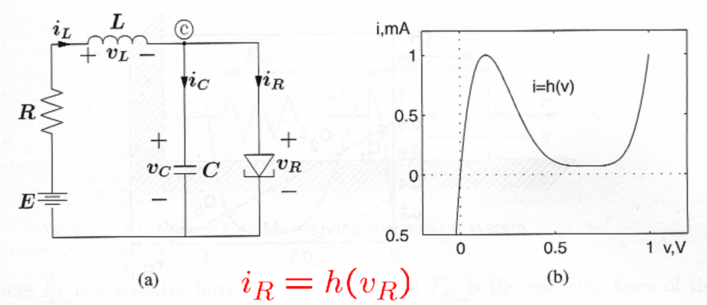
Multiple Equilibria (2.3)

- For linear systems,
  - det $A \neq 0$ (A has no zero eigenvalues), $\dot{x} = Ax$ has an isolated equilibrium point at $x = 0$.
  - det $A = 0$, the system has a continuum of equilibrium points.
  - There are the only possible patterns.
- For nonlinear systems,
  - it can have multiple isolated equilibrium points.
- the tunnel-diode circuit
- the pendulum euqation

Tunnel-Diode Circuit



$i_R = h(v_R)$

Figure 1.2: (a) Tunnel-diode circuit; (b) Tunnel-diode v_R–i_R characteristic.

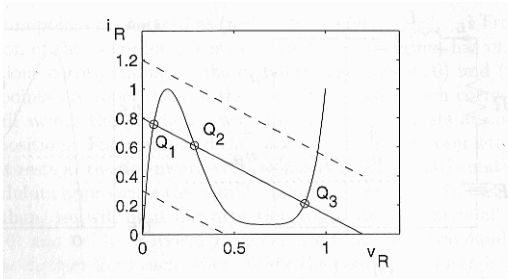


Figure 1.3: Equilibrium points of the tunnel-diode circuit.

Kirchhoff's current/voltage law:

$$\begin{aligned} i_C + i_R - i_L &= 0 \quad \text{(KCL)} \\ v_C - E + Ri_L + v_L &= 0 \quad \text{(KVL)} \end{aligned}$$

State model:

- state: $x_1 = v_C, x_2 = i_L$, and
- input: $u = E$,
- $i_C = C\frac{dv_C}{dt}$, $v_L = L\frac{di_L}{dt}$

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u] \end{aligned}$$

Equilibrium points:

$$\begin{aligned} 0 &= -h(x_1) + x_2 \\ 0 &= -x_1 - Rx_2 + u \end{aligned}$$

That is, the roots of:

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$

Tunnel-Diode Circuit - 1

- **Example 2.1**:
  State Model:

$$\begin{aligned}\dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u]\end{aligned}$$

- Assume that the circuit parameters are:

$$u = 1.2V, R = 1.5k\Omega, C = 2pF, L = 5\mu H$$

- time t in nanoseconds
  $x_2, h(x_1)$ in mA

- State Model:

$$\begin{aligned}\dot{x}_1 &= 0.5[-h(x_1) + x_2] \\ \dot{x}_2 &= 0.2[-x_1 - 1.5x_2 + 1.2]\end{aligned}$$

and

$$\begin{aligned}h(x_1) &= 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 \\ &\quad -226.31x_1^4 + 83.72x_1^5\end{aligned}$$

- Equilibrium Points: (let $\dot{x}_1 = \dot{x}_2 = 0$)
  (0.063,0.758), (0.285,0.61),(0.884,0.21)

Tunnel-Diode Circuit - 2

- **Example 2.3**:
  Tunnel-Diode Circuit:
  The Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

- Evaluated at E.P. Q_1, Q_2, Q_3:

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (-3.57, -0.33)$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (1.77, -0.25)$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad (-1.33, -0.4)$$

$$V_1 = \begin{bmatrix} -.99 & 0.15 \\ -0.06 & -0.99 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.99 & -0.23 \\ -0.09 & 0.97 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} -0.98 & -0.43 \\ -0.19 & -0.89 \end{bmatrix},$$

Tunnel-Diode Circuit - 3

- Q_1 is a stable node

  Q_2 is a sddle point

  Q_3 is a stable node

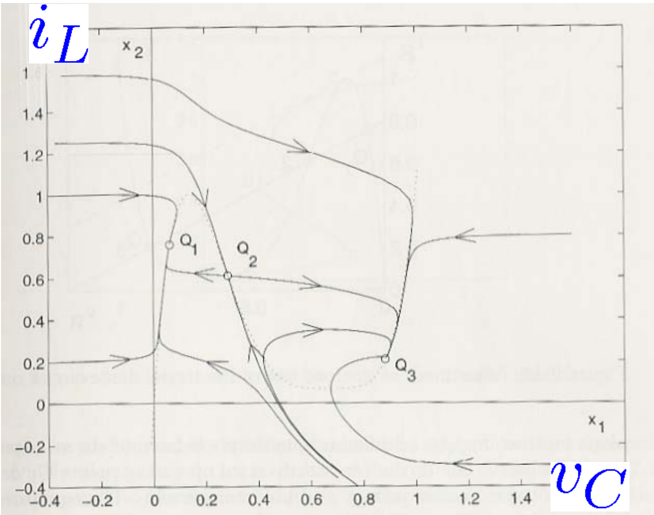
- The two special trajectories,

  which approach Q_2,

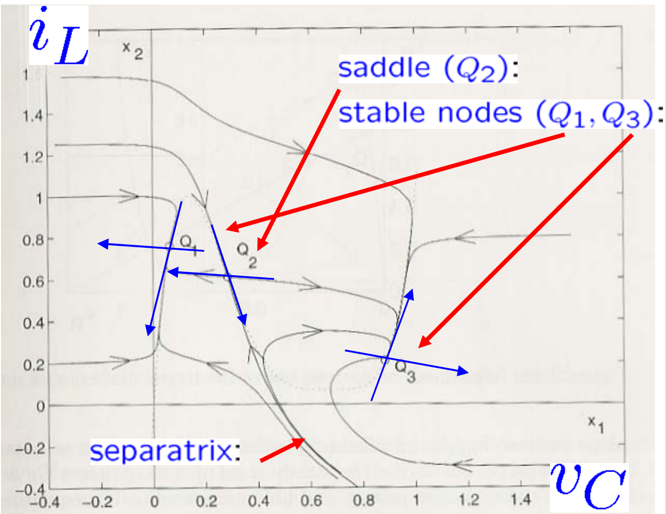
  are the stable trajectories of the saddle.

  They form a curve that divides the plane into two halves.

  Which is called a separatrix.



Tunnel-Diode Circuit - 4



Tunnel-Diode Circuit - 5

- The separatrix partitions the plane into two regions of different qualitative behavior.

- In an experimental setup, we shall observe one of the two steady-state operating points Q_1 or Q_3, depending on the initial capacitor voltage and inductor current.

- The equilibrium point at Q_2 is never observed in practice because the ever-present physical noise would cause the trajectories to diverge from Q_2 even if it were possible to set up the exact initial conditions corresponding to Q_2.

Tunnel-Diode Circuit - 6

- The tunnel-diode circuit is refered as a bistable circuit, because it has two steady-state operating points.

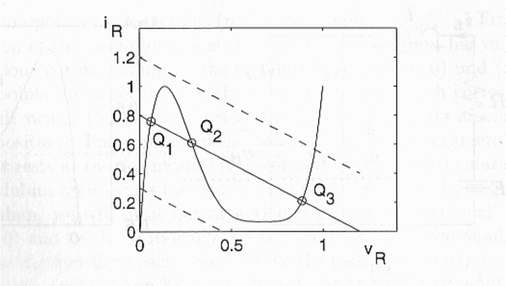
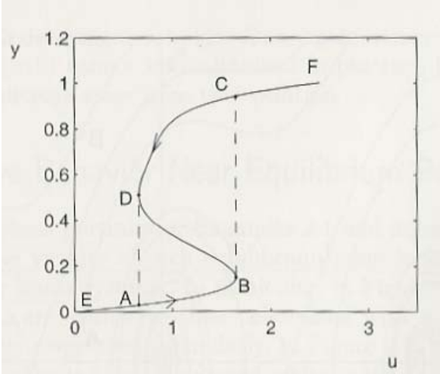


Figure 1.3: Equilibrium points of the tunnel-diode circuit.

- Used in computer memory,
  $Q_1 \rightarrow'' 0''$
  $Q_3 \rightarrow'' 1''$

- Triggering from Q_1 to Q_3 or vice versa is achieved by a triggering signal of sufficiently amplitude and duration that allows the trajectory to move to the other side of the separatrix.

- Hysteresis characteristics:



Pendulum

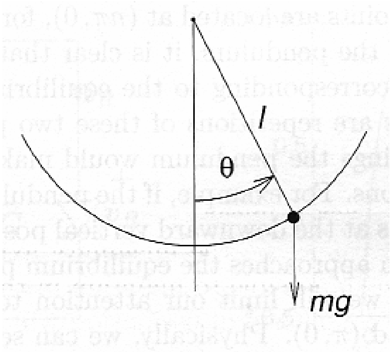
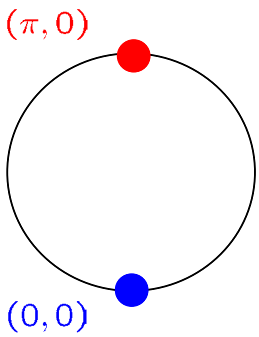


Figure 1.1: Pendulum.



Using Newton's Second Law,
Write the equation of motion
in the tangential direction:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

State model (let $x_1 = \theta, x_2 = \dot{\theta}$):

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

Equilibrium points (let $\dot{x}_1 = \dot{x}_2 = 0$):

$$\begin{aligned}0 &= x_2 \\ 0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

Equilibrium points are $(n\pi, 0), n = 0, \pm 1, \pm 2, ...,$
or, physically, $(0, 0)$ and $(\pi, 0)$.

Question? Which one is stable or unstable?

Pendulum Equation w/ Friction - 1

- **Example 2.2**:
  State model:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10\sin x_1 - x_2\end{aligned}$$

- (0,0): or $(0, 0), (2\pi, 0), (-2\pi, 0)$, etc.
  a stable focus.

- (π,0): or $(\pi, 0), (-\pi, 0)$, etc.
  a saddle.

- This picture is repeated periodically.

- Trajectories approach different E.P.,
  corresponding to # of full swings.

Pendulum Equation w/ Friction - 2

- A and B have the same initial position,
  but different speeds.
  So, different initial conditions.

- A oscillates with decaying amplitude.
  B has more initial kinectic energy.
  B makes a full swing
  before to oscillate with decaying amplitude.

- The unstable E.P. $(\pi, 0)$
  cannot be maintained in practice.

Pendulum Equation w/ Friction - 3

- **Example 2.4**:
  Pendulum Equation:
  The Jacobian matrix:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}$$

- Evaluated at E.P. $(0,0), (\pi, 0)$:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad (-0.5 \pm j3.12)$$

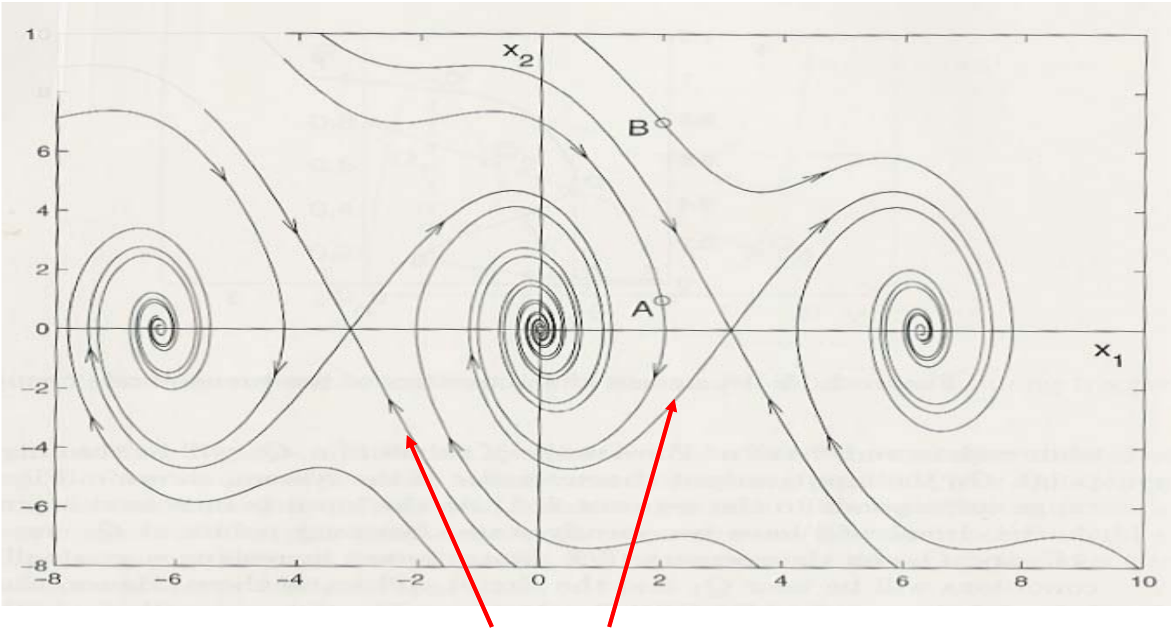
$$A_2 = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \quad (-3.7, 2.7)$$

$$V_1 = \begin{bmatrix} -0.05 - j0.30 & -0.05 + j0.30 \\ 0.95 & 0.95 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} -0.35 & -0.26 \\ 0.94 & -0.97 \end{bmatrix},$$

- $(0,0)$ is a stable focus
  $(\pi, 0)$ is a sddle point

Pendulum Equation w/ Friction - 4



separatrices:

Qualitative Behavior Near E.P. - 1

- Phase portraits of Tunnel-Diode Circuit and Pendulum Equation shows that the qualitative behavior in the vicinity of each E.P. looks just like those for linear systems.

- Tunnel-Diode circuit:
  The trajectories near Q_1, Q_2, Q_3 are similar to those associated with a stable node, saddle point, and stable node, respectively.

- Pendulum:
  The trajectories near $(0, 0), (\pi, 0)$ are similar to those associated with a stable focus and saddle point, respectively.

Qualitative Behavior Near E.P. - 2

- In this section,
  we analyzed the behavior near the E.P.
  w/o drawing the phase portrait.

- Except for some special cases,
  the qualitative behavior of a nonlinear system near an E.P. can be determined via linearization with respect to that point.

A Center - 1

- **Example 2.5**:
  A case of E.P. is a Center:
  The system:

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= \quad x_1 - \mu x_2(x_1^2 + x_2^2)\end{aligned}$$

  has an E.P. at the origin.
  The linearized state equation at the origin has eigenvalues $\pm j$.
  A center E.P.

- The qualitative behavior of the nonlinear system:

$$\begin{aligned}x_1 &= r\cos\theta & \dot{r} &= -\mu r^3\\ x_2 &= r\sin\theta & \dot{\theta} &= 1\end{aligned}$$

  - a stable focus when $\mu > 0$
  - an unstable focus when $\mu < 0$

- A system oscillates

  when it has a nontrivial periodic solution:

  $$x(t+T) = x(t), \forall t \leq 0, \text{ for some } T > 0$$

- The image of a periodic solution

  in the phase portrait

  is a closed trajectory,

  which is usually called

  a periodic orbit or a closed orbit.

- In 2nd-order linear system: Oscillation

  - with eigenvalues $\pm j\beta$,

  - $x = 0$ is a center,

  - the solution:

  $$z_1(t) = r_0 \cos(\beta t + \theta_0),$$

  $$z_2(t) = r_0 \sin(\beta t + \theta_0),$$

  where

  $$r_0 = \sqrt{z_1^2(0) + z_2^2(0)},$$

  $$\theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right],$$

  - the harmonic oscillator

Limit Cycles: Harmonic Oscillator - 3

- Two fundamental problems
  with the linear oscillator:
  1. robustness:
     perturbation will destroy the oscillation.
     the linear oscillator is
     not structurally stable.
  2. the amplitude of oscillation is
     dependent on the initial conditions.

- It is possible to build
  physical nonlinear oscillators
  such that
  1. the nonlinear oscillator is
     structurally stable.
  2. the amplitude of oscillation
     (at steady state)
     is independent of initial conditions.

Limit Cycles: Energy Approach - 1

- The negative-resistance oscillator:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \epsilon h'(x_1)x_2\end{aligned}$$

  the system has only one E.P.
  at $x_1 = x_2 = 0$.

- The Jacobian matrix:

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\epsilon h'(0) \end{bmatrix}$$

- Since $h'(0) < 0$, the origin is
  either an unstable node
  or unstable focus,
  depending on the value of $\epsilon h'(0)$.

Limit Cycles: Energy Approach - 2

- All trajectories starting near the origin would diverge away from it and head toward infinity.

- The resistive element is "active", and supplies energy.

- The total energy stored in the capacitor and inductor at any time t is given by:

$$E = \frac{1}{2}Cv_C^2 + \frac{1}{2}Li_L^2$$

where $\quad v_C = x_1, i_L = -h(x_1) - \frac{1}{\epsilon}x_2,$

$$\epsilon = \sqrt{L/C}$$

Limit Cycles: Energy Approach - 3

- Rewrite the energy expression as:

$$E = \frac{1}{2}C\left\{x_1^2 + \left[\epsilon h(x_1) + x_2\right]^2\right\}$$

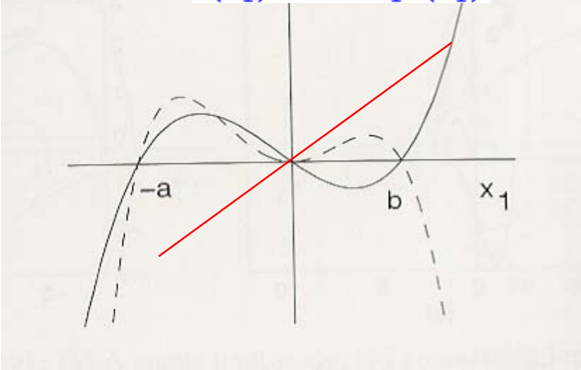
- The rate of change of energy is given by:

$$\begin{aligned}
\dot{E} &= C\Big\{x_1\dot{x}_1 + [\epsilon h(x_1) + x_2] \\
&\qquad [\epsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\Big\} \\
&= C\Big\{x_1 x_2 + [\epsilon h(x_1) + x_2] \\
&\qquad [\epsilon h'(x_1)x_2 - x_1 - \epsilon h'(x_1)x_2]\Big\} \\
&= C\Big[x_1 x_2 - \epsilon x_1 h(x_1) - x_1 x_2\Big] \\
&= -\epsilon C x_1 h(x_1)
\end{aligned}$$

Limit Cycles: Closed Orbit

- Near the origin, the trajectory gains energy since for small $|x_1|$, $x_1 h(x_1)$ is negative.

$h(x_1)$ and $-x_1 h(x_1)$:



- Also, the trajectory gains energy within the strip $-a < x_1 \leq b$, and loses energy outside the strip.

- A stationary oscillation will occur if, along a trajectory, the net exchange of energy over one cycle is zero.

- Such a trajectory will be a closed orbit. The negative-resistance oscillator has an isolated closed orbit.

Limit Cycles: Van der Pol Equation - 1

- **Example 2.6** Van der Pol equation:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

- For $\epsilon = 0.2, 1.0, 5.0$ are shown in Figs.

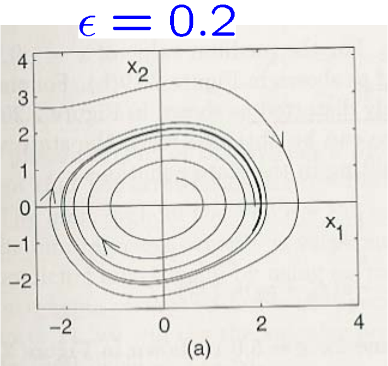
- There is a unique closed orbit that attracts all trajectories starting off the orbit.

- The closed orbit of $\epsilon = 0.2$ (small value) is a smooth orbit that is closed to a circle of radius 2.

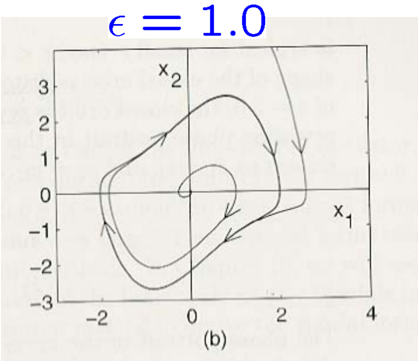
- For medium value of ϵ (=1.0), the circular shape of the closed orbit is distorted.

- For large value of ϵ (=5.0), the closed orbit is severely distorted.

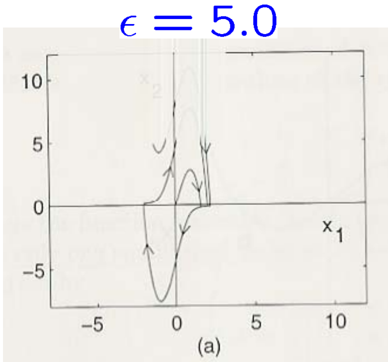
$\epsilon = 0.2$



$\epsilon = 1.0$



$\epsilon = 5.0$



Stable & Unstable Limit Cycles

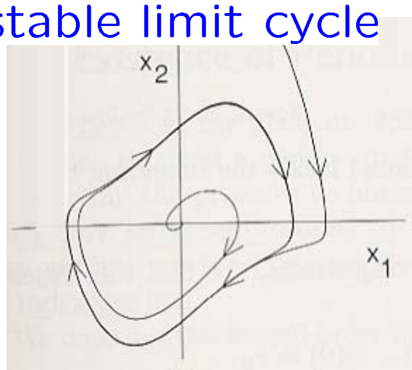
- In the case of harmonic oscillator, there is a continuum of closed orbit.
- In the Van der Pol example, there is only one isolated periodic orbit.
- An isolated periodic or bit is called a limit cycle.
- stable and unstable limit cycles:

stable:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

unstable:
$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = x_1 - \epsilon(1 - x_1^2)x_2$$

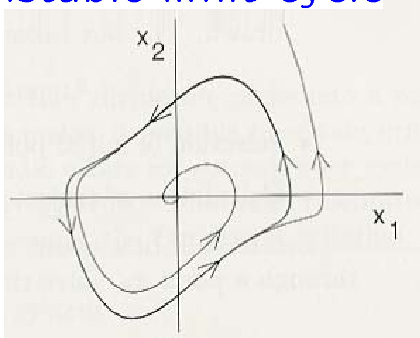
- Two special forms:
  - $\epsilon \to 0$: the averaging method
  - $\epsilon \to \infty$: the singular perturbation method

stable limit cycle



(a)

unstable limit cycle



(b)

Existence of Periodic Orbits (2.6)

- Periodic orbits in the plane are special that they divide the plane into a region inside the orbit and a region outside it.
- This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems.
- The most celebrated of these criteria are the Poincaré-Bendixson theorem, the Bendixson criterion, and the index method.

Poincaré-Bendixson Theorem - 1

- Theorem (Poincaré-Bendixson):

  Let γ^+ be a bounded positive semiorbit of

  $\dot{x} = f(x)$, i.e., $\gamma^+(y) = \{\phi(t, y) \mid 0 \leq t < \infty\}$

  and L^+ be its positive limit set.

  If L^+ contains no e.p.,

  then it is a periodic orbit.

Poincaré-Bendixson Theorem - 2

- Lemma 2.1, Presence of Limit Cycles (Poincaré-Bendixson Criterion):

  Consider $\dot{x} = f(x)$ and let M be a closed bounded subset of the plane, such that

  - M contains no e.p., or contains only one e.p. such that the Jcaobian matrix $[\partial f / \partial x]$ at this point has eigenvalues with positive real parts. (Hence, the e.p. is unstable focus or node.)

  - Every trajectory starting in M stays in M for all future time.

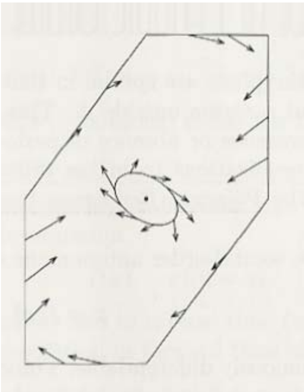
  Then, M contains a periodic orbit of $\dot{x} = f(x)$.

Poincaré-Bendixson Theorem - 3

- Intuition:
  Bounded trajectories in the plane
  will have to approach
  periodic orbits or equilibrium points
  as time tends to infinity.

- If M contains no e.p.,
  then it must contain a periodic orbit.

- If M contains only one e.p.
  that satisfies the stated conditions,
  then in the vicinity of that point
  all trajecotries will be moving away from it.



- Therefore, we can choose
  a simple closed curve around the e.p.
  such that the vector field on the curve
  points outward.