Existence of Periodic Orbits (2.6)

- Periodic orbits in the plane are special that they divide the plane into a region inside the orbit and a region outside it.
- This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems.
- The most celebrated of these criteria are the Poincaré-Bendixson theorem, the Bendixson criterion, and the index method.

Poincaré-Bendixson Theorem - 1

- Theorem (Poincaré-Bendixson):

  Let γ^+ be a bounded positive semiorbit of $\dot{x} = f(x)$, i.e., $\gamma^+(y) = \{\phi(t, y) \mid 0 \leq t < \infty\}$ and L^+ be its positive limit set.

  If L^+ contains no e.p., then it is a periodic orbit.

Poincaré-Bendixson Theorem - 2

- Lemma 2.1, Presence of Limit Cycles (Poincaré-Bendixson Criterion):
  Consider $\dot{x} = f(x)$ and
  let M be a closed bounded subset
  of the plane,
  such that

  - M contains no e.p.,
    or contains only one e.p.
    such that the Jcaobian matrix $[\partial f / \partial x]$
    at this point has eigenvalues
    with positive real parts.
    (Hence, the e.p. is unstable focus or node.)
  - Every trajectory starting in M
    stays in M for all future time.

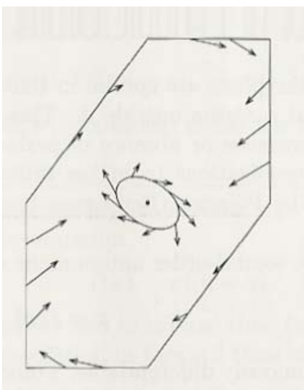
  Then, M contains a periodic orbit
  of $\dot{x} = f(x)$.

Poincaré-Bendixson Theorem - 3

- Intuition:
  Bounded trajectories in the plane
  will have to approach
  periodic orbits or equilibrium points
  as time tends to infinity.

- If M contains no e.p.,
  then it must contain a periodic orbit.

- If M contains only one e.p.
  that satisfies the stated conditions,
  then in the vicinity of that point
  all trajecotries will be moving away from it.



- Therefore, we can choose
  a simple closed curve around the e.p.
  such that the vector field on the curve
  points outward.

Geometric Interpretation - 1

- Consider a simple closed curve
  defined by $V(x) = c$,
  where $V(x)$ is continuously differentiable.

- The vector field $f(x)$
  at a point x on the curve
  points inward
  if the inner product of $f(x)$ and
  the gradient vector $\nabla V(x)$ is negative;
  that is,

$$f(x) \cdot \nabla V(x) = \frac{\partial V}{\partial x_1}(x) f_1(x) + \frac{\partial V}{\partial x_2}(x) f_2(x) < 0$$

- The vector field $f(x)$ points outward
  if $f(x) \cdot \nabla V(x) > 0$.

- It is tangent to the curve
  if $f(x) \cdot \nabla V(x) = 0$.

Geometric Interpretation - 2

- Trajectories can leave a set
  only if the vector field points outward
  at some points on its boundary.

- For a set of the form $M = \{V(x) \leq c\}$,
  for some $c > 0$,
  trajectories are trapped inside M
  if $f(x) \cdot \nabla V(x) \leq 0$
  on the boundary $V(x) = c$.

- For annular region of the form
  $M = \{W(x) \geq c_1 \text{ and } V(x) \leq c_2\}$,
  for some $c_1 > 0, c_2 > 0$
  trajectories are trapped inside M
  if $f(x) \cdot \nabla V(x) \leq 0$ on $V(x) = c_2$
  and $f(x) \cdot \nabla W(x) \geq 0$ on $W(x) = c_1$.

Harmonic Oscillator - 1

- **Example 2.7**:
- Consider the harmonic oscillator:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

the annual region $M = \{c_1 \leq V(x) \leq c_2\}$,
where $V(x) = x_1^2 + x_2^2$ and $c_2 > c_1 > 0$.

- The set M is closed, bounded, and free of e.p., since the only e.p. is at (0,0).

- Trajectories are trapped inside M since $f(x) \cdot \nabla V(x) = 0$ everywhere.

- By PBC, there is a periodic orbit in M.

Harmonic Oscillator - 2

- **Example 2.7**:
- PBC assures the existence of a periodic orbit, but not its uniqueness.

- Harmonic oscillator has a continuum of periodic orbits in M.

Example 2.8 - 1

- **Example 2.8**: The system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

has a unique e.p. at (0,0).

- the Jacobian matrix:

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1x_2 \\ -2 - 2x_1x_2 & 1 - x_1^2 - 3x_2^2 \end{bmatrix}_{x=0}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

has eigenvalues $1 \pm j\sqrt{2}$.

Example 2.8 - 2

- Let $M = \{V(x) \leq c\}$,
  where $V(x) = x_1^2 + x_2^2$ and $c > 0$.

- M is closed, bounded, and
  contains only one e.p. at which
  the Jacobian matrix has eigenvalues with
  positive real parts.

- On the surface $V(x) = c$, we have:

$$\begin{aligned} & \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 \\ = \; & 2x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)] \\ & +2x_2[-2x_1 + x_2 - x_2(x_1^2 + x_2^2)] \\ = \; & 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1x_2 \\ \leq \; & 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) \\ = \; & 3c - 2c^2 \end{aligned}$$

Example 2.8 - 3

- where we used the fact that
  $|2x_1x_2| \leq x_1^2 + x_2^2$.

- By choosing $c \geq 1.5$,
  we can ensure that all trajectories are
  trapped inside M.

- Hence, by PBC,
  there is a periodic orbit in M.

Negative-Resistance Oscillator - 1

- **Example 2.9**:
  The negative-resistance oscillator:

$$\ddot{v} + \epsilon h'(v)\dot{v} + v = 0$$

  where ϵ is a positive constant
  h satisfies the conditions:

$$h(0) = 0, h'(0) < 0,$$
$$\lim_{v \to \infty} h(v) = \infty, \lim_{v \to -\infty} h(v) = -\infty$$

- To simplify the analysis,
  we impose the additional requirements:

$$h(v) = -h(-v),$$
$$h(v) < 0 \text{ for } 0 < v < a,$$
$$h(v) > 0 \text{ for } v > a$$

Negative-Resistance Oscillator - 2

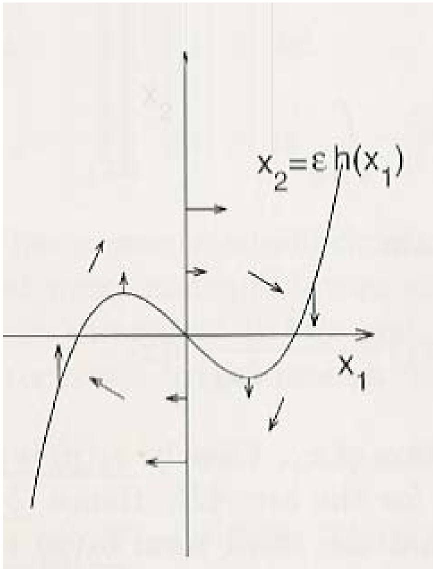
- Choose the state variables as:

$$x_1 = v, \text{ and } x_2 = \dot{v} + \epsilon h(v)$$

- The state model as:

$$\begin{aligned} \dot{x}_1 &= x_2 - \epsilon h(x_1) \\ \dot{x}_2 &= -x_1 \end{aligned}$$

which has a unique e.p. at the origin.



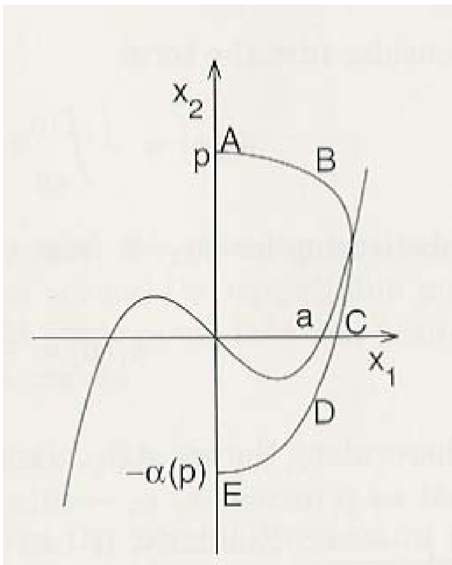
- First, by looking at the vector field, we can show that every nonequlilibrium solution rotates around the e.p. in the clockwise direction.

- Divide the state plane into four regions by the intersection of

$$x_2 - \epsilon h(x_1) = 0 \text{ and } x_1 = 0$$

Negative-Resistance Oscillator - 3

- It is not difficulty to see a solution from $A = (0, p)$ to $E = (0, -\alpha(p))$.



- We can show that if p is chosen large enough, then $\alpha(p) < p$.

- Consider the function:

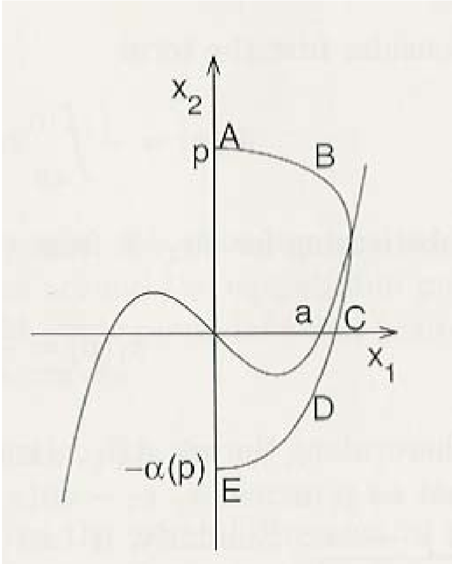
$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

- To show that $\alpha(p) < p$, it is enough to show that $V(E) - V(A) < 0$, since $V(E) - V(A) = \frac{1}{2}[\alpha^2(p) - p^2] := \delta(p)$

Negative-Resistance Oscillator - 4

- The derivative of $V(x)$ is given by

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 - \epsilon x_1 h(x_1) - x_1x_2 \\ &= -\epsilon x_1 h(x_1) \end{aligned}$$

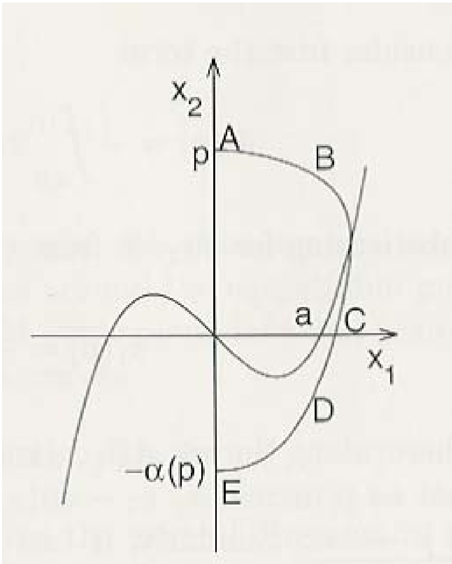


- Thus, $\dot{V}$ is positive for $x_1 < a$ and negative for $x_1 > a$.

Negative-Resistance Oscillator - 5

- Now,

$$\delta(p) = V(E) - V(A) = \int_{AE} \dot{V}(x(t))dt$$



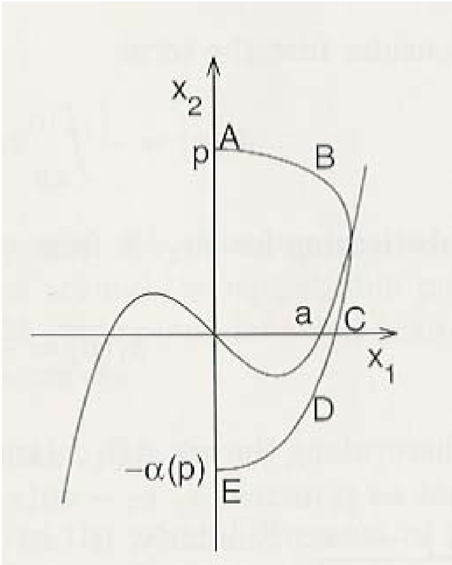
- If p is small, the whole arc will lie inside the strip $0 < x_1 < a$. Then, $\delta(p)$ will be positive.

- As p increases, a piece of the arc will lie outside the strip (BCD).

Negative-Resistance Oscillator - 6

- Divide the integral into three parts:

$$\delta(p) = \delta_1(p) + \delta_2(p) + \delta_3(p)$$
$$\delta_1(p) = \int_{AB} \dot{V}(x(t))dt$$
$$\delta_2(p) = \int_{BCD} \dot{V}(x(t))dt$$
$$\delta_3(p) = \int_{DE} \dot{V}(x(t))dt$$



Negative-Resistance Oscillator - 7

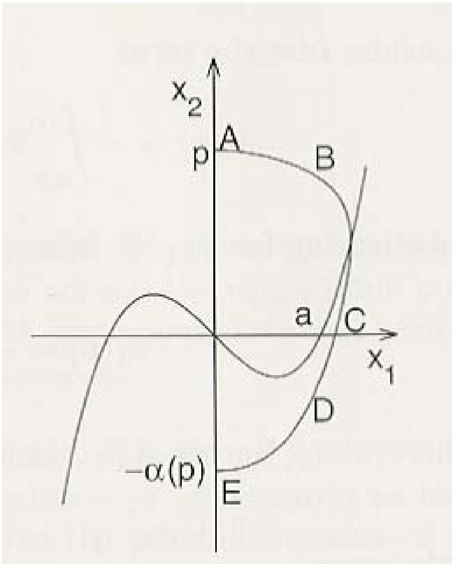
- Consider the first term:

$$\delta_1(p) = -\int_{AB} \epsilon x_1 h(x_1) dt = -\int_{AB} \epsilon x_1 h(x_1) \frac{dt}{dx_1} dx_1$$

- Substituting for dx_1/dt
  from the state model, we obtain:

$$\delta_1(p) = -\int_{AB} \epsilon x_1 h(x_1) \frac{1}{x_2 - \epsilon h(x_1)} dx_1$$

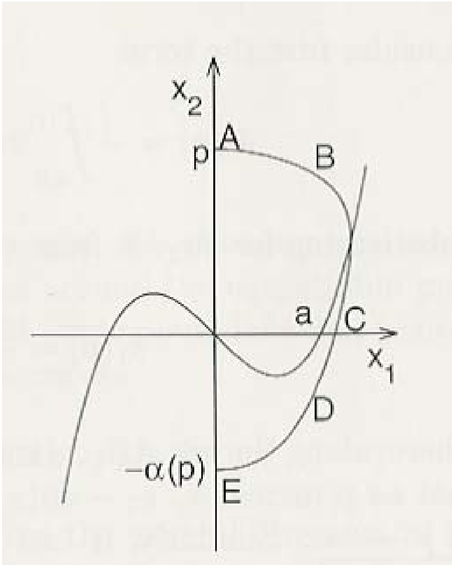
  where along the arc AB,
  x_2 is a given function of x_1.



- Clearly, $\delta_1(p)$ is positive.

Negative-Resistance Oscillator - 8

- As p increases,
  $x_2 - \epsilon h(x_1)$ increases for the acr AB.



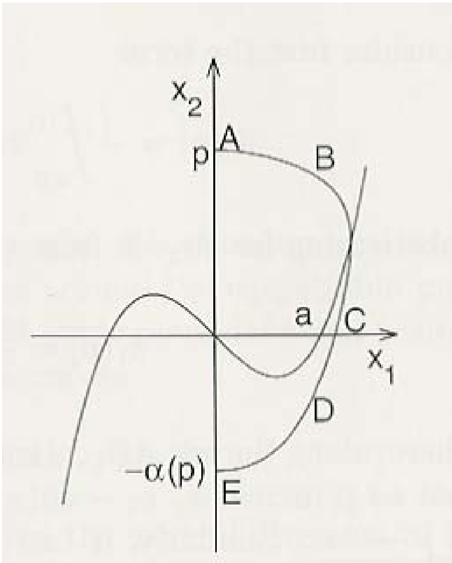
- Hence, $\delta_1(p)$ decreases as $p \to \infty$.

- Similarly, $\delta_3(p)$ is positive and decreases as $p \to \infty$.

Negative-Resistance Oscillator - 9

- Consider the second term:

$$\delta_2(p) = -\int_{BCD} \epsilon x_1 h(x_1) dt = -\int_{BCD} \epsilon x_1 h(x_1) \frac{dt}{dx_2} dx_2$$



- Substituting for dx_2/dt from the state model, we obtain:

$$\delta_2(p) = \int_{BCD} \epsilon h(x_1) dx_2$$

  where along the arc BCD, x_1 is a given function of x_2.

- Since $h(x_1) > 0$ and $dx_2 < 0$, the integral is negative.

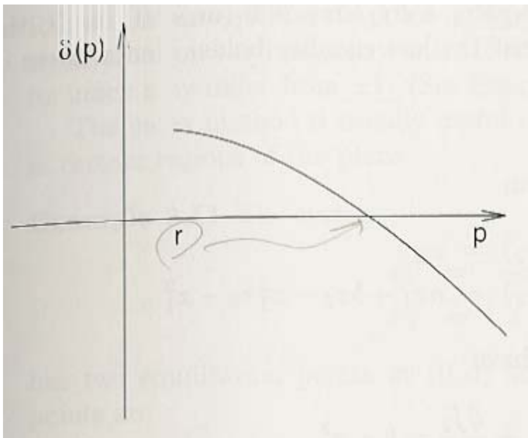
Negative-Resistance Oscillator - 10

- As p increases,
  the acr $ABCDE$ moves to the right and the domain of integration for $\delta_2(p)$ increases.

- $\delta_2(p)$ decreases, as p increases and evidently $\lim_{p\to\infty} \delta_2(p) = -\infty$.

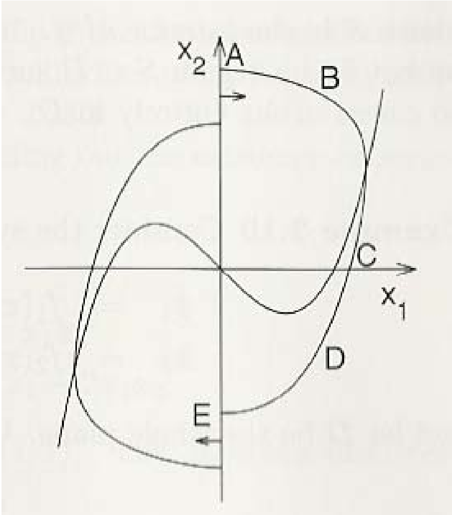
- In summary:
  - $\delta(p) > 0$, if $p < r$, for some $r > 0$.
  - $\delta(p)$ decreases monotonically to $-\infty$ as $p \to \infty, p \geq r$

- From the Fig.
  by choosing p large enough,
  $\delta(p)$ is negative,
  hence, $\alpha(p) < p$.



Negative-Resistance Oscillator - 11

- Since $h(\cdot)$ is an odd function,
  due to its symmetry,
  if (x_1, x_2) is a solution,
  then so is $(-x_1, -x_2)$. See Fig.



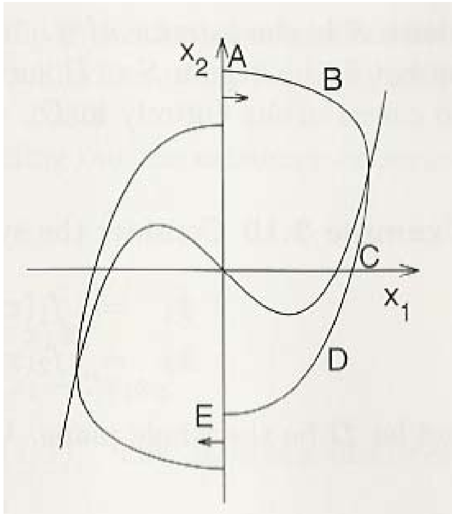
- Let M be the region enclosed by this closed curve.

- Then very trajectory starting in M at $t = 0$ will remain inside for all $t \geq 0$.

- Because
  (1) the directions of the vector fields on the x_2-axis segments and
  (2) uniqueness of solutions
  (trajectories do not intersect each other).

Negative-Resistance Oscillator - 12

- M is closed, bounded, and has a unique e.p. at the origin.



- The Jacobian matrix at the origin:

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\epsilon h'(0) \end{bmatrix}$$

has eigenvalues wiht positive real parts since $h'(0) < 0$.
By PBC, there is a closed orbit in M.

- This closed orbit is unique iff $\alpha(p) = p$.
Only one value of p, see Fig.

- Every nonequilibrium solution spirals toward the unique closed orbit.

Bendixson Criterion - 1

- To rule out the existence of periodic orbits:

- Lemma 2.2, Absence of Limit Cycles (Bendixson Criterion)
If, on a simply connected region D of the plane,
the express $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$
is not identically zero and
does not change sign,
then $\dot{x} = f(x)$ has no periodic orbits
lying entirely in D.

Bendixson Criterion - 2

- Proof:
- On any orbit of $\dot{x} = f(x)$,
  we have $dx_2/dx_1 = f_2/f_1$.
  Therefore, on any closed orbit γ, we have
  $$\int_\gamma f_2(x_1,x_2)dx_1 - f_1(x_1,x_2)dx_2 = 0$$
  This implies, by Green's theorem, that
  $$\int\int_S (\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2})dx_1 dx_2 = 0$$
  where S is the interior of γ.

- If $\partial f_1/\partial x_1 + \partial f_2/\partial x_2 > 0$ or (< 0) on D,
  then we cannot find a region $S \subset D$
  such that the last equality holds.
  Hence, there can be no closed orbits entirely in D.

Bendixson Criterion - 3

- **Example 2.10**: Consider the system:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,x_2) = x_2 \\ \dot{x}_2 &= f_2(x_1,x_2) = ax_1 + bx_2 - x_1^2 x_2 - x_1^3\end{aligned}$$

  and let D be the whole plane.

- We have
  $$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = b - x_1^2$$
  Hence, there can be no periodic orbits
  if $b < 0$.

Poincaré Index of An E.P. - 1

- Consider $\dot{x} = f(x)$
- let C be a simple closed curve
  not passing through any of its E.P.
- Consider the orientation of the vector field
  $f(x)$ at a point $p \in C$.
- Letting p traverse C
  in the counterclockwise direction,
  the vector $f(x)$ rotates continuously and,
  upon returning to the original position,
  must have rotated an angle $2\pi k$
  for some integer k,
  where the angle is measured
  counterclockwise.

Poincaré Index of An E.P. - 2

- The integer k is called
  the index of the closed curve C.
- If C is chosen
  to encircle a single isolated E.P. $\bar{x}$,
  then k is called the index of $\bar{x}$.

Poincaré Index of An E.P. - 3

- Lemma 2.3:

(a) The index of
a node, a focus, or a center is $+1$.

(b) The index of
a (hyberbolic) saddle is -1.

(c) The index of
a closed orbit is $+1$.

(d) The index of
a closed curve not encircling any e.p.
is 0.

(e) The index of a closed curve
is equal to the sum of the indices
of the e.p. within it.

Poincaré Index of An E.P. - 4

- Colollary 2.1:

Inside any periodic orbit γ,

there must be at least one e.p.

Suppose the e.ps. inside γ are hyperbolic,

then if N is the number of nodes and foci

and S is the number of saddles,

it must be that $N - S = 1$.

Poincaré Index of An E.P. - 5

- An e.p. is hyperbolic

  if the Jacobian at that point has

  no eigenvalues on the imaginary axis.

- If the e.p. is not hyperbolic,

  then its index may differ from ± 1.

- The index method is usually useful

  in ruling out the existence of periodic orbits

  in certain regions of the plane.

Poincaré Index of An E.P. - 6

- **Example 2.11**: The system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2\end{aligned}$$

has two e.p. at (0,0) and (1,1).

The Jacobian matrices at these points are

$$\left[\frac{\partial f}{\partial x}\right]_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix};$$
$$\left[\frac{\partial f}{\partial x}\right]_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Hence, (0,0) is a saddle,

while (1,1) is a stable focus.

The only combination of e.p.

that can be encircled by a periodic orbit

is a single focus.