(x)$ is then computed from the integral

$$\begin{aligned} V(x) &= \int_0^x g^T(y) dy \\ &= \int_0^x \sum_{i=1}^n g_i(y) dy_i \end{aligned}$$

- The integration is taken over any path joining the origin to x.

- Usually, this is done along the axes, that is,

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(y_1, 0, ..., 0) dy_1 \\ &+ \int_0^{x_2} g_2(x_1, y_2, 0, ..., 0) dy_2 \\ &+ \cdots \\ &+ \int_0^{x_n} g_n(x_1, x_2, ..., x_{n-1}, y_n) dy_n \end{aligned}$$

Variable Gradient Method - 4

- By leaving some parameters of g(x) undetermined, one would try to choose them to ensure that $V(x)$ is positive definite.

- ...

$$\begin{aligned} \nabla V(x) &= g(x) \Rightarrow \frac{\partial g}{\partial x} \text{ is symmetric} \\ \dot{V}(x) &= g^{\top}(x) f(x) \end{aligned}$$

Example 4.5 - 1

- Example 4.5

  Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 \end{aligned}$$

  where $a > 0$,
  $h(\cdot)$ is locally Lipschitz, $h(0) = 0$, and
  $yh(y) > 0$ for all $y \neq 0, y \in (-b, c)$
  for some positive constants b and c.

- The pendulum equation is a special case of this system.

Example 4.5 - 2

- To apply the variable gradient method, we want to choose a second-order vector $g(x)$ that satisfies

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

$$\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0,$$

for $x \neq 0$ and

$$V(x) = \int_0^x g^T(y)dy > 0, \text{ for } x \neq 0$$

- Let us try

$$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$$

where the scalar functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$, and $\delta(\cdot)$ are to be determined.

Example 4.5 - 3

- To satisfy the symmetry requirement, we must have

$$\beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2$$

$$= \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$$

- The derivative $\dot{V}(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 \\ &- a\delta(x)x_2^2 - \delta(x)x_2h(x_1) \\ &- \gamma(x)x_1h(x_1) \end{aligned}$$

Example 4.5 - 4

- To cancel cross-product terms, we choose

$$\alpha(x)x_1 - \alpha\gamma(x)x_1 - \delta(x)h(x_1) = 0$$

so that

$$\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$$

- To simplify our choices, let us take $\delta(x) = \delta =$ constant, $\gamma(x) = \gamma =$ constant, and $\beta(x) = \beta =$ constant.

- Then, $\alpha(x)$ depends only on x_1, and the symmetry requirement is satisfied by choosing $\beta = \gamma$.

Example 4.5 - 5

- The expression for $g(x)$ reduces to

$$g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

- By integration, we obtain

$$\begin{aligned} V(x) &= \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]dy_1 \\ &\quad + \int_0^{x_2} (\gamma x_1 + \delta y_2)dy_2 \\ &= \frac{1}{2}a\gamma x_1^2 + \delta \int_0^{x_1} h(y)dy + \gamma x_1 x_2 + \frac{1}{2}\delta x_2^2 \\ &= \frac{1}{2}x^T P x + \delta \int_0^{x_1} h(y)dy \end{aligned}$$

where $P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$

- Choosing $\delta > 0$ and $0 < \gamma < a\delta$ ensures that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite.

Example 4.5 - 6

- For example,
  taking $\gamma = ak\delta$ for $0 < k < 1$
  yields the Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)dy$$

  which satisfies conditions (4.2) and (4.4)
  of Theorem 4.1 over the domain D,
  $D = \{x \in R^2 \mid -b < x_1 < c\}$.

- So, the origin is asymptotically stable.

Region of Attraction - 1 (§4.1)

- Region of attraction
  Region of asymptotic stability
  Domain of attraction
  Basin

- When the origin $x = 0$ is
  asymptotically stable,
  we are often interested in determining
  how far from the origin
  the trajectory can be and
  still converge to the origin
  as t approaches ∞.

- Let $\phi(t; x)$ be the solution of (4.1)
  that starts at initial state x at time $t = 0$.

Region of Attraction - 2

- Then, the region of attraction is defined as the set of all points x such that $\phi(t; x)$ is defined for all $t \geq 0$ and $\lim_{t\to\infty} \phi(t; x) = 0$.

- Finding the exact region of attraction analytically might be difficult or even impossible.

- However, Lyapunov functions can be used to estimate the sets contained in the region of attraction.

Region of Attraction - 3

- From the proof of Theorem 4.1, we see that if there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D and, if $\Omega_c = \{x \in R^n \mid V(x) \leq c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$.

- Thus, Ω_c is an estimate of the region of attraction.

- The estimate may be conservative, that is, it may be much smaller than the actual region of attraction.

Region of Attraction - 4

- In Section 8.2, we will solve examples on estimating the region of attraction and see some ideas to enlarge the estimates.

- Here, we want to pursue another question: Under what conditions will the region of attraction be the whole space R^n?

Region of Attraction is R^n - 1

- The region of attraction be the whole space R^n?

- For any initial state x, the trajectory $\phi(t;x)$ approaches the origin as $t \to \infty$, no matter how large $||x||$ is.

- If an asymptotically stable E.P. at the origin has this property, it is said to be globally asymptotically stable.

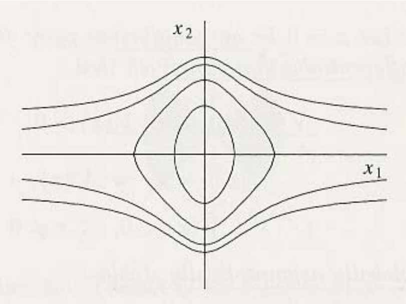
Region of Attraction is R^n - 2

- From the proof of Theorem 4.1, for the global asymptotic stability if $x \in R^n$ can be included in the interior of a bounded set Ω_c. That is, $D = R^n$; but, is that enough?

- The problem is that for large c, the set Ω_c need not be bounded.

- For example, consider the function

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



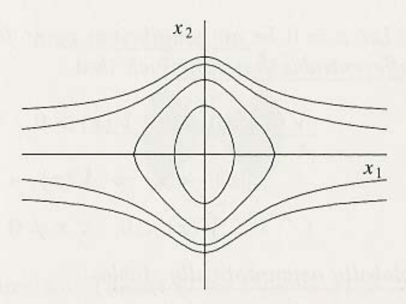
- Fig. 4.4 shows the surfaces $V(x) = c$ for various positive values of c.

Region of Attraction is R^n - 3

- For small c, the surface $V(x) = c$ is closed; hence, Ω_c is bounded since it is contained in a closed ball B_r for some $r > 0$.

- This is a consequence of the continuity and positive definiteness of $V(x)$.

- As c increases, a value is reached after which the surface $V(x) = c$ is open and Ω_c is unbounded.



Radially Unbounded - 1

- For Ω_c to be
  in the interior of a ball B_r,
  c must satisfy $c < \inf_{||x|| \geq r} V(x)$

- If
  $$l = \lim_{r \to \infty} \inf_{||x|| \geq r} V(x) < \infty$$
  Then Ω_c will be bounded if $c < l$.

- In the preceding example,
  $$\begin{aligned} l &= \lim_{r \to \infty} \min_{||x||=r} \left[\frac{x_1^2}{1+x_1^2} + x_2^2 \right] \\ &= \lim_{|x_1| \to \infty} \frac{x_1^2}{1+x_1^2} = 1 \end{aligned}$$

- Thus, Ω_c is bounded only for $c < 1$.

Radially Unbounded - 2

- Thus, Ω_c is bounded only for $c < 1$.

- An extra condition that ensures that
  Ω_c is bounded for all values of $c > 0$ is
  $$V(x) \to \infty \text{ as } ||x|| \to \infty$$

- A function satisfying this condition
  is said to be radially unbounded.

Globally Asymptotically Stable - 1 (§4.1)

- Barbashin-Krasovskii Theorem:
  radial boundedness
  for globally asymptotically stability.

- Theorem 4.2
- Let $x = 0$ be an E.P. for (4.1).

- Let $V : R^n \to R$
  be a continuously differentiable function
  such that

$$V(0) = 0 \text{ and } V(x) > 0,\ \forall x \neq 0 \quad (4.5)$$

$$||x|| \to \infty \Rightarrow V(x) \to \infty \quad (4.6)$$

$$\dot{V}(x) < 0,\ \forall x \neq 0 \quad (4.7)$$

then $x = 0$ is
globally asymptotically stable.

Globally Asymptotically Stable - 2

- Proof:
- Given any point $p \in R^n$,
  let $c = V(p)$.

- Condition (4.6) implies that
  for any $c > 0$, there is $r > 0$
  such that
  whenever $||x|| > r$,
  $V(x) > c$

- Thus, $\Omega_c \subset B_r$,
  which implies that Ω_c is bounded.

- The rest of the proof is similar to
  that of Theorem 4.1.

Example 4.6 - 1

- Example 4.6
- Consider again the system of Example 4.5, but this time, assume that the condition $yh(y) > 0$ holds for all $y \neq 0$.
- The Lyapunov function

$$V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)dy$$

is positive definite for all $x \in R^2$ and radially unbounded.

- The derivative

$$\dot{V}(x) = -a\delta(1-k)x_2^2 - a\delta k x_1 h(x_1)$$

is negative definite for all $x \in R^2$ since $0 < k < 1$.

- Therefore, the origin is globally asymptotically stable.

Example 4.6 - 2

- If the origin $x = 0$ is a globally asymptotically stable E.P. of a system, then it must be the unique E.P. of the system.
- For if there were another E.P. $\bar{x}$, the trajectory starting at $\bar{x}$ would remain at $\bar{x}$, $\forall t \geq 0$; hence, it would not approach the origin, which contradicts the claim that the origin is globally asymptotically stable.
- Therefore, global asymptotic stability is not studied for multiple equilibria systems like the pendulum equation.

Instability Theorem - 1 (§4.1)

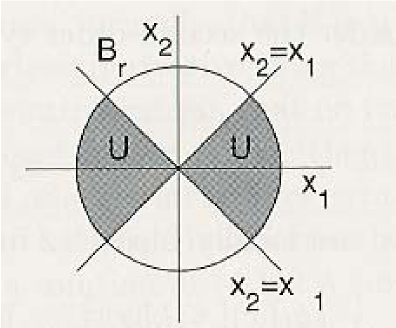
- Let $V : D \to R$ be a continuously differentiable function on a domain $D \subset R^n$ that contains the origin $x = 0$.

- Suppose $V(0) = 0$ and there is a point x_0 arbitrarily close to '0' such that $V(x_0) > 0$.

- Choose $r > 0$ such that the ball $B_r = \{x \in R^n \mid ||x|| \le r\}$ is contained in D, and let

$$U = \{x \in B_r \mid V(x) > 0\} \quad (4.8)$$

- The set U is a nonempty set contained in B_r.



Instability Theorem - 2

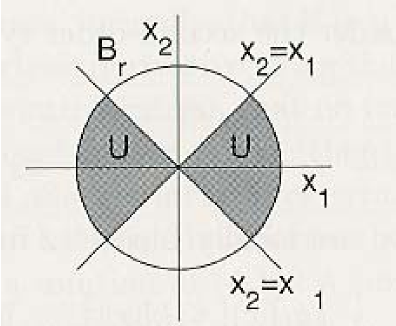
- Its boundary is the surface $V(x) = 0$ and the sphere $||x|| = r$.

- Since $V(0) = 0$, the origin lies on the boundary of U inside B_r.

- Notice that U may contain more than one component.

- For example, Figure 4.5 shows the set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

- The set U can be always constructed provided that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 arbitrarily close to the origin.



Theorem 4.3: Chetaev's Theorem - 1

- Chetaev's Theorem:
- Theorem 4.3:
- Let $x = 0$ be an E.P. for (4.1).
- Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$, and $V(x_0) > 0$ for some x_0 with arbitrarily small $||x_0||$.
- Define a set U as in (4.8) and suppose that $\dot{V}(x) > 0$ in U.
- Then, $x = 0$ is unstable.

Theorem 4.3: Chetaev's Theorem - 2

- Proof:
- The point x_0 is in the interior of U and $V(x_0) = a > 0$.
- The trajectory $x(t)$ starting at $x(0) = x_0$ must leave the set U.
- To see this point, notice that as long as $x(t)$ is inside U, $V(x(t)) \geq a$, since $\dot{V}(x) > 0 \in U$.
- Since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \geq a\} = \{x \in B_r \text{ and } V(x) \geq a\}$.

  Let $\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \geq a\}$

Theorem 4.3: Chetaev's Theorem - 3

- Then, $\gamma > 0$ and

$$\begin{aligned} V(x(t)) &= V(x_0) + \int_0^t \dot{V}(x(s))ds \\ &\geq a + \int_0^t \gamma ds = a + \gamma t \end{aligned}$$

- This inequality shows that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U.

- Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \geq a$.

- Hence, it must leave U through the sphere $||x|| = r$.

- Because this can happen for an arbitrarily small $||x_0||$, the origin is unstable.

Example 4.7 - 1

- Example 4.7:
- Consider the second-order system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $g_{1,2}(\cdot)$ are locally Lipschitz functions that satisfy the inequalities

$$|g_1(x)| \leq k||x||_2^2,$$

$$|g_2(x)| \leq k||x||_2^2$$

in a neighborhood D of the origin.

- These inequalities imply that $g_1(0) = g_2(0) = 0$.

- Hence, the origin is an E.P.

Example 4.7 - 2

- Consider the function

$$V(x) = \frac{1}{2}(x_1^2 - x_2^2)$$

- On the line $x_2 = 0$, $V(x) > 0$ at points arbitrarily close to the origin.

- The set U is shown in Figure 4.5.

- The derivative of V(x) along the trajectories of the system is given by

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

Example 4.7 - 3

- The magnitude of the term $x_1 g_1(x) - x_2 g_2(x)$ satisfies the inequality

$$\begin{aligned} |x_1 g_1(x) - x_2 g_2(x)| &\le \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \\ &\le 2k ||x||_2^3 \end{aligned}$$

- Hence,

$$\dot{V}(x) \ge ||x||_2^2 - 2k||x||_2^3 = ||x||_2^2 (1 - 2k||x||_2)$$

- Choosing r such that $B_r \subset D$ and $r < 1/(2k)$, it is seen that all the conditions of Thm 4.3 are satisfied.

- Therefore, the origin is unstable.