Nonlinear Systems Analysis

Lecture 5

Appendix B & Section 3.1

Contraction Mapping Theorem + Existence & Uniqueness

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Outline

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Introduction - 1

- Fundamental properties of the solutions of ODEs:
  existence,
  uniqueness,
  continuous dependence on initial conditions
  and continuous dependence on parameters.

- Starting an experiment at t_0,
  we expect the system will move and
  its states will be defined at $t > t_0$.

- With a deterministic system,
  we expect that
  we can repeat the experiment exactly,
  i.e. get same motion and same state
  at $t > t_0$.

Introduction - 2

- To obtain this prediction,
  the initial-value problem
  $\dot{x} = f(t, x),\ x(t_0) = x_0$
  must have a unique solution.

- The existence and uniqueness
  can be ensured
  by imposing some constraints on $f(t, x)$.

- The key constraint is
  the Lipschitz condition:
  $||f(t, x) - f(t, y)|| \leq L||x - y||$
  for all (t, x) and (t, y)
  in some neighborhood of (t_0, x_0).

Introduction - 3

- An essential factor in the validity of any math model is the continuous dependence of its solutions on the data of the problem.

- The data are
  the initial state x_0,
  the initial time t_0, and
  the $f(t,x)$.

- Arbitrarily small errors in the data will not result in large errors in the solutions.

Introduction - 4

- Sensitivity equations
  to describe the effect
  of small parameter variations
  on the performance of the system.

- Comparison principle
  to bound the solution
  of a scalar differential inequality
  $\dot{v} \leq f(t,v)$
  by the solution of $\dot{u} = f(t,u)$.

Banach Space - 1 (App. B; page 653)

- Linear Vector Spaces:
- A linear vector space χ over the field R
  is a set of elements $x, y, z, \ldots$ called vectors
  such that for any two vectors $x, y \in \chi$
- the sum $x + y$ is defined,
  - $x + y \in \chi$,
  - $x + y = y + x$,
  - $(x + y) + z = x + (y + z)$,
- and there is zero vector $0 \in \chi$
  - such that $x + 0 = x$ for all $x \in \chi$.

Banach Space - 2

- For any numbers $\alpha, \beta \in R$,
  the scalar multiplication αx is defined,
  - $\alpha x \in \chi$,
  - $1 \cdot x = x$,
  - $0 \cdot x = 0$,
  - $(\alpha\beta)x = \alpha(\beta x)$,
  - $\alpha(x + y) = \alpha x + \alpha y$, and
  - $(\alpha + \beta)x = \alpha x + \beta x$, for all $x, y \in \chi$.

Banach Space - 3

- Normed Linear Spaces:
- A linear space χ is a normed linear space
  if, to each vector $x \in \chi$,
  there is a real-valued norm $||x||$
  that satisfies:
- $||x|| \geq 0$ for all $x \in \chi$, with $||x|| = 0$ iff $x = 0$.
- $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in \chi$.
- $||\alpha x|| = |\alpha| \; ||x||$ for all $\alpha \in R$ and $x \in \chi$.

Banach Space - 4

- Convergence:
- A sequence $\{x_k\} \in \chi$, a normed linear space,
  converges to $x \in \chi$
  if $||x_k - x|| \to 0$ as $k \to \infty$.
- Closed Set:
- A set $S \subset \chi$ is closed
  iff every convergent sequence
  with elements in S has its limit in S.
- Cauchy Sequence:
- A sequence $\{x_k\} \in \chi$ is said to be
  a Cauchy sequence
  if $||x_k - x_m|| \to 0$ as $k, m \to \infty$.

Banach Space - 5

- Banach Space:
- A normed linear space χ is complete if every Cauchy sequence in χ converges to a vector in χ.
- A complete normed linear space is a Banach space.

Contraction Mapping Theorem - 1 (App. B; page 655)

- Theorem B.1 (Contraction Mapping):
- Let S be a closed subset of a Banach space χ and let T be a mapping that maps S into S.
- Suppose that

$$||T(x) - T(y)|| \leq \rho ||x - y||,$$

$$\forall x, y \in S, 0 \leq \rho < 1$$

then
  - there exists a unique vector $x^* \in S$ satisfying $x^* = T(x^*)$.
  - x^* can be obtained by the method of successive approximation, starting from any arbitrary initial vector in S.

Contraction Mapping Theorem - 2

- Proof:
- Select an arbitrary $x_1 \in S$ and
  define the sequence $\{x_k\}$ by $x_{k+1} = T(x_k)$.
  Since T maps S into S, $x_k \in S, \quad \forall k \geq 1$.

- Show that $\{x_k\}$ is Cauchy:
  We have

$$
\begin{aligned}
||x_{k+1} - x_k|| &= ||T(x_k) - T(x_{k-1})|| \\
&\leq \rho ||x_k - x_{k-1}|| \\
&\leq \rho^2 ||x_{k-1} - x_{k-2}|| \\
&\leq \dots \\
&\leq \rho^{k-1} ||x_2 - x_1||
\end{aligned}
$$

Contraction Mapping Theorem - 3

- It follows that

$$
\begin{aligned}
||x_{k+r} - x_k|| &\leq ||x_{k+r} - x_{k+r-1}|| \\
&\quad + ||x_{k+r-1} - x_{k+r-2}|| + \dots \\
&\quad + ||x_{k+1} - x_k|| \\
&\leq [\rho^{k+r-2} + \rho^{k+r-3} + \dots + \rho^{k-1}] \\
&\quad ||x_2 - x_1|| \\
&\leq \rho^{k-1} \sum_{i=0}^{\infty} \rho^i ||x_2 - x_1|| \\
&= \frac{\rho^{k-1}}{1-\rho} ||x_2 - x_1||
\end{aligned}
$$

The RHS tends to zero as $k \to \infty$.
Thus, the sequence is Cauchy.

- Because χ is a Banach space,
  $x_k \to x^* \in \chi$ as $k \to \infty$.
- Moreover, since S is closed, $x^* \in S$.

Contraction Mapping Theorem - 4

- Show that $x^* = T(x^*)$:
- For any $x_k = T(x_{k-1})$, we have

$$\begin{aligned} ||x^* - T(x^*)|| &\leq ||x^* - x_k|| + ||x_k - T(x^*)|| \\ &\leq ||x^* - x_k|| + \rho ||x_{k-1} - x^*|| \end{aligned}$$

By choosing k large enough,
the RHS can be made arbitrarily small.
Thus, $||x^* - T(x^*)|| = 0$, i.e., $x^* = T(x^*)$.

Contraction Mapping Theorem - 5

- Show that x^* is the unique fixed point of T in S.
- Suppose that x^* and y^* are fixed points.
  Then,

$$||x^* - y^*|| = ||T(x^*) - T(y^*)|| \leq \rho ||x^* - y^*||$$

Since $\rho < 1$, we have $x^* = y^*$.

- QED.

- T maps S into S.

- T is a contraction mapping over S.

Existence and Uniqueness (3.1)

- Sufficient conditions
  for the existence and uniqueness
  of the solution
  of the initial-value problem (3.1).

- A solution of (3.1) over the interval $[t_0, t_1]$,
  a continuous function $x : [t_0, t_1] \to R^n$
  such that
  $\dot{x}(t)$ is defined and
  $\dot{x} = f(t, x(t))$ for all $t \in [t_0, t_1]$.

- If $f(t, x)$ is continuous in t and x,
  then the solution $x(t)$ will be
  continuously differentiable.

- If $f(t, x)$ is continuous in x,
  but only piecewise continuous in t,
  then a solution $x(t)$ could only be
  piecewise continuously differentiable.

Local Existence and Uniqueness - 1 (3.1)

- A ball: $B_r(x_0) = \left\{ x \in R^n \;\middle|\; ||x - x_0|| \le r \right\}$

- Theorem 3.1
  (Local Existence and Uniqueness)
  Let $f(t, x)$ be piecewise continuous in t
  and satisfy the Lipschitz condition

$$||f(t, x) - f(t, y)|| \le L||x - y||$$

  $\forall x, y \in B_r(x_0), \forall t \in [t_0, t_1]$.
  Then, there exists some $\delta > 0$
  such that $\dot{x} = f(t, x)$ with $x(t_0) = x_0$
  has a unique solution over $[t_0, t_0 + \delta]$.

Local Existence and Uniqueness - 2

- Proof:
- First, $x(t)$ satisfies both the following eqns:

$$\dot{x} = f(t, x),\ x(t_0) = x_0 \quad (C.1)$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \quad (C.2)$$

$$\Rightarrow \qquad x(t) = (Tx)(t)$$

So, we will focus on the discussion of the 2nd one.

- View its RHS as a mapping of the continuous function $x : [t_0, t_1] \rightarrow R^n$, Denote it by $(Tx)(t)$, Write it as $x(t) = (Tx)(t)$ Note that $(Tx)(t)$ is continuous in t. A solution of it is a fixed point of the mapping T that maps x into Tx.

Local Existence and Uniqueness - 3

- Existence of a fixed pint can be established by using the contraction mapping theorem. We need to define a Banach space χ and a closed set $S \subset \chi$ such that T maps S into S and is a contraction over S.

- Let $\chi = C[t_0, t_0 + \delta]$, (set of all cont. fun) with norm $||x||_C = \max_{t \in [t_0, t_0+\delta]} ||x(t)||$ and $S = \{x \in \chi \mid ||x - x_0||_C \leq r\}$

- We restrict the choice of δ to satisfy $\delta \leq t_1 - t_0$ so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$.

Local Existence and Uniqueness - 4

- Notice that $||x(t)||$ dentoes a norm on R^n, while $||x||_C$ denotes a norm on χ.

- Also, B is a ball in R^n, while S is a ball in χ.

- By definition, T maps χ into χ.

- To show that T maps S into S, write

$$
\begin{aligned}
(Tx)(t) - x_0 &= \int_{t_0}^{t} f(s, x(s))ds \\
&= \int_{t_0}^{t} [f(s, x(s)) - f(s, x_0) \\
&\quad + f(s, x_0)]ds
\end{aligned}
$$

Local Existence and Uniqueness - 5

- By piecewise continuity of f, we know that $f(t, x_0)$ is bounded on $[t_0, t_1]$. Let $h = \max_{t \in [t_0, t_1]} ||f(t, x_0)||$.

- Using the Lipschitz condition and the fact that for each $x \in S$, $||x(t) - x_0|| \leq r, \forall t \in [t_0, t_0 + \delta]$, we obtain

$$
\begin{aligned}
||(Tx)(t) - x_0|| &\leq \int_{t_0}^{t} \Big[||f(s, x(s)) - f(s, x_0)|| \\
&\quad + ||f(s, x_0)|| \Big] ds \\
&\leq \int_{t_0}^{t} \Big[L||x(s) - x_0|| + h \Big] ds \\
&\leq \int_{t_0}^{t} (Lr + h) ds \\
&= (t - t_0)(Lr + h) \\
&\leq \delta(Lr + h)
\end{aligned}
$$

Local Existence and Uniqueness - 6

- And

$$\begin{aligned} ||Tx - x_0||_C &= \max_{t \in [t_0, t_0+\delta]} ||(Tx)(t) - x_0| \\ &\le \delta(Lr + h) \quad \le r \end{aligned}$$

- Hence, choosing $\delta \le r/(Lr + h)$ ensures that T maps S into S.

- To show that T is a contraction mapping over S:

Local Existence and Uniqueness - 7

- Let $x, y \in S$, consider

$$\begin{aligned} &||(Tx)(t) - (Ty)(t)|| \\ &= \left|\left| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))] ds \right|\right| \\ &\le \int_{t_0}^{t} ||f(s, x(s) - f(s, y(s))|| ds \\ &\le \int_{t_0}^{t} L||x(s) - y(s)|| ds \\ &\le \int_{t_0}^{t} ds L||x - y||_C \end{aligned}$$

- Therefore, for $\delta \le \frac{\rho}{L}$,

$$||Tx - Ty||_C \le L\delta||x - y||_C \le \rho||x - y||_C$$

- Choosing $\rho < 1$ and $\delta \le \rho/L$ ensures that T is a contraction mappint over S.

Local Existence and Uniqueness - 8

- By the contraction mapping theorem,
  we can conclude that
  if δ is chosen to satisfy

  $$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr+h}, \frac{\rho}{L}\right\} \quad \text{for } \rho < 1$$

  then (C.2) will have
  a unique solution in S.

- Our final goad is to establish
  uniqueness of the solution
  among all continuous functions $x(t)$,
  that is, uniqueness in χ.

- It turns out that
  any solution of (C.2) in χ will lie in S.

Local Existence and Uniqueness - 9

- Note that
  since $x(t_0) = x_0$ is inside the ball B,
  any continuous solution $x(t)$ must
  lie inside B for some interval of time.

- Suppose that
  $x(t)$ leaves the ball B and
  let $t_0 + \mu$ be the first time
  $x(t)$ intersects the boundary of B.
  Then, $||x(t_0+\mu) - x_0|| = r$.

- On the other hand, for all $t \leq t_0 + \mu$,

$$\begin{aligned} ||x(t) - x_0|| &\leq \int_{t_0}^{t} \Big[\, ||f(s,x(s)) - f(s,x_0)|| \\ &\quad + \; ||f(s,x_0)|| \,\Big]\, ds \\ &\leq \int_{t_0}^{t} \Big[L||x(s) - x_0|| + h\Big]\, ds \\ &\leq \int_{t_0}^{t} (Lr + h) ds \end{aligned}$$

Local Existence and Uniqueness - 10

- Therefore,

$$r = ||x(t_0 + \mu) - x_0|| \le (Lr + h)\mu$$

$$\Rightarrow \mu \ge \frac{r}{Lr + h} \ge \delta$$

- Hence,
  the solution $x(t)$ cannot leave the set B
  within the time interval $[t_0, t_0 + \delta]$,
  which implies that
  any solution in χ lies in S.

- Consequently,
  uniqueness of the solution in S
  implies uniqueness in χ.

- QED

Lipschitz in x - 1 (3.1)

- A function is Lipschitz in x

- Lipschitz constant: L

- Local Lipschitz:
  A function $f(x)$ is said to be
  local Lipschitz on a domain
  (open and connected set) $D \subset R^n$
  if each point of D has a neighborhood D_0
  such that
  f satisfies the Lipschitz condition (3.2)
  for all points in D_0
  with some Lipschitz constant L_0.

Lipschitz in x - 2

- A local Lipschitz function on a domain D is not necessarily Lipschitz on D, since the Lipschitz condition may not hold uniformly (with the same constant L) for all points in D.

- A local Lipschitz function on a domain D is Lipschitz on every compact (cloesed and bounded) subset of D.

- A function $f(x)$ is said to be globally Lipschitz if it is Lipschitz on R^n.

Lipschitz Property & Continuity - 1 (3.1)

- The following lemma shows how a Lipschitz constant can be calculated using knowledge of $[\partial f/\partial x]$.
- Lemma 3.1
- Let $f : [a,b] \times D \to R^m$ be continuous for some domain $D \subset R^n$.
- Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b] \times D$.
- For a convex subset $W \subset D$, if there is a constant $L \geq 0$ such that

$$||\frac{\partial f}{\partial x}(t,x)|| \leq L$$

on $[a,b] \times W$, then

$$||f(t,x) - f(t,y)|| \leq L||x-y||$$

for all $t \in [a,b], x, y \in W$.

Lipschitz Property & Continuity - 2

- The Lipschitz property is stronger than continuity.

- If $f(x)$ is Lipschitz on W, then it is uniformly continuous on W (Exercise 3.20).
  The converse is not true.

Lipschitz Property & Continuity - 3

- The following lemma shows that the Lipschitz property is weaker than continuous differentiability.

- Lemma 3.2
  If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times D$, for some domain $D \subset R^n$, then f is local Lipschitz in x on $[a,b] \times D$.

- Lemma 3.3
  If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times R^n$, then f is globally Lipschitz in x on $[a,b] \times R^n$ iff $[\partial f/\partial x]$ is uniformly bounded on $[a,b] \times R^n$.

Global Existence and Uniqueness - 1 (3.1)

- Theorem 3.2
  (Global Existence and Uniqueness)
  Suppose that
  $f(t,x)$ is piecewise continuous in t
  and satisfies

  $$||f(t,x) - f(t,y)|| \leq L||x-y||$$

  $\forall x, y \in R^n, \forall t \in [t_0, t_1]$.
  Then, the state equation $\dot{x} = f(t,x)$,
  with $x(t_0) = x_0$,
  has a unique solution over $[t_0, t_1]$.

Global Existence and Uniqueness - 2

- Local Lipschitz property of a function is
  basically a smoothness requirement.
  It is implied by continuous differentiability.
  Except for discontinuous nonlinearities,
  it is reasonable to expect models
  of physical sysems
  to have locally Lipschitz RHS functions.

- Global Lipshitz property is restrictive.

- The following theorem shows that
  global existence and uniqueness only needs
  the local Lipschitz property of f
  at the expense of having to know
  more about the solution of the system.

Global Existence and Uniqueness - 3

- Theorem 3.3
  (Global Existence and Uniqueness)
- Let $f(t,x)$ be piecewise continuous in t
  and local Lipschitz in x
  for all $t \geq t_0$ and all x in a domain $D \subset R^n$.
- Let W be a compact subset of D, $x_0 \in W$,
  and suppose it is known that
  every solution of $\dot{x} = f(t,x), x(t_0) = x_0$
  lies entirely in W.
- Then, there is a unique solution
  that is defined for all $t \geq t_0$.