Nonlinear Systems Analysis

Lecture 7

Sections 3.2, 3.2, 3.4

Dependence on Data, Sensitivity Analysis, Comparison Principle

Feng-Li Lian

NTU-EE

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Outline

- Introduction (L5)
- Banach Space (L5)
- Contraction Mapping Theorem (L5)
- Existence and Uniqueness (L6)
- Continuous Dependence on Initial Conditions and Parameters (L6)
- Differentiability of Solutions and Sensitivity Equations (L7)
- Comparison Principle (L7)

Continuous Dependence on Data - 1 (3.2)

- Here, we discuss the dependence of the solution of (3.1) on the initial state x_0, and the RHS function $f(t, x)$.

- Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$.

Continuous Dependence on Data - 2

- Dependence on x_0:
- $B_\delta(y_0) = \{x \in R^n | \; ||x - y_0|| < \delta\}$

- Given $\epsilon > 0$, there is $\delta > 0$ such that for all z_0 in $B_\delta(y_0)$, $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $||z(t) - y(t)|| < \epsilon$ for all $t \in [t_0, t_1]$.

Continuous Dependence on Data - 3

- Dependence on $f(t,x)$:
- Assume that f depends continuously on a set of constant parameters; that is, $f = f(t,x,\lambda)$, where $\lambda \in R^p$.
- Let $x(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ defined on $[t_0,t_1]$, with $x(t_0,\lambda_0) = x_0$.
- The solution is said to depend continuously on λ if for any $\epsilon > 0$, there is $\delta > 0$ such that for all λ in $B_\delta(\lambda_0)$, $\dot{x} = f(t,x,\lambda)$ has a unique solution $x(t,\lambda)$ defined on $[t_0,t_1]$, with $x(t_0,\lambda) = x_0$, and satisfies $||x(t,\lambda) - x(t,\lambda_0)|| < \epsilon$ for all $t \in [t_0,t_1]$.

Gronwall-Bellman Inequality - 1 (App. A, page 651)

- Lemma A.1: (Gronwall-Bellman Inequality)
- Let $\lambda : [a,b] \to R$ be continuous and $\mu : [a,b] \to R$ be cont. and nonnegative.
- If a continuous function $y : [a,b] \to R$ satisfies

$$y(t) \le \lambda(t) + \int_a^t \mu(s)y(s)ds$$

for $a \le t \le b$,

then on the same interval

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp^{[\int_s^t \mu(\tau)d\tau]} ds$$

Gronwall-Bellman Inequality - 2

- In particular,
  if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \leq \lambda \exp[\int_a^t \mu(\tau)d\tau]$$

- If, in addition,
  $\mu(t) \equiv \mu \geq 0$ is a constant, then

$$y(t) \leq \lambda \exp[\mu(t-a)]$$

Closeness of Solutions - 1 (3.2)

- Theorem 3.4:

- Let $f(t,x)$ be
  piecewise continuous in t and
  Lipschitz in x on $[t_0, t_1] \times W$
  with a Lipschitz constant L,
  where $W \subset R^n$ is an open connected set.

- Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y), \ y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z), \ z(t_0) = z_0$$

such that $y(t), z(t) \in W$ for all $t \in [t_0, t_1]$.

Closeness of Solutions - 2

- Suppose that

$$||g(t,x)|| \leq \mu, \quad \forall (t,x) \in [t_0, t_1] \times W$$

for some $\mu > 0$.

- Then, $\forall t \in [t_0, t_1]$

$$||y(t) - z(t)|| \leq ||y_0 - z_0|| \exp^{[L(t-t_0)]} + \frac{\mu}{L}\left\{\exp^{[L(t-t_0)]} - 1\right\}$$

Closeness of Solutions - 3

- Proof:
- The solutions $y(t)$ and $z(t)$ are given by

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$
$$z(t) = z_0 + \int_{t_0}^{t} [f(s, z(s)) + g(s, z(s))]ds$$

- Substracting the two equations and taking norms yield

$$\begin{aligned}||y(t) - z(t)|| &\leq ||y_0 - z_0|| \\ &+ \int_{t_0}^{t} ||f(s, y(s)) - f(s, z(s))||ds \\ &+ \int_{t_0}^{t} ||g(s, z(s))||ds \\ &\leq \gamma + \mu(t - t_0) \\ &+ \int_{t_0}^{t} L||y(s) - z(s)||ds\end{aligned}$$

where $\gamma = ||y_0 - z_0||$.

Closeness of Solutions - 4

- By the Gronwall-Bellman inequality (Lemma A.1) the function $||y(t) - z(t)||$ results in

$$\begin{aligned} ||y(t) - z(t)|| &\le \gamma + \mu(t - t_0) \\ &+ \int_{t_0}^{t} L[\gamma + \mu(s - t_0)] \\ &\exp[L(t - s)]ds \end{aligned}$$

- Integrating the RHS by parts, we obtain

$$\begin{aligned} ||y(t) - z(t)|| &\le \gamma + \mu(t - t_0) \\ &- \gamma - \mu(t - t_0) \\ &+ \gamma \exp[L(t - t_0)] \\ &+ \int_{t_0}^{t} \mu \exp[L(t - s)]ds \\ &= \gamma \exp[L(t - t_0)] \\ &+ \frac{\mu}{L}\{\exp[L(t - t_0)] - 1\} \end{aligned}$$

which completes the proof of the theorem.

Dependence on Initial States & Parameters - 1 (3.2)

- Theorem 3.5:
- Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0, t_1] \times D \times \{||\lambda - \lambda_0|| \le c\}$ where $D \subset R^n$ is an open connected set.
- Let $y(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in D$.
- Suppose $y(t, \lambda_0)$ is defined and belongs to D for all $t \in [t_0, t_1]$.

Dependence on Initial States & Parameters - 2

- Then, given $\epsilon > 0$, there is $\delta > 0$ such that if

$$||z_0 - y_0|| < \delta \text{ and } ||\lambda - \lambda_0|| < \delta$$

then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$
defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$,
and $z(t, \lambda)$ satisfies

$$||z(t, \lambda) - y(t, \lambda_0)|| < \epsilon, \ \forall t \in [t_0, t_1]$$

Dependence on Initial States & Parameters - 3

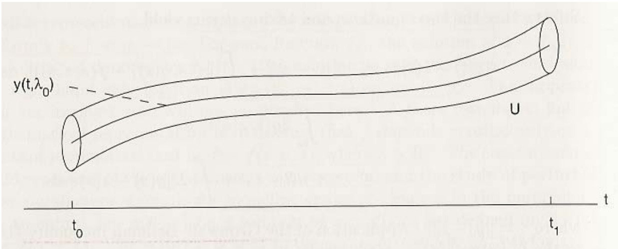
- Proof:

- By continuity of $y(t, \lambda_0)$ in t and the compactness of $[t_0, t_1]$, $y(t, \lambda_0)$ is bounded on $[t_0, t_1]$.

- Define a "tube" U around the solution $y(t, \lambda_0)$ by

$$U = \{(t, x) \in [t_0, t_1] \times R^n \mid ||x - y(t, \lambda_0)|| \leq \epsilon\}$$

- Suppose that $U \subset [t_0, t_1] \times D$;
if not, replace ϵ by $\epsilon_1 < \epsilon$
that is smaller enough to ensure that
$U \subset [t_0, t_1] \times D$ and
continue the proof with ϵ_1.



Dependence on Initial States & Parameters - 4

- The set U is compact;
  hence, $f(t, x, \lambda)$ is Lipschitz in x on U
  with a Lipschitz constant, says, L.

- By continiuity of f in λ
  for any $\alpha > 0$,
  there is $\beta > 0$ (with $\beta < c$)
  such that

$$||f(t, x, \lambda) - f(t, x, \lambda_0)|| < \alpha,$$

$$\forall (t, x) \in U, \forall ||\lambda - \lambda_0|| < \beta$$

- Take $\alpha < \epsilon$ and $||z_0 - y_0|| < \alpha$.

Dependence on Initial States & Parameters - 5

- By the local existence and uniqueness thm,
  there is a unique solution $z(t, \lambda)$
  on some time interval $[t_0, t_0 + \Delta]$.

- The solution starts insdie the tube U,
  and as long as it remains in the tube,
  it can be extended.

- By choosing a small enough α,
  the solution remains in U for all $t \in [t_0, t_1]$.

- In particular,
  we let τ be the first time
  the solution leaves the tube and
  show that we can make $\tau > t_1$.

Dependence on Initial States & Parameters - 6

- On the time interval $[t_0, \tau]$, the conditions of Thm 3.4 are satisfied with $\mu = \alpha$.

- Hence,

$$\begin{aligned} ||z(t,\lambda) - y(t,\lambda_0)|| &< \alpha \exp^{[L(t-t_0)]} \\ &\quad + \frac{\alpha}{L}\{\exp^{[L(t-t_0)]} - 1\} \\ &< \alpha(1 + \frac{1}{L}) \exp^{[L(t-t_0)]} \end{aligned}$$

- Choosing $\alpha \leq \epsilon L \exp^{[-L(t_1 - t_0)]} / (1 + L)$ ensures that the solution $z(t,\lambda)$ cannot leave the tube during the interval $[t_0, t_1]$.

Dependence on Initial States & Parameters - 7

- Therefore, $z(t,\lambda)$ is defined on $[t_0, t_1]$ and satisfies $||z(t,\lambda) - y(t,\lambda_0)|| < \epsilon$.

- Take $\delta = \min\{\alpha, \beta\}$ completes the proof of the theorem.

- QED

Differentiability of Solutions - 1 (3.3)

- Suppose that
  $f(t, x, \lambda)$ is continuous in (t, x, λ) and
  has continuous first partial derivatives
  wrt x and λ
  for all $(t, x, \lambda) \in [t_0, t_1] \times R^n \times R^p$.

- Let λ_0 be a nominal value of λ, and
  suppose that
  the nominal state equation $\dot{x} = f(t, x, \lambda_0)$,
  with $x(t_0) = x_0$
  has a unique solution $x(t, \lambda_0)$ over $[t_0, t_1]$.

- From Thm 3.5,
  for all λ sufficiently close to λ_0,
  that is, $||\lambda - \lambda_0||$ sufficiently small,
  $\dot{x} = f(t, x, \lambda)$, with $x(t_0) = x_0$
  has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$
  that is close to the nominal solution $x(t, \lambda_0)$.

Differentiability of Solutions - 2

- The continuous differentiability of f
  wrt x, λ
  implies the additional property that
  the solution $x(t, \lambda)$ is differentiable
  wrt λ near λ_0.

- To see that, write

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) ds$$

- Take partial derivatives wrt λ yields

$$\begin{aligned} x_\lambda(t, \lambda) &= \frac{\partial x(t, \lambda)}{\partial \lambda} \\ &= \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) x_\lambda(s, \lambda) \right. \\ &\quad \left. + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds \end{aligned}$$

Differentiability of Solutions - 3

- Differentiating wrt t,
  it can be seen that $x_\lambda(t,\lambda)$ satisfies

$$\frac{\partial}{\partial t} x_\lambda(t,\lambda) = A(t,\lambda) x_\lambda(t,\lambda) + B(t,\lambda)$$

$$x_\lambda(t_0,\lambda) = 0 \quad (3.4)$$

$$A(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial x}\right|_{x=x(t,\lambda)}$$

$$B(t,\lambda) = \left.\frac{\partial f(t,x,\lambda)}{\partial \lambda}\right|_{x=x(t,\lambda)}$$

Sensitivity Equation - 1 (3.3)

- For λ sufficiently close to λ_0,
  the matrices $A(t,\lambda)$ and $B(t,\lambda)$ are
  defined on $[t_0, t_1]$.
  Hence, $x_\lambda(t,\lambda)$ is defined
  on the same interval.

- At $\lambda = \lambda_0$,
  the RHS of (3.4) depends only on
  the nominal solution $x(t,\lambda_0)$.

- Let $S(t) = x_\lambda(t,\lambda_0)$;
  then $S(t)$ is the unique solution of

  $\dot{S} = A(t,\lambda_0)S(t) + B(t,\lambda_0), S(t_0) = 0$ (3.5)

- $S(t)$ is called the sensitivity function, and
  (3.5) is called the sensitivity equation.

Sensitivity Equation - 2

- Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions.

- For small $||\lambda - \lambda_0||$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$:

$$x(t, \lambda) = x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + HOT$$

Or, $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$

Sensitivity Equation - 3

- Procedure for calculating $S(t)$:
  - Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$
  - Evaluate the Jacobian matrices

$$A(t, \lambda_0) = \left.\frac{\partial f(t, x, \lambda)}{\partial x}\right|_{x = x(t,\lambda_0), \lambda = \lambda_0}$$

$$B(t, \lambda_0) = \left.\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right|_{x = x(t,\lambda_0), \lambda = \lambda_0}$$

  - Solve the sensitivity equation (3.5) for $S(t)$.

- Alternative approach for calculating $S(t)$:

$$\dot{x} = f(t, x, \lambda_0), \quad x(t_0) = x_0,$$

$$\dot{S} = \left[\frac{\partial f(t, x, \lambda)}{\partial x}\right]_{\lambda = \lambda_0} S + \left[\frac{\partial f(t, x, \lambda)}{\partial \lambda}\right]_{\lambda = \lambda_0}$$

$$S(t_0) = 0$$

which is solved numerically.

Comparison Principle - 1 (3.4)

- Sometimes we only want to compute the bounds of $x(t)$ without solving it.
- The Gronwall-Bellman Inequality is a tool. Another tool is the comparison lemma.
- Consider a differential inequality
  $\dot{v}(t) \leq f(t, v(t))$
  and a differential equation
  $\dot{u}(t) = f(t, u(t))$.

Comparison Principle - 2

- And two facts:
  - If $v(t)$ is differentiable at t, then $D^+v(t) = \dot{v}(t)$.
  - If $\frac{1}{h}|v(t+h) - v(t)| \leq g(t,h)$, $\forall h \in (0, b]$ and $\lim_{h \to 0^+} g(t,h) = g_0(t)$ then $D^+v(t) \leq g_0(t)$.

    The limit $h \to 0^+$ means that h approaches zero from above.

    upper RH derivative:

$$D^+v(t) = \lim_{h \to 0^+} \sup \frac{v(t+h) - v(t)}{h}$$

Comparison Principle - 3

- Lemma 3.4: (Comparison Lemma)
- Consider $\dot{u} = f(t,u)$, $u(t_0) = u_0$
  where $f(t,u)$ is
  continuous in t and
  locally Lipschitz in u,
  for all $u \in J \subset R$.
- Let $[t_0, T)$ (T could be infinity)
  be the maximal interval of existence
  of the solution $u(t)$,
  and suppose $u(t) \in J$ for all $t \in [t_0, T)$.
- Let $v(t)$ be a continuous function
  whose upper RH derivative $D^+v(t)$
  satisfies the differential inequality
  $D^+v(t) \leq f(t, v(t))$, $v(t_0) \leq u_0$
  with $v(t) \in J$ for all $t \in [t_0, T)$.
  Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Example 3.8 - 1 (3.4)

- Example 3.8:
- Consider the scalar D.E.

$$\dot{x} = f(x) = -(1 + x^2), \ x(0) = a$$

  has a unique solution on $[0, t_1)$,
  for some $t_1 > 0$,
  because $f(x)$ is local Lipschitz.
- Let $v(t) = x^2(t)$.
- $v(t)$ is differentiable and
  its derivative is given by

$$\dot{v} = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t)$$

Example 3.8 - 2

- Hence,
  $v(t)$ satisfies the differential inequality

$$\dot{v}(t) \leq -2v(t), \; v(0) = a^2$$

- Let $u(t)$ be the solution of the D.E.

$$\dot{u} = -2u, \; u(0) = a^2,$$

$$\Rightarrow u(t) = a^2 e^{-2t}$$

- Then, by the comparison lemma,
  the solution $x(t)$ is defined for all $t \geq 0$
  and satisfies

$$|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \; \forall t \geq 0$$