Nonlinear Systems Analysis

Lecture 3

Sections 2.1, 2.3, 2.2

Qualitative Behaviors & Perturbed System Analysis & Multiple Equilibria

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Sep04 – Jan05

Outline

- Nonlinear State Model
- Second-Order Systems
- Qualitative Behavior of Linear Systems
  - Characteristics of eigenvalues
- Qualitative Behavior Near Equilibrium Points
  - Linearization, Jacobian Matrix
- Multiple Equilibria
  - Tunnel-diode circuit, pendulum
- Perturbed Linear Systems

Nonlinear State Models - 1

- In this course,
  we consider dynamical systems
  modeled by a finite number of coupled first-order ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, ..., x_n, u_1, ..., u_p) \\ \dot{x}_2 &= f_2(t, x_1, ..., x_n, u_1, ..., u_p) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, ..., x_n, u_1, ..., u_p) \end{aligned}$$

and

$$\begin{aligned} y_1 &= h_1(t, x_1, ..., x_n, u_1, ..., u_p) \\ y_2 &= h_2(t, x_1, ..., x_n, u_1, ..., u_p) \\ &\vdots \\ y_q &= h_q(t, x_1, ..., x_n, u_1, ..., u_p) \end{aligned}$$

- Or, let state, input, output be:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, y = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}, f, h,$$

Nonlinear State Models - 2

- Unforced state equation:

$$\dot{x} = f(t, x)$$

  - $u = 0$,
  - $u = \gamma(t)$: a given function of time,
  - $u = \gamma(x)$: a given function of the state
  - $u = \gamma(t, x)$
- Autonomous or time invariant systems:

$$\dot{x} = f(x)$$

- Nonautonomous or time varying systems:
- Equilibrium points:
  A point $x = x^*$ in the state space
  is said to be an equilibrium point
  if it has the property that
  whenever the state of system starts at x^*,
  it will remain at x^* for all future time.
- For autonomous systems,
  the equilibrium points are
  the roots of the equation $f(x) = 0$.
  - isolated equilibrium points,
  - continuum of equilibrium points

Second-Order Systems - 1

- In Chapter 2, we first study second-order autonomous systems.

- Solutions of second-order systems
  → easy visualization in 2-D plane

- Key points:
  1. The behavior of a nonlinear system near equilibrium points;
  2. The phenomenon of nonlinear oscillation;
  3. Bifurcation.

Second-Order Systems - 2

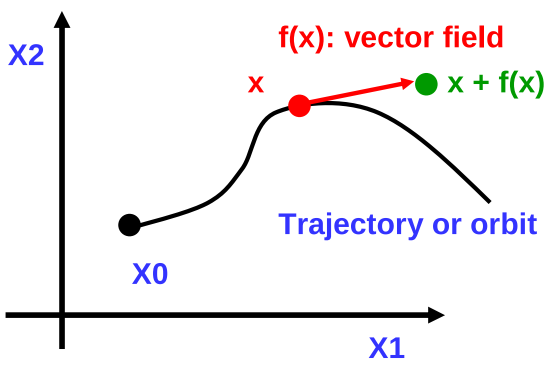
- State model of a 2nd-order autonomous system:

$$\dot{x}_1 = f_1(x_1, x_2) \qquad \text{or} \qquad \dot{x} = f(x)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

- Initial state, solution:

$$x(0) = x_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

State plane or phase plane



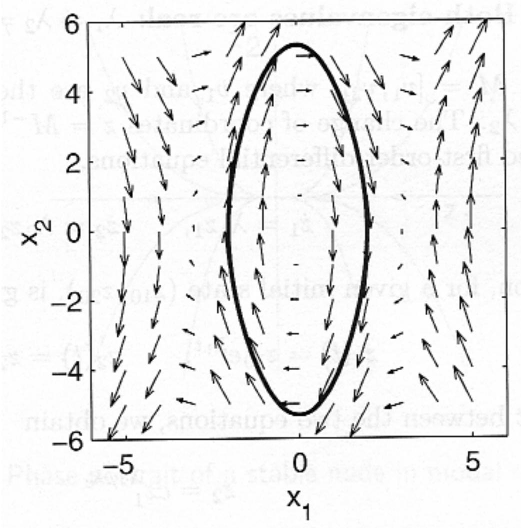
$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 2x_1^2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$x + f(x) = \begin{pmatrix} 1+2 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Vector Field, Phase Portrait - 1

- Vector field diagram:
  Repeat the above at every point.
  Example (pendulum equation w/o friction)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10\sin x_1\end{aligned}$$

see Fig. 2.2



Vector Field, Phase Portrait - 2

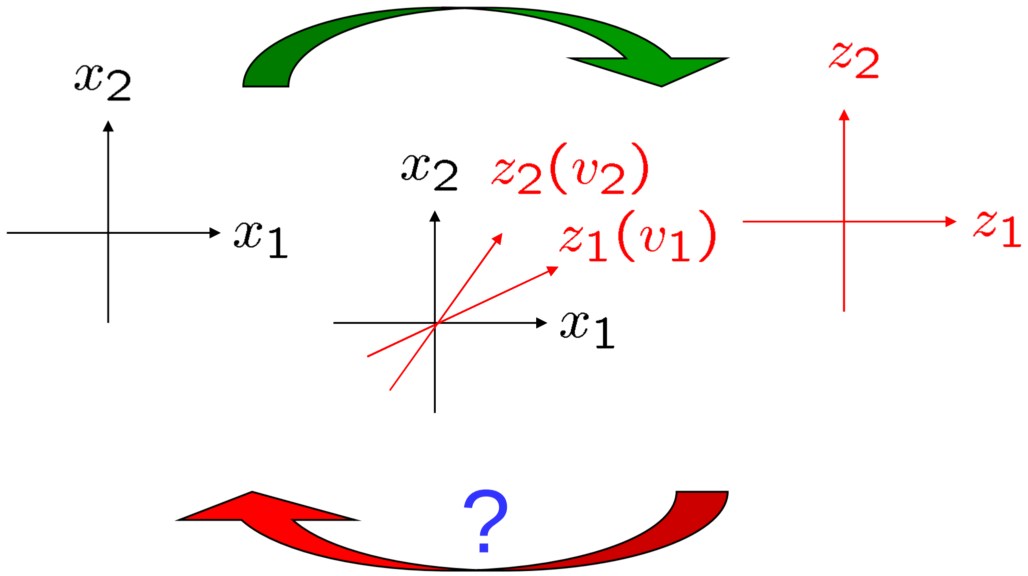
- The length of the arrow at a given point is proportional to the length of $f(x)$, that is, $\sqrt{f_1^2(x) + f_2^2(x)}$
- The vector field at a point is tangent to the trajectory through that piont.
- So, $x_0 \to f(x_0) \to x_a \to f(x_a) \to x_b$
- Phase portrait of (2,1)-(2.2): the family of all trajectories or solution curves.
- Since the time t is suppressed in a trajectory, a trajectory gives only the qualitative, but not quantitative, behavior of the associated solution.

Qualitative Behavior of Linear Systems (2.1)

Linearization
at the E.P.

Change of Coordinate
$z = M^{-1}x$
$J_r = M^{-1}AM$

- Nonlinear Systems: $\dot{x} = f(x)$ $\Longrightarrow$
- Linear Systems: $\dot{x} = Ax$ $\Longrightarrow$
- In z-coordinate: $\dot{z} = J_r z$



Qualitative Behavior of Linear Systems

- Consider the linear time-invariant systems:

$$\dot{x} = Ax, \quad A \in R^{2\times 2}$$

- Solution for a given initial state x_0 is:

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

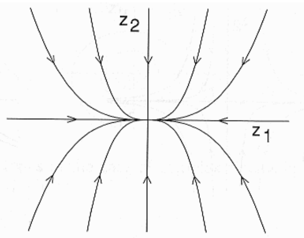
  where
  - J_r is the real Jordan form of A,
  - M is a real nonsingular matrix such that $M^{-1}AM = J_r$.

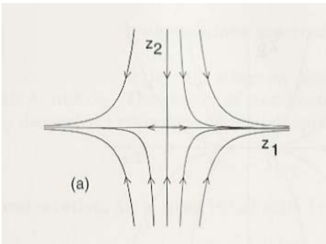
- Depending on the eigenvalues $\lambda_{1,2}$ of A, the real Jordan form may take one of three forms:
  - $\lambda_{1,2}$ are real and distinct: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
  - $\lambda_{1,2}$ are real and equal: $\begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}$, k is either 0 or 1.
  - $\lambda_{1,2} = \alpha \pm j\beta$ are complex: $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

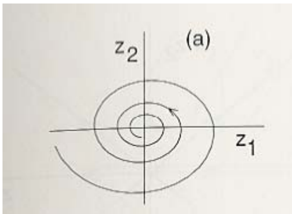
Qualitative Behavior of Linear Systems

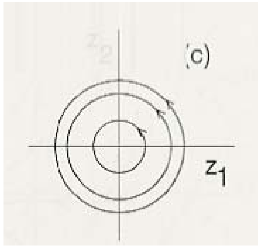
- Consider the following 4 cases:
  1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$.
  2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$.
  3. Nonzero multiple eigenvalues: $\lambda_1 = \lambda_2 = \lambda \neq 0$.
  4. One or both eigenvalues are zero.

- Qualitative behavior of equilibrium points:
  - node: stable and unstable
  - saddle:
  - focus: stable and unstable
  - center:









Case 1: Both eigenvalues are REAL: λ_1 \= λ_2 \= 0

- $M = [v_1, v_2]$, v_i is the real eigenvectors associated with λ_i.

- By the chage of coordinates $z = M^{-1}x$, it becomes a system of two decoupled first-order differential equations:

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2$$

- Solution (with initial state (z_{10}, z_{20})) is:

$$z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

- Eliminating t, we obtain:

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$$

- The phase portrait of the system is given by the family of curves generated from the above equation.

- The shape of the phase portrait depends on the signs of λ_i.

Case 1.1: Both eigenvalues are NEGATIVE - 1

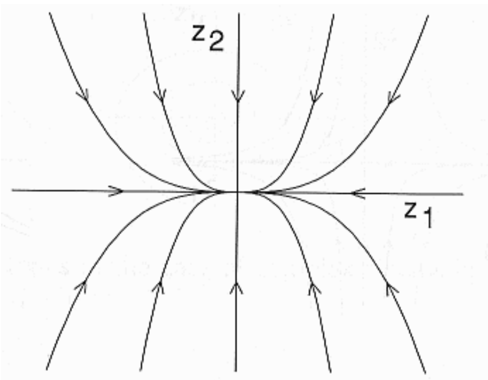
- λ_1, λ_2 are negative (let $\lambda_2 < \lambda_1 < 0$):
  - Both e^{λ_i} tend to zero as $t \to \infty$.
  - e^{λ_2} tends to zero faster than e^{λ_1} because $\lambda_2 < \lambda_1 < 0$.
  - $e^{\lambda_2}(v_2)$: fast eigenvalue (eigenvector), $e^{\lambda_1}(v_1)$: slow eigenvalue (eigenvector).
  - $\lambda_2/\lambda_1 > 1$
  - The slope of the curve:

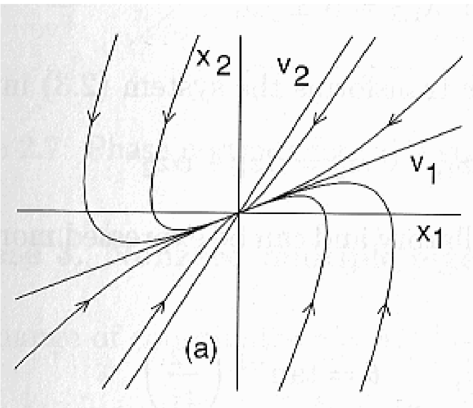
$$\frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

  - $(\lambda_2/\lambda_1) - 1 > 0$:
    the slope $\to 0$ as $|z_1| \to 0$,
    (tangent to the z_1 axis),
    the slope $\to \infty$ as $|z_1| \to \infty$
    (parallel to the z_2 axis).

Case 1.1: Both eigenvalues are NEGATIVE - 2

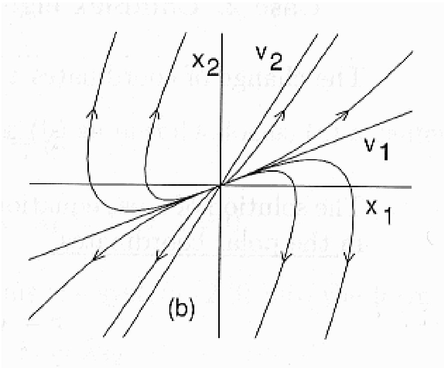
- λ_1, λ_2 are negative (let $\lambda_2 < \lambda_1 < 0$):
  - In the $x_1 - x_2$ plane, trajecotry tagent to the slow eigenvector v_1 as approach the origin, trajecotry parallel to the fast eigenvector v_2 as far from the origin.
  - $x = 0$ is called a stable node.





Case 1.2: Both eigenvalues are POSITIVE

- λ_1, λ_2 are positive:
  - The phase portrait will retain the charcter of the case of stable node,
  - but with the trajectory directions reversed,
  - since $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially as t increases.
  - $x = 0$ is an unstable node.

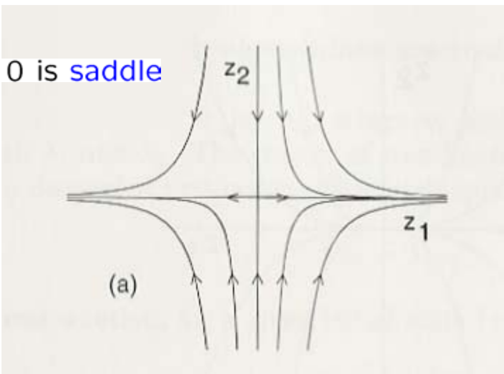


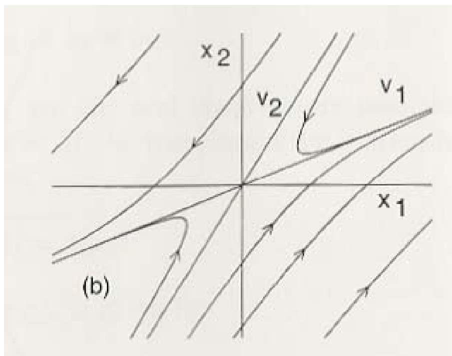
Case 1.3: Eigenvalues have OPPOSITE signs

- λ_1, λ_2 have oppositive signs:
  (let $\lambda_2 < 0 < \lambda_1$)
  - As $t \to 0$, $e^{\lambda_1 t} \to \infty$, while $e^{\lambda_2 t} \to 0$
  - λ_2: the stable eigenvalue
    λ_1: the unstable eigenvalue
  - v_2: the stable eigenvector
    v_1: the unstable eigenvector

tagent to the z_1-axis as $|z_1| \to \infty$
tagent to the z_2-axis as $|z_1| \to 0$

$x = 0$ is saddle





two trajecotries along z_2-axis are stable
two trajecotries along z_1-axis are unstable

Case 2: COMPLEX eigenvalues - 1

- $\lambda_{1,2} = \alpha \pm j\beta$
- By the chage of coordinates $z = M^{-1}x$:

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \quad \dot{z}_2 = \beta z_1 + \alpha z_2$$

- the solution is oscillatory
- in the polar coordinate:

$$r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

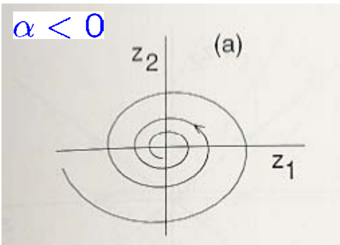
$$\dot{r} = \alpha r \quad \text{and} \quad \dot{\theta} = \beta$$

- Solution for a given initial state (r_0, θ_0) is:

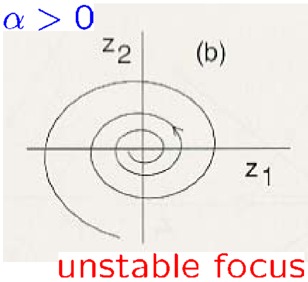
$$r(t) = r_0 e^{\alpha t}, \quad \theta(t) = \theta_0 + \beta t$$

- that is, a logarithmic spiral in the $z_1 - z_2$ plane

Case 2: COMPLEX eigenvalues - 2

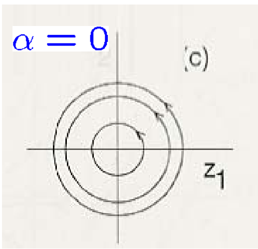


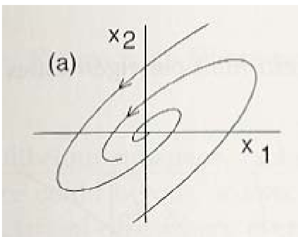
stable focus

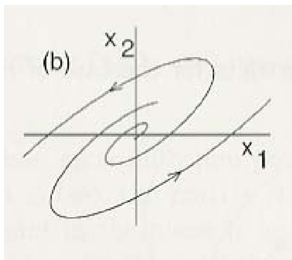


unstable focus

center







Case 3: NONZERO multiple eigenvalues - 1

- $\lambda_1 = \lambda_2 = \lambda \neq 0$

- By the chage of coordinates $z = M^{-1}x$:

$$\dot{z}_1 = \lambda z_1 + k z_2, \quad \dot{z}_2 = \lambda z_2$$

- Solution for a given initial state (z_{10}, z_{20}) is:

$$z_1(t) = e^{\lambda t}(z_{10} + k z_{20} t), \quad z_2(t) = e^{\lambda t} z_{20}$$

- Eliminating t, we obtain:

$$z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln\left(\frac{z_2}{z_{20}}\right) \right]$$

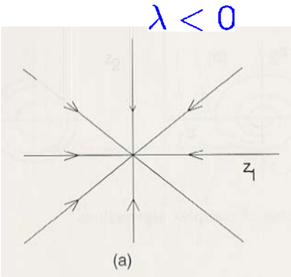
- Do not have the asymptotic slow-fast behavior

- The global qualitative behavior of the system is determined by the type of equilibrium point. This is a characteristic of linear systems, but not of nonlinear systems.

Case 3: NONZERO multiple eigenvalues - 2

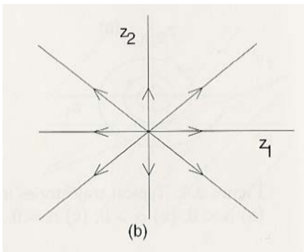
$k = 0$

$\lambda < 0$



stable node

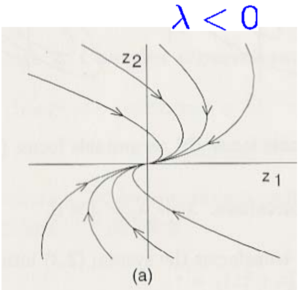
$\lambda > 0$



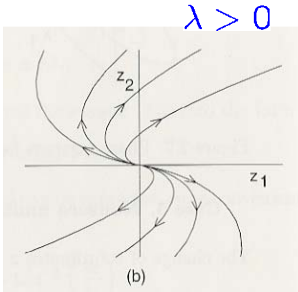
unstable node

$k = 1$

$\lambda < 0$



$\lambda > 0$



Case 4: One or both eigenvalues are ZERO - 1

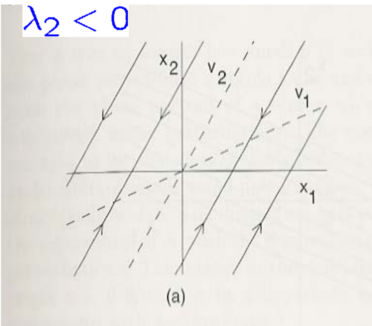
- key characters:
  - the phase portrait is degenate
  - A has a nontrivial null space
  - equilibrium point → equilibrium subspace
- $\lambda_1 = 0, \lambda_2 \neq 0$ (& dim of null space = 1):
  - v_1 spans the null space of A
  - transformed system:

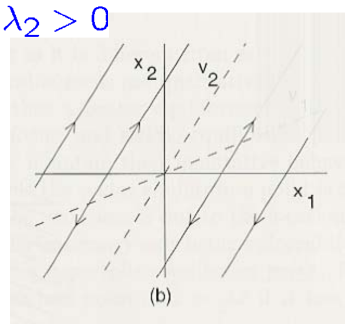
$$\dot{z}_1 = 0, \quad \dot{z}_2 = \lambda_2 z_2$$

- Solution is:

$$z_1(t) = z_{10}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

- $e^{\lambda_2 t}$: grow or decay, depending on the sign of λ_2





Case 4: One or both eigenvalues are ZERO - 2

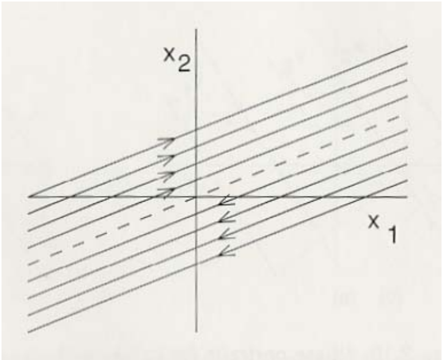
- $\lambda_1 = 0, \lambda_2 = 0$ (& dim of null space = 1):
  - transformed system:

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = 0$$

- Solution is:

$$z_1(t) = z_{10} + z_{20}t, , \quad z_2(t) = z_{20}$$

- $z_{20}t$: increase or decrease, depending on the sign of z_{20}



Linearization at E.P. - 1 (2.3)

- Consider the state model:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- E.P.: $p = (p_1, p_2)$.

- f_1, f_2 are continuously differentiable.

- Expand the RHS into its Taylor series about p:

$$\dot{x}_1 = f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + H.O.T.$$

$$\dot{x}_2 = f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + H.O.T.$$

where

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x_1=p_1, x_2=p_2}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x_1=p_1, x_2=p_2},$$

$$a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{p_1, p_2}, \qquad a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{p_1, p_2}$$

Linearization at E.P. - 2

- Since (p_1, p_2) is an E.P., we have $f_1(p_1, p_2) = f_2(p_1, p_2) = 0$

- Let $y_1 = x_1 - p_1, y_2 = x_2 - p_2$ analyze the trajectory near (p_1, p_2).

- New state equation:

$$\begin{aligned}\dot{y}_1 = \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + H.O.T.\\ \dot{y}_2 = \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + H.O.T.\end{aligned}$$

that is, if we only consider a sufficiently small neighborhood of the E.P. (H.O.T.=0)

$$\begin{aligned}\dot{y}_1 &= a_{11}y_1 + a_{12}y_2\\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2\end{aligned} \qquad \dot{y} = Ay$$

where

$$A = \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Jacobian Matrix

- $\frac{\partial f}{\partial x}$ is called the Jacobian matrix of $f(x)$
  A is the Jacobian matrix evaluated at the E.P. p.

- If the origion of the linearized state equation is

(1) a stable/unstable node with distinct eigenvalues,

(2) a stable/unstable focus, or

(3) a saddle point,

- Then in a small neighborhood of the E.P., the trajectories of the nonlinear state equation
  will behave like

(1) a stable/unstable node,

(2) a stable/unstable focus, or

(3) a saddle point.

Perturbed Linear System -> Nonlinear System - 1

- How conclusive the linearization approach is depends to a great extent on how the various qualitative phase portraits of a linear system persis under perturbations.

- Let's examine the special case of linear perturbations.

- Suppose A has distinct eigenvalues
  consider $A + \Delta A$
  ΔA: 2×2 real matrix
  its elememts have arbitrarily small magnitudes.

- From the purterbation theory of matrices, the eigenvalues of a matrix depend continuously on its parameters.

Perturbed Linear System -> Nonlinear System - 2

- That is, given an $\epsilon > 0$,
  exist a corresponding $\delta > 0$
  the magnitude of the perturbation $< \delta$,
  the eigenvalues of $A + \Delta A$ will lie in B_ϵ,
  $B_\epsilon =$ open discs of radius ϵ centered at the
  the eigenvalues of A.

- Hence, after arbitrarily small perturbations,
  eigenvalues of A
  in open RHP remain in open RHP
  in open LHP remain in open LHP

- However, when perturbated,
  eigenvalues on the imaginary axis might
  go into either the RHP or LHP.

Perturbed Linear System -> Nonlinear System - 3

- If the equilibrium points $x = 0$ of $\dot{x} = Ax$ is
  a node, focus, or saddle point,
  then the equilibrium point $x = 0$ of
  $\dot{x} = (A + \Delta A)x$ will be of the same type
  for sufficiently small perturbations.

- It is quite different
  if the equilibrium point is a center.

- The node, focus, and saddle equilibrium
  points are said to be structurally stable,
  while the center equilibrium point is not.

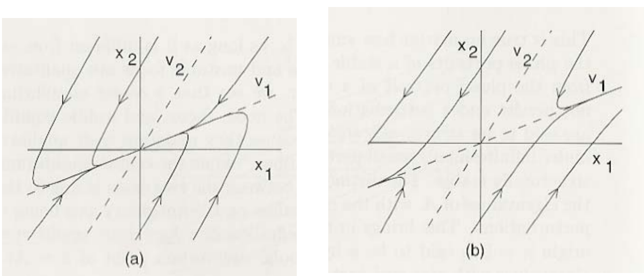
- Hyperbolic equilibrium point:
  If A has no eigenvalues with zero real part,
  $x = 0$ is said to be a hyperbolic equilibrium
  point of $\dot{x} = Ax$.

Perturbed Linear System - 1

- A has multiple nonzero real eigenvalues:
  - Infinitesimally small perturbations $\rightarrow$ a pair of complex eigenvalues.
  - A stable or unstable node would either remain a stable or unstable node or become a stable or unstable focus.
- A has eigenvalues at zero:
  - Perturbations will move these eigenvalues away from zero, resulting a major change in the phase portrait.

Perturbed Linear System - 2

- Only one eigenvalue at zero:
  - $\lambda = 0 \rightarrow \lambda_1 = \mu$
    μ is a positive or negative real number.
  - $|\lambda_1| = |\mu|$ is much smaller than $|\lambda_2|$.
  - $\rightarrow$ a node or saddle point, depending on the signs of λ_2 and μ.
  - Since $|\lambda_1| << |\lambda_2|$, $e^{\lambda_2 t}$ changes much faster than $e^{\lambda_1 t}$ a typical phase portraits of a node and a saddle. See Fig.
  - $\mu < 0$: $x = 0$ a stable node
  - $\mu > 0$: $x = 0$ a saddle point.



Perturbed Linear System - 3

- Both eigenvalues are zeros:
  - more dramatic
  - consider the 4 possible cases of the Jordan form:

    $$\begin{bmatrix} 0 & 1 \\ -\mu^2 & 0 \end{bmatrix}, \begin{bmatrix} \mu & 1 \\ -\mu^2 & \mu \end{bmatrix}, \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} \mu & 1 \\ 0 & -\mu \end{bmatrix}$$

  - μ: positive or negative perturbation parameter
  - The equilibrium points in these 4 cases are a center, a focus, a node and a saddle point, respectively.